2 Natural Deduction for finite-valued logics

Segerberg presented a general completeness proof for propositional logics. For this purpose, a deductive system was defined in a way that its rules were rules for an arbitrary k-place Boolean operator in a given propositional logic. Each of those rules corresponds to a row on the operator's truth-table. This chapter extends Segerberg's idea to finite-valued propositional logic. We maintain the idea of defining a deductive system whose rules correspond to rows of truth-tables, but instead of having n types of rules (one for each truth-value), we use a bivalent representation that makes use of the technique of separating formulas as defined by Carlos Caleiro and João Marcos.

2.1 Introduction

Many-valued logic has been extensively studied since the early papers of Łukasiewicz, who proposed a many-valued logic as an answer to traditional philosophical questions posed to the usual bi-valued logic by the nature of contingent futures, especially those related to logical determinism, as Aristotle's famous sea-battle problem. By introducing a value that is neither true nor false, the so called $\frac{1}{2}$ or 'possible' truth value, Łukaciewicz proposes a way to escape from the traps of logical determinism. He also defined n-valued ($n > 3$) logics as natural extensions of his 3-valued logic presented in 1922. Of course one can discuss the real impact of Łukaciewicz proposed solution to logical determinism, but there's no doubt that his proposal of a more general truth-valued propositional logic is important by itself: he introduced the concept of logical matrix, a concept used by whoever tried to define and use many-valued logics. Emil Post wrote on many-valued logic in his thesis. His motivation was mathematical and was related to functional completeness of m-valued truth- functions, the so called Post's theorem stating necessary and sufficient conditions for a set of 2-valued truth-functions to be complete. Bochvar (1939) and Kleene (1938) independently used many-valued logic as a way to analyse logical paradoxes, by means of the possible assignment of a 'paradoxical' truth value to propositions. Another very important result in the area was obtained by Gödel with his proof that intuitionistic logic cannot be taken as a finitely valued logic.

Apart from its philosophical and mathematical importance, many-valued logics have provided a vast field of study in model theory and proof-theory. The definition of a complete and sound deductive system for a class of many-valued logics can certainly be seen as a contribution for this vast field. As possible applications of the results presented here, it's worth mentioning the use of many- valued logics in computer science to deal with problems of epistemic gaps, paradoxical knowledge and degrees of believe. In section 2.6 we briefly review some recent results related to our approach.

In his 1983 paper, "Arbitrary truth-value functions and Natural Deduction" (19), Segerberg presented a general completeness proof for propositional logic based on Henkin's method. For "general" we mean that any fragment of the full propositional logic is a particular case of this theorem. For this purpose, Segerberg defined a deductive system whose rules are rules for an arbitrary k-place Boolean operator in a given propositional logic. The rules are defined in a way that each of them corresponds to a line on the operator's truth-table. In order to define this system, each line of the operator's truth-table were classified according to the operator's truth-value. Corresponding to each of this classification, the rules are of two kinds, one when the main operator is of type 0, in which case the rules correspond to an elimination rule, and another when the main operator is of type 1, in which case the rules correspond to an introduction rule.

This chapter extends Segerberg's idea to finite-valued propositional logics. We maintain the idea of defining a deductive system whose rules correspond to lines of truth-tables, but instead of having n types of rules (one for each truth-value), we use a method provided by Carlos Caleiro and João Marcos (6) to reduce many-valued semantics to a bivalent one. The truth-values of many-valued logics can be separated in two classes: the class of designated values and the class of undesignated values and to define this reduction we need first to define a function t that goes from a set of truth-values to $\{0, 1\}$ by taking designated values to 1 and undesignated values to 0 is defined. Then, we have to find a one-place formula $\theta(P)$ such that $t(\theta(P))$ receives different values when different values in the same class are assigned to $\theta(P)$, i.e., the class to which the truth-value of $\theta(P)$ belongs when $v(P) = v_i$ is different from the class to which the truth-value of $\theta(P)$ belongs when $v(P) = v_j$, for $j \neq i$. With the aid of these formulas, known as *separating formulas*, a bivalent semantic can be constructed from a many-valued one.

This bivalent semantics allows us to construct the truth-table of a finite-valued k-place operator with only two truth-values. On the other hand, besides the variables of which the operator depends, the truth-value function depends also on separating formulas. Then, we take Segerberg's idea and define a deductive system by corresponding each line of the truth-table to a rule. This deductive system must have rules that cope with rules whose main formula are formulas of the form $\theta(\bigstar(A_1, \ldots, A_k))$, $\bigstar$ a k-place operator and A_i, $1 \leq i \leq k$, a formula.

2.2
The system $\mathbf{N_{mv}}$

Let $\mathcal{L}$ be a finite-valued propositional logic. We define now the system we are going to call $\mathbf{N_{mv}}$.

Let $\bigstar$ be a k-place finite-value operator and $A_1, \ldots A_k$ be formulas in $\mathcal{L}$ and let $\theta^1(P)$, $\theta^2(P)$, $\ldots$, $\theta^s(P)$ be the separating formulas defined to reduce the many-valued semantic of $\mathcal{L}$ to a bivalent semantic. We write $\theta^0(P)$ to represent the proposition P. For each $1 \leq r \leq k$, let $< I_r, J_r >$, be a partition of $\{0, 1, \ldots, s\}$. For $i_r \in I_r$ and $j_r \in J_r$, $\bigstar(A_1, \ldots, A_k)$ can be classified in two types with respect to the partitions:

type 0 $\bigstar(A_1, \ldots, A_k), \{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}$

type 1 $\{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}, \bigstar(A_1, \ldots, A_k)$

Note that not all partitions are used as each of the n truth-values of $\mathcal{L}$ is represented by only one of those partitions and there exists more than n partitions of $\{0, 1, \ldots, s\}$.

If $\bigstar(A_1, \ldots, A_k)$ is of type 0 with respect to the partitions $< I_r, J_r >$, for all $1 \leq r \leq k$, and $i_r \in I_r$ and $j_r \in J_r$, then,

$$\frac{\bigstar(A_1, \ldots, A_k) \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{array}{c}[\theta^{j_r}(A_r)] \\ \vdots \\ C(\text{all } j_r \in J_r)\end{array}}{C}\ (\langle I_r, J_r\rangle,\ 0)$$

If $\bigstar(A_1, \ldots, A_k)$ is of type 1 with respect to the partitions $< I_r, J_r >$, for $1 \leq r \leq k$ and $i_r \in I_r$ and $j_r \in J_r$, then,

$$\frac{\begin{array}{c}[\bigstar(A_1, \ldots, A_k)] \\ \vdots \\ C\end{array} \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{array}{c}[\theta^{j_r}(A_r)] \\ \vdots \\ C(\text{all } j_r \in J_r)\end{array}}{C}\ (\langle I_r, J_r\rangle,\ 1)$$

What was stated so far does not consist of the whole system. It can happen that one of the $A's$ above is of the form $\bigstar'(A'_1 \ldots A'_{k'})$, and then we need rules to cope with $\theta^t(\bigstar'(A'_1 \ldots A'_{k'}))$. As with $\bigstar(A_1, \ldots, A_k)$, $\theta^l(\bigstar(A_1, \ldots, A_k))$ for $0 \leq l \leq s$, is classified in two types with respect to the partitions. For all $1 \leq r \leq k$, and $i_r \in I_r$ and $j_r \in J_r$, $\bigstar(A_1, \ldots, A_k)$:

type 0 $\theta^l(\bigstar(A_1, \ldots, A_k)), \{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}$

type 1 $\{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}, \theta^l(\bigstar(A_1, \ldots, A_k))$

According to this classification, we define the following rules:

If $\theta^l(\bigstar(A_1, \ldots, A_k))$, $1 \leq l \leq s$, is of type 0 with respect to the partitions $< I_r, J_r >$, then,

$$\frac{\theta^l(\bigstar(A_1, \ldots, A_k)) \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{array}{c}[\theta^{j_r}(A_r)] \\ \vdots \\ C(\text{all } j_r \in J_r)\end{array}}{C} \; (\langle I_r, J_r \rangle,\, 0)$$

where $1 \leq r \leq k$.

If $\theta^l(\bigstar(A_1, \ldots, A_k))$, $1 \leq l \leq s$, is of type 1 with respect to the partitions $< I_r, J_r >$,

$$\frac{\begin{array}{c}[\theta^l(\bigstar(A_1, \ldots, A_k))] \\ \vdots \\ C\end{array} \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{array}{c}[\theta^{j_r}(A_r)] \\ \vdots \\ C(\text{all } j_r \in J_r)\end{array}}{C} \; (\langle I_r, J_r \rangle,\, 1)$$

where $1 \leq r \leq k$

Note that, as we are using $\theta^0(P)$ to designate P, by putting $l = 0$ in this last pair of formulas we get the first.

As said above, not all partitions are used to represent a truth-value. Let $\langle I, J \rangle$ be a partition of $\{0, 1, \ldots, s\}$ that do not represent any truth-value and P a formula in $\mathcal{L}$. According to those partitions, we define the following rules, that we call U-rules:

$$\frac{\theta^i(P)(\text{all } i \in I) \quad \begin{array}{c}[\theta^j(P)] \\ \vdots \\ C(\text{all } j \in J)\end{array}}{C} \; (\langle I, J \rangle,\, \text{U})$$

for every $i \in I$ and $j \in J$.

2.3 Soundness and Completeness

In this section we prove soundness and completeness with respect to the usual semantics for finitely many-valued logics. The completeness theorem is proved with the use of Hintikka sets and is an adaptation of the proof given in (19). As a matter of illustration we provide a natural deduction system for **Ł₃** as an instance of our approach. This example is shown in subsection 3.1.

Let Γ be a consistent set of formulas. We define the *degree of a formula* A, $G(A)$, to be the number of occurrences in A of logical symbols.

The formulas $\theta^i(A_r)$, $0 \leq i \leq s$ and $1 \leq r \leq k$, and $\theta^l(\bigstar(A_1,\ldots,A_k))$, l a number between 0 and s, are *sub-formulas* of $\theta^l(\bigstar(A_1,\ldots,A_k))$. Consequently, $G(\theta^i(A_r)) < G(\theta^l(\bigstar(A_1,\ldots,A_k)))$.

Theorem 2.3.1 *(Soundness) If* $\Gamma \vdash A$*, then* $\Gamma \vDash A$*.*

Proof: Assume that $\Gamma \vdash A$. We want to prove that

$$\Gamma \vDash A$$

Let v be any truth-value assignment for the set of boolean atoms such that $\bar{v}(C) = 1$ for all $C \in \Gamma$. The proof is by induction on the complexity of derivations. Let Π be a derivation of A from Γ and suppose the theorem holds for every derivation less complex than Π.

If A is the conclusion of an application of $(\langle I_r, J_r\rangle, 0)$, then, by induction hypotheses, there are sets of formulas Γ', $\Gamma_{ir}, \Gamma_{jr} \subset \Gamma$ such that

1. $\Gamma' \vDash \theta^l(\bigstar(A_1,\ldots,A_k))$
2. $\Gamma_{ir} \vDash \theta^{i_r}(A_r)$, for all $i_r \in I_r$
3. $\Gamma_{jr}, \theta^{j_r}(A_r) \vDash A$, for all $j_r \in J_r$

By 1. and 2., we have that

1'. $\bar{v}(\theta^l(\bigstar(A_1,\ldots,A_k))) = 1$

2'. $\bar{v}(\theta^{i_r}(A_r)) = 1$, for all $i_r \in I_r$

Suppose that $\bar{v}(A) = 0$. Then, by 3.,

3'. $\bar{v}(\theta^{j_r}(A_r)) = 0$, for all $j_r \in J_r$

It follows from 1'., 2'. and 3'. that $\theta^l(\bigstar(A_1,\ldots,A_k))$ is of type 1 with respect to $\langle I_r, J_r\rangle$, which is contrary to our assumption. Consequently, $\bar{v}(A) = 1$, i.e., $\Gamma \vDash A$.

The case were A is the conclusion of an application of $(\langle I_r, J_r\rangle, 1)$ is analogous.

Suppose A is the conclusion of an application of $(\langle I, J\rangle, U)$. Then, the partition $\langle I, J\rangle$ does not define any truth-value of $\mathcal{L}$. By induction hypotheses, there exist $\Gamma_i, \Gamma_j \subseteq \Gamma$ such that

1. $\Gamma_i \vDash \theta^i(B)$, for all $i \in I$

2. $\Gamma_j, \theta^j(B) \vDash A$, for all $j \in J$

Then,

1'. $\bar{v}(\theta^i(B)) = 1$, for all $i \in I$

Suppose that $\bar{v}(A) = 0$. By 2.,

2'. $\bar{v}(\theta^j(B)) = 0$, for all $j \in J$.

The information given by 1'. and 2'. is unobtainable from the initial truth-values of $\mathcal{L}$. A tuple so defined does not represent any truth-value in $\mathcal{L}$.

Hence, $\bar{v}(A) = 1$ and $\Gamma \vDash A$. ■

Theorem 2.3.2 *(Completeness) If* $\Gamma \vDash A$, *then* $\Gamma \vdash A$.

Proof: Suppose $\Gamma \nvdash A$. Then there exists Γ^* such that $\Gamma \subseteq \Gamma^*, \Gamma^* \nvdash A$ and $\Gamma^* \cup \{B\} \vdash A$, $B \notin \Gamma^*$. Define a particular truth-value assignment v for the set of atoms: $v(p) = 1$ iff $p \in \Gamma^*$. The theorem now is reduced to the claim

$$\bar{v}(B) = 1 \text{ iff } B \in \Gamma^*.$$

With this condition we show that $\Gamma^* \nvDash A$, for $A \notin \Gamma^*$ ($\Gamma^* \nvdash A$).

The proof is by induction on the degree of B. The basic step of the induction is taken care of by the definition of v. Suppose B is of the form $\theta^l(\bigstar(A_1, \ldots, A_k))$.For each $1 \leq r \leq k$, put $I_r = \{i_r : \theta^{i_r}(A_r) \in \Gamma^*\}$ and $J_r = \{j_r : \theta^{j_r}(A_r) \notin \Gamma^*\}$. Then, $\langle I_r, J_r\rangle$ are partitions of $\{0, 1, \ldots, k\}$. By assumption,

1. $\theta^{i_r}(A_r) \in \Gamma^*$, that is, $\Gamma^* \vdash \theta^{i_r}(A_r)$, for each $i_r \in I_r$

2. $\theta^{j_r}(A_r) \notin \Gamma^*$, that is, $\Gamma^* \cup \{\theta^{j_r}(A_r)\} \vdash A$, for each $j_r \in J_r$

If one of those partitions, say $\langle I_t, J_t\rangle$, does not define a truth-value of $\mathcal{L}$, then by applying $(\langle I_t, J_t\rangle, U)$, by 1. and 2., we get that $\Gamma^* \vdash A$, what goes against our initial supposition. Hence, all partitions as defined above define truth-values of $\mathcal{L}$.

By the induction hypothesis, $\bar{v}(\theta^{i_r}(A_r)) = 1$, for all $i_r \in I_r$ and $\bar{v}(\theta^{j_r}(A_r)) = 1$, for all $j_r \in J_r$. Consequently, $\bar{v}(\theta^l(\bigstar(A_1, \ldots, A_k))) = 1$ iff $\theta^l(\bigstar(A_1, \ldots, A_k))$ is of type 1 with respect to the partitions. Hence, we need to prove that

$\bar{v}(\theta^l(\bigstar(A_1, \ldots, A_k))) = 1$ *iff* $\theta^l(\bigstar(A_1, \ldots, A_k))$ *is of type 1 with respect to the partitions* $\langle I_r, J_r \rangle$, *for all* $1 \leq r \leq k$.

Suppose that $\theta^l(\bigstar(A_1, \ldots, A_k)) \in \Gamma^*$. Then

3. $\Gamma^* \vdash \theta^l(\bigstar(A_1, \ldots, A_k))$

If $\theta^l(\bigstar(A_1, \ldots, A_k))$ were of type 0 with respect to the partitions, then $\Gamma^* \vdash A$ (by 1., 2. and 3.) what is against our assumption. Consequently, $\theta^l(\bigstar(A_1, \ldots, A_k))$ is of type 1 with respect to the partitions

Suppose that $\theta^l(\bigstar(A_1, \ldots, A_k)) \notin \Gamma^*$. Then

3'. $\Gamma^* \cup \{\theta^l(\bigstar(A_1, \ldots, A_k))\} \vdash A$

If $\theta^l(\bigstar(A_1, \ldots, A_k))$ were of type 1 with respect to the partitions, then $\Gamma^* \vdash A$ (by 1., 2. and 3'.), what is against our assumption. Consequently, $\theta^l(\bigstar(A_1, \ldots, A_k))$ is of type 0 with respect to the partitions. ■

2.3.1 Example: the trivalent logic Ł$_3$

The truth-value set of the Ł$_3$ logic is composed of three elements $0, i, 1$, where 0 and i are undesignated values and 1 is a designated value. Hence, we can define a function t that takes both 0 and i to 0 and 1 to 1. For implication, the truth-table is as follows

P	Q	$P \to Q$
1	1	1
1	i	i
1	0	0
i	1	1
i	i	1
i	0	i
0	1	1
0	i	1
0	0	1

A separating formula for $\mathbf{Ł_3}$ can be $\theta(P) = \neg P \rightarrow P$, for when P has truth-value 0, $\theta(P)$ has truth-value 0 (undesignated) and when P has truth-value i, $\theta(P)$ has truth-value 1 (designated). In this case, we only need one separating formula.

Now that we have the function t and the separating formula, we can reduce the semantic of $\mathbf{Ł_3}$ to a bivalent semantic. Let v be the truth-value function of $\mathbf{Ł_3}$. Instead of writing $v(P \rightarrow Q) = 1$ when $v(P) = 1$ and $v(Q) = 1$, we will write $v(P \rightarrow Q) = 1$ when $v(P) = 1, v(\theta(P)) = 1, v(Q) = 1$ and $v(\theta(Q)) = 1$; instead of writing $v(P \rightarrow Q) = i$ when $v(P) = 1$ and $v(Q) = i$, we will write $v(P \rightarrow Q) = 0$ when $v(P) = 1, v(\theta(P)) = 1, v(Q) = 0$ and $v(\theta(Q)) = 1$ and so on. Thus, the bivalent truth-table is as follows

P	$\theta(P)$	Q	$\theta(Q)$	$P \rightarrow Q$
1	1	1	1	1
1	1	0	1	0
1	1	0	0	0
0	1	1	1	1
0	1	0	1	1
0	1	0	0	0
0	0	1	1	1
0	0	0	1	1
0	0	0	0	1

According to the definition of $\mathbf{N_{mv}}$, the above truth-table yields nine rules:

$$\dfrac{P \rightarrow Q \quad P \quad \theta(P) \quad \begin{matrix}[Q] \\ \vdots \\ C\end{matrix} \quad \theta(Q)}{C}\,(1) \qquad \dfrac{P \rightarrow Q \quad P \quad \theta(P) \quad \begin{matrix}[Q] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\theta(Q)] \\ \vdots \\ C\end{matrix}}{C}\,(2)$$

$$\dfrac{P \rightarrow Q \quad \begin{matrix}[P] \\ \vdots \\ C\end{matrix} \quad \theta(P) \quad \begin{matrix}[Q] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\theta(Q)] \\ \vdots \\ C\end{matrix}}{C}\,(3) \qquad \dfrac{\begin{matrix}[P \rightarrow Q] \\ \vdots \\ C\end{matrix} \quad P \quad \theta(P) \quad Q \quad \theta(Q)}{C}\,(4)$$

$$\frac{\begin{array}{ccccc}[P \to Q] & [P] & & & \\ \vdots & \vdots & & & \\ C & C & \theta(P) & Q & \theta(Q)\end{array}}{C}\,(5) \qquad \frac{\begin{array}{ccccc}[P \to Q] & [P] & & [Q] & \\ \vdots & \vdots & & \vdots & \\ C & C & \theta(P) & C & \theta(Q)\end{array}}{C}\,(6)$$

$$\frac{\begin{array}{ccccc}[P \to Q] & [P] & [\theta(P)] & & \\ \vdots & \vdots & \vdots & & \\ C & C & C & Q & \theta(Q)\end{array}}{C}\,(7) \qquad \frac{\begin{array}{ccccc}[P \to Q] & [P] & [\theta(P)] & [Q] & \\ \vdots & \vdots & \vdots & \vdots & \\ C & C & C & C & \theta(Q)\end{array}}{C}\,(8)$$

$$\frac{\begin{array}{ccccc}[P \to Q] & [P] & [\theta(P)] & [Q] & [\theta(Q)] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C & C & C & C & C\end{array}}{C}\,(9)$$

Now, it can happen that the formula P is of the form $A \to B$, and then there is need of rules that cope with $\theta(A \to B)$. As the truth-value of $\theta(A \to B)$ depends solely on the truth-value of $A \to B$, we can construct the truth-table of $A \to B$ to construct the truth-table of $\theta(A \to B)$.

A	$\theta(A)$	B	$\theta(B)$	$A \to B$	$\theta(A \to B)$
1	1	1	1	1	1
1	1	0	1	0	1
1	1	0	0	0	0
0	1	1	1	1	1
0	1	0	1	1	1
0	1	0	0	0	1
0	0	1	1	1	1
0	0	0	1	1	1
0	0	0	0	1	1

The construction of the rules are analogously to what was made above. We have the following rules:

$$\dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad A \quad \theta(A) \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \theta(B)}{C}\,1', \qquad \dfrac{\theta(A\to B) \quad A \quad \theta(A) \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(B)]\\ \vdots\\ C\end{matrix}}{C}\,2'$$

$$\dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \theta(A) \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(B)]\\ \vdots\\ C\end{matrix}}{C}\,3', \qquad \dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad A \quad \theta(A) \quad B \quad \theta(B)}{C}\,4'$$

$$\dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \theta(A) \quad B \quad \theta(B)}{C}\,5', \qquad \dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \theta(A) \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \theta(B)}{C}\,6'$$

$$\dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(A)]\\ \vdots\\ C\end{matrix} \quad B \quad \theta(B)}{C}\,7', \qquad \dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(A)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \theta(B)}{C}\,8'$$

$$\dfrac{\begin{matrix}[\theta(A\to B)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(A)]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[\theta(B)]\\ \vdots\\ C\end{matrix}}{C}\,9'$$

Moreover, as the pair $\langle 1, 0\rangle$ does not define any truth-value, we have a single U-rule:

$$\dfrac{P \qquad \begin{matrix}[\theta(P)]\\ \vdots\\ C\end{matrix}}{C}\,U$$

As an example of how the system works, we are going to show the proof of two lemmas of Ł$_3$, viz. $A \to (B \to A)$ and $((A \to B) \to ((B \to C) \to (A \to C)))$. Because of the size of the proofs, we need to prove separately sub-derivations of these proofs (and sub-derivations of these sub-derivations). Whenever we prove a relation $\text{ł}_n\,[x] : \Gamma \vdash C$ we will say that Π^n_X is a proof of C from Γ.

Lemma 2.3.1 *1)* $\text{ł}_1\,[P] : P \vdash \theta(P)$

2) $\text{ł}_2 \begin{bmatrix} P \\ Q \end{bmatrix} : \theta(P), Q, \theta(Q) \vdash P \to Q$

3) $\text{ł}_3 \begin{bmatrix} P \\ Q \end{bmatrix} : P, Q, \theta(Q) \vdash P \to Q$

4) $\text{ł}_4 \begin{bmatrix} Q \\ P \end{bmatrix} : Q \vdash P \to Q$

5) $\text{ł}_5 \begin{bmatrix} Q \\ P \end{bmatrix} : Q \vdash \theta(P \to Q)$

6) $\text{ł}_6 \begin{bmatrix} P \\ Q \end{bmatrix} : \theta(P), \theta(Q) \vdash \theta(P \to Q)$

7) $\text{ł}_7 \begin{bmatrix} Q \\ P \end{bmatrix} : \theta(Q) \vdash \theta(P \to Q)$

8) $\text{ł}_8 \begin{bmatrix} P \\ Q \end{bmatrix} : P, P \to Q \vdash Q$

9) $\text{ł}_9 \begin{bmatrix} P \\ Q \end{bmatrix} : P, \theta(P \to Q) \vdash \theta(Q)$

10) $\text{ł}_{10} \begin{bmatrix} P \\ Q \end{bmatrix} : \theta(P), P \to Q \vdash \theta(Q)$

11) $\text{ł}_{11} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : P, P \to Q, Q \to R \vdash P \to R$

12) $\text{ł}_{12} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : \theta(P), P \to Q, Q \to R \vdash P \to R$

13) $\text{ł}_{13} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : \theta(R), P \to Q, Q \to R \vdash P \to R$

14) $\text{ł}_{14} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : P \to Q, Q \to R \vdash P \to R$

15) $\text{ł}_{15} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : P \to Q, \theta(Q \to R) \vdash \theta(P \to R)$

16) $\text{ł}_{16} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} : \theta(P \to Q), Q \to R \vdash \theta(P \to R)$

Proof:

1)

$$\dfrac{P \qquad [\theta(P)]^U}{\theta(P)}U$$

2)

$$\dfrac{[P \to Q]^5 \qquad \dfrac{[P \to Q]^4 \quad [P]^5 \quad \theta(P) \quad Q \quad \theta(Q)}{P \to Q}\,4 \qquad \theta(P) \quad Q \quad \theta(Q)}{P \to Q}5$$

3)

$$\dfrac{\begin{array}{c}[\theta(P)]^U\ Q\ \theta(Q)\\ \Pi^2\\ P \to Q\end{array} \qquad P}{P \to Q}{}_U$$

4)

$$\dfrac{Q \qquad \dfrac{[P \to Q]^7 \qquad \begin{array}{c}[P]^7\ \ Q\ \ [\theta(Q)]^U\\ \Pi^3\\ P \to Q\end{array} \qquad \begin{array}{c}[\theta(P)]^7\ \ Q\ \ [\theta(Q)]^U\\ \Pi^2\\ P \to Q\end{array} \qquad Q \quad [\theta(Q)]^U}{P \to Q}7}{P \to Q}{}_U$$

5)

$$\dfrac{\begin{array}{c}Q\\ \Pi^4_{Q,P}\\ P \to Q\end{array} \qquad [\theta(P \to Q)]^U}{\theta(P \to Q)}U$$

6)

$$
\dfrac{[\theta(P \to Q)]^{6'} \qquad \dfrac{[P]^{6'} \quad \theta(P) \quad \theta(P \to Q) \quad \begin{matrix} [\theta(P \to Q)]^{1'} \\ [Q]^{1'} \\ \Pi^5_{Q,P} \\ \theta(Q) \end{matrix}}{\theta(P \to Q)}\,1\text{'} \qquad \theta(P) \quad \theta(P \to Q) \quad \begin{matrix} [Q]^{6'} \\ \Pi^5_{Q,P} \\ \theta(Q) \end{matrix}}{\theta(P \to Q)}\,6'
$$

7)

$$
\dfrac{[\theta(P \to Q)]^{8'} \qquad \dfrac{[P]^{8'} \quad \begin{matrix} [\theta(P)]^{U} \quad \theta(Q) \\ \Pi^6 \\ \theta(P \to Q) \end{matrix}}{\theta(P \to Q)}\,\mathrm{U} \qquad \begin{matrix} [\theta(P)]^{8'} \quad \theta(Q) \\ \Pi^6 \\ \theta(P \to Q) \end{matrix} \qquad \begin{matrix} [Q]^{8'} \\ \Pi^4_{Q,P} \\ P \to Q \quad \theta(Q) \end{matrix}}{\theta(P \to Q)}\,8'
$$

8)

$$
\dfrac{\dfrac{P \to Q \quad P \quad [\theta(P)]^{U} \qquad \dfrac{\begin{matrix} P \to Q \quad P \quad [\theta(P)]^{U} \\ [Q]^1 \quad [\theta(Q)]^3 \end{matrix}}{Q}\,1 \qquad \dfrac{\dfrac{\begin{matrix} P \\ P \to Q \quad P \quad [\theta(P)]^{U} \\ [Q]^1 \quad [\theta(Q)]^{U} \end{matrix}}{Q}\,1 \qquad [Q]^3}{Q}\,\mathrm{U}}{Q}\,3}{Q}\,U
$$

9)

$$
\dfrac{P \qquad \dfrac{\theta(P \to Q) \quad P \quad [\theta(P)]^{U'} \qquad \dfrac{[Q]^{2'} \quad [\theta(Q)]^{U}}{\theta(Q)}\,\mathrm{U} \qquad [\theta(Q)]^{2'}}{\theta(Q)}\,2\text{'}}{\theta(Q)}\,U'
$$

10)

$$\dfrac{P \to Q \qquad \dfrac{\begin{array}{c} P \to Q \quad [P]^3 \\ \Pi^8 \\ Q \end{array} \qquad [\theta(Q)]^U}{\theta(Q)}\,\mathrm{U} \qquad \theta(P) \qquad \dfrac{[Q]^3 \quad [\theta(Q)]^U}{\theta(Q)}\,\mathrm{U} \qquad [\theta(Q)]^3}{\theta(Q)}\,3$$

11) Easily derived from ł$_8 \begin{bmatrix} P \\ Q \end{bmatrix}$, ł$_8 \begin{bmatrix} Q \\ R \end{bmatrix}$ and ł$_4 \begin{bmatrix} R \\ P \end{bmatrix}$

12)

$$\dfrac{[P \to R]^6 \qquad \begin{array}{c} [P]^6 \\ P \to Q \quad Q \to R \\ \Pi^{11} \\ P \to R \end{array} \qquad \theta(P) \quad \begin{array}{c} [R]^6 \\ \Pi^4_{R,P} \\ P \to R \end{array} \qquad \begin{array}{c} \theta(P) \quad P \to Q \\ \Pi^{10}_{P,Q} \\ \theta(Q) \qquad\qquad Q \to R \\ \Pi^{10}_{Q,R} \\ \theta(R) \end{array}}{P \to R}\,6$$

13)

$$\dfrac{[P \to R]^8 \qquad \begin{array}{c} [P]^8 \\ P \to Q \quad Q \to R \\ \Pi^{11} \\ P \to R \end{array} \quad \begin{array}{c} [\theta(P)]^8 \\ P \to Q \quad Q \to R \\ \Pi^{12} \\ P \to R \end{array} \quad \begin{array}{c} [R]^8 \\ \Pi^4_{R,P} \\ P \to R \end{array} \quad \theta(R)}{P \to R}\,8$$

14)

$$\dfrac{\begin{array}{c} [P]^9 \\ P \to Q \quad Q \to R \\ \Pi^{11} \\ P \to R \end{array} \quad \begin{array}{c} [P \to R]^9 \\ [\theta(P)]^9 \\ P \to Q \quad Q \to R \\ \Pi^{12} \\ P \to R \end{array} \quad \begin{array}{c} [R]^9 \\ \Pi^4_{R,P} \\ P \to R \end{array} \quad \begin{array}{c} [\theta(R)]^9 \\ P \to Q \quad Q \to R \\ \Pi^{13} \\ P \to R \end{array}}{P \to R}\,9$$

15) e 16) Similar to what was already shown.

∎

Theorem 1 $Ł_1 : \ A \to (B \to A)$

Proof:

- $\text{ł}_\alpha : A \vdash A \to (B \to A)$ is the composition of $\text{ł}_4 \begin{bmatrix} A \\ B \end{bmatrix}$ and $\text{ł}_4 \begin{bmatrix} B \to A \\ A \end{bmatrix}$
- $\text{ł}_\beta : \theta(A), \theta(B \to A) \vdash A \to (B \to A)$

$$\dfrac{[A \to (B \to A)]^7 \quad \begin{array}{c} [A]^7 \\ \Pi_\alpha \\ A \to (B \to A) \end{array} \quad \theta(A) \quad \begin{array}{c} [B \to A]^7 \\ \Pi^4_{B \to A, A} \\ A \to (B \to A) \end{array} \quad \theta(B \to A)}{A \to (B \to A)}\, 7$$

- $\text{ł}_\gamma : \theta(B \to A) \vdash A \to (B \to A)$

$$\dfrac{\begin{array}{c} [A]^8 \\ \Pi_\alpha \\ A \to (B \to A) \end{array} \quad \begin{array}{c} [A \to (B \to A)]^8 \\ [\theta(A)]^8 \;\; \theta(B \to A) \\ \Pi_\beta \\ A \to (B \to A) \end{array} \quad \begin{array}{c} [B \to A]^8 \\ \Pi^4_{B \to A, A} \\ A \to (B \to A) \end{array} \;\; \theta(B \to A)}{A \to (B \to A)}\, 8$$

- $\text{ł}_\delta : \theta(A) \vdash A \to (B \to A)$ is the composition of $\text{ł}_6 \begin{bmatrix} A \\ B \end{bmatrix}$ and ł_γ.

  Finally,

$$\dfrac{\begin{array}{c} [A]^8 \\ \Pi_\alpha \\ A \to (B \to A) \end{array} \;\; \begin{array}{c} [\theta(A)]^8 \\ \Pi_\delta \\ A \to (B \to A) \end{array} \;\; \begin{array}{c} [A \to (B \to A)]^8 \\ [B \to A]^8 \\ \Pi^4_{B \to A, A} \\ A \to (B \to A) \end{array} \;\; \begin{array}{c} \theta(B \to A) \\ \Pi_\gamma \\ A \to (B \to A) \end{array}}{A \to (B \to A)}\, 8$$

■

Theorem 2 $\text{Ł}_2 : \;(A \to B) \to ((B \to C) \to (A \to C))$

Proof: We put $\varphi = (B \to C) \to (A \to C)$

- $\text{ł}_\alpha : A \to B, B \to C \vdash (B \to C) \to (A \to C)$ is the composition of $\text{ł}_{14} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ and $\text{ł}_4 \begin{bmatrix} A \to C \\ B \to C \end{bmatrix}$.

– $\text{ł}_\beta : A \to B, \theta(B \to C) \vdash (B \to C) \to (A \to C)$

$$\dfrac{[\varphi]^6 \quad \begin{array}[b]{c} A \to B \ \ [B \to C]^6 \\ \Pi_\alpha \\ \varphi \end{array} \quad \theta(B \to C) \quad \begin{array}[b]{c} [A \to C]^6 \\ \Pi^4_{A \to C, B \to C} \\ \varphi \end{array} \quad \begin{array}[b]{c} A \to B \ \ \theta(B \to C) \\ \Pi^{15}_{A,B,C} \\ \theta(A \to C) \end{array}}{\varphi}\,6$$

– $\text{ł}_\gamma : A \to B, \theta(A \to C) \vdash (B \to C) \to (A \to C)$

$$\dfrac{[\varphi]^8 \quad \begin{array}[b]{c} A \to B \ \ [B \to C]^8 \\ \Pi_\alpha \\ \varphi \end{array} \quad \begin{array}[b]{c} A \to B \ \ [\theta(B \to C)]^8 \\ \Pi_\beta \\ \varphi \end{array} \quad \begin{array}[b]{c} [A \to C]^8 \\ \Pi^4_{A \to C, B \to C} \\ \varphi \end{array} \quad \theta(A \to C)}{\varphi}\,8$$

– $\text{ł}_\delta : A \to B \vdash (A \to B) \to ((B \to C) \to (A \to C))$

$$\begin{array}{c} [\varphi]^9 \\ \dfrac{\begin{array}[b]{c} A \to B \ \ [B \to C]^9 \\ \Pi_\alpha \\ \varphi \end{array} \quad \begin{array}[b]{c} A \to B \ \ [\theta(B \to C)]^9 \\ \Pi_\beta \\ \varphi \end{array} \quad \begin{array}[b]{c} [A \to C]^9 \\ \Pi^4_{A \to C, B \to C} \\ \varphi \end{array} \quad \begin{array}[b]{c} A \to B \ \ [\theta(A \to C)]^9 \\ \Pi_\gamma \\ \varphi \end{array}}{\varphi}\,9 \\ \Pi^4_{\varphi, A \to B} \\ (A \to B) \to \varphi \end{array}$$

– $\text{ł}_\epsilon : \theta(A \to B), B \to C \vdash \theta((B \to C) \to (A \to C))$ is the composition of $\text{ł}_{16} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ and $\text{ł}_7 \begin{bmatrix} A \to C \\ B \to C \end{bmatrix}$.

– $\text{ł}_\zeta : \theta(A \to B), \theta(B \to C) \vdash \theta((B \to C) \to (A \to C))$

$$\dfrac{[\theta(\varphi)]^{3'} \quad \begin{array}[b]{c} \theta(A \to B) \ \ [B \to C]^{3'} \\ \Pi_\epsilon \\ \theta(\varphi) \end{array} \quad \theta(B \to C) \quad \begin{array}[b]{c} [A \to C]^{3'} \\ \Pi^5_{A \to C, B \to C} \\ \theta(\varphi) \end{array} \quad \begin{array}[b]{c} [\theta(A \to C)]^{3'} \\ \Pi_\gamma \\ \theta(\varphi) \end{array}}{\theta(\varphi)}\,3'$$

– $\text{ł}_\eta : \theta(A \to B) \vdash \theta((B \to C) \to (A \to C))$

$$\begin{array}{c} [\theta(\varphi)]^{9'} \\ \dfrac{\begin{array}[b]{c} \theta(A \to B) \ \ [B \to C]^{9'} \\ \Pi_\epsilon \\ \theta(\varphi) \end{array} \quad \begin{array}[b]{c} \theta(A \to B) \ \ [\theta(B \to C)^{9'} \\ \Pi_\zeta \\ \theta(\varphi) \end{array} \quad \begin{array}[b]{c} [A \to C]^{9'} \\ \Pi^5_{A \to C, B \to C} \\ \theta(\varphi) \end{array} \quad \begin{array}[b]{c} [\theta(A \to C)]^{9'} \\ \Pi_\gamma \\ \theta(\varphi) \end{array}}{\theta(\varphi)}\,9' \end{array}$$

– $\text{ł}_\iota : \theta(A \to B) \vdash (A \to B) \to ((B \to C) \to (A \to C))$

$$\dfrac{[(A\to B)\to\varphi]^6 \quad \begin{array}{c}[A\to B]^6\\ \Pi_\delta\\ (A\to B)\to\varphi\end{array} \quad \theta(A\to B) \quad \begin{array}{c}[\varphi]^6\\ \Pi^4_{A\to B,\varphi}\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}\theta(A\to B)\\ \Pi_\eta\\ (A\to B)\to\varphi\end{array}}{(A\to B)\to\varphi}6$$

– $\text{ł}_\kappa : \theta(\varphi) \vdash (A \to B) \to ((B \to C) \to (A \to C))$

$$\dfrac{[(A\to B)\to\varphi]^8 \quad \begin{array}{c}[A\to B]^8\\ \Pi_\delta\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}[\theta(A\to B)]^8\\ \Pi_\iota\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}[\varphi]^8\\ \Pi^4_{A\to B,\varphi}\\ (A\to B)\to\varphi\end{array} \quad \theta(\varphi)}{(A\to B)\to\varphi}8$$

Finally,

$$\dfrac{[(A\to B)\to\varphi]^9 \quad \begin{array}{c}[A\to B]^9\\ \Pi_\delta\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}[\theta(A\to B)]^9\\ \Pi_\iota\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}[\varphi]^9\\ \Pi^4_{A\to B,\varphi}\\ (A\to B)\to\varphi\end{array} \quad \begin{array}{c}[\theta(\varphi)]^9\\ \Pi_\kappa\\ (A\to B)\to\varphi\end{array}}{(A\to B)\to\varphi}9$$

■

2.4 Normalization

In this section we prove the normalization of $\mathbf{N_{mv}}$. We start with some basic notions.

A *maximal segment* in a derivation is a sequence $A_1, \ldots, A_n$ of occurrences of formulas of the same form such that A_1 is the conclusion of an application of a rule of type 1 and A_n is the major premiss of an application of a rule of type 0.

A *maximal formula* is a maximal segment whose length is 1.

The *degree of a formula* A, $G(A)$, is the number of occurrences in A of logical symbols. The *degree of a segment* is the degree of the formula that occurs in the segment.

The *degree of a derivation* Π, $G(\Pi)$, is the highest degree of a maximal segment in Π. If Π does not have maximal segments, then $G(\Pi) = 0$.

The premiss $\theta^l(\bigstar(A_1, \ldots, A_k))$ of a rule of type 0 and the premises $\theta^i(P)$ of a U-rule are called *major premises*. The premises C of a rule of type 0, of a U-rule and of a rule of type 1 are called c-premises. The other premises are called *minor premises.*

The premiss $\theta^l(\bigstar(A_1, \ldots, A_k))$ discharged by a rule of type 1 is called *major discharged premiss.* The other premises discharged by a rule of type 1 are called *minor discharged premises.*

A *branch* of a derivation Π is a sequence $A_1, \ldots, A_n$ of formula occurrences in Π such that

1. A_1 is a top-formula in Π;
2. A_{i+1} stands immediately above A_i;
3. A_n is either the first formula occurrence in the sequence that is a minor premise or the end-formula of Π if there is no such minor premise in the sequence.

The *order* of a branch is defined as follows: a branch that ends with the end-formula of the derivation has order 0. A branch that ends with a minor premise is assigned the order $n+1$ if the side formula that is a major premise belongs to a branch of order n.

A *critical derivation* in $\mathbf{N_{mv}}$ is a derivation Π such that, if $G(\Pi) = g$, then the last inference of Π has a maximal premiss with degree g, and for every sub-derivation Σ of Π, $G(\Sigma) < G(\Pi)$.

A derivation Π is in *U-form* if the major premise of any application of a rule of type 0 in Π is not the conclusion of an application of U-rule and the conclusion of any application of a rule of type 1 in Π is not a major premise of an application of U-rule.

A *simplified derivation* is a derivation where no major premiss of a rule of type 1 is a major discharged premiss of a rule of type 1.

A derivation is *normal* if it does not have maximal segments and it is a simplified derivation in U-form.

Lemma 2.4.1 *Every derivation* Π *can be transformed into a derivation in U-form.*

Proof: Let $\langle I_r, J_r\rangle$, $0 \leq r \leq k$ be a partition of $s+1$ and $\langle I'_a, J'_a\rangle$ be a partition that does not represent any truth-value.

Case 1: The conclusion of a U-rule is the major premiss of a rule of type 0.

Then Π is of the form

$$
\dfrac{\dfrac{\begin{array}{c}\Pi_{i'a}\\ \theta^{i'_a}(B_a)\end{array} \quad \begin{array}{c}[\theta^{j'_a}(B_a)]^U\\ \Pi_{j'a}\\ \theta^l(\bigstar(A_1,\ldots,A_k))\end{array}}{\theta^l(\bigstar(A_1,\ldots,A_k))}\,U \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{array} \quad \begin{array}{c}[\theta^{j_r}(A_r)]^0\\ \Pi_{jr}\\ C\end{array}}{C}\,0
$$

Π can be transformed into a derivation of the form

$$
\dfrac{\begin{array}{c}\Pi_{i'a}\\ \theta^{i'_a}(B_a)\end{array} \quad \dfrac{\begin{array}{c}[\theta^{j'_t}(B_t)]^U\\ \Pi_{j't}\\ \theta^l(\bigstar(A_1,\ldots,A_k))\end{array} \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{array} \quad \begin{array}{c}[\theta^{j_r}(A_r)]^0\\ \Pi_{jr}\\ C\end{array}}{C}\,0\text{, for each } j'_t \in J'_a}{C}\,U
$$

Case 2: A major premiss of a U-rule is the conclusion of a rule of type 1. Then Π is of the form

$$
\dfrac{\dfrac{\begin{array}{c}[\theta^l(\bigstar(A_1,\ldots,A_k))]^1\\ \Pi_l\\ C\end{array} \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(B_r)\end{array} \quad \begin{array}{c}[\theta^{j_r}(B_r)]^1\\ \Pi_{jr}\\ C\end{array}}{C}\,1 \quad \begin{array}{c}\Pi_{i'a}\\ \theta^{i'_a}(B_a)\end{array} \quad \begin{array}{c}[\theta^{j'_a}(B_a)]^U\\ \Pi_{j'a}\\ D\end{array}}{D}\,U
$$

where C is of the form $\theta^{i'_t}(A_t)$, for some $i'_t \in I'_a$. Π can be transformed into a derivation of the form

$$
\dfrac{\dfrac{\begin{array}{c}[\theta^l(\bigstar(A_1,\ldots,A_k))]^1\\ \Pi_l\\ C\end{array} \quad \begin{array}{c}\Pi_{i'a}\\ \theta^{i'_a}(B_a)\end{array} \quad \begin{array}{c}[\theta^{j'_a}(B_a)]^U\\ \Pi_{j'a}\\ D\end{array}}{D}\,U \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(B_r)\end{array} \quad \begin{array}{c}[\theta^{j_t}(B_t)]^1\\ \Psi_{jt}\\ D\end{array}}{D}\,1
$$

where $\Psi_{jt} \equiv \dfrac{\begin{array}{c}\theta^{j_t}(B_t)\\ \Pi_{jt}\\ C\end{array} \quad \begin{array}{c}\Pi_{i'a}\\ \theta^{i'_a}(B_a)\end{array} \quad \begin{array}{c}[\theta^{j'_a}(B_a)]^U\\ \Pi_{j'a}\\ D\end{array}}{D}\,U$, for all $j_t \in J_r$

■

Lemma 2.4.2 *Every derivation can be transformed into a simplified derivation.*

Proof: Let Π be a derivation of the form

$$
\dfrac{\dfrac{[\theta^{l}(\bigstar(A_1,\ldots,A_k))]^1 \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{array} \quad \begin{array}{c}[\theta^{j_r}(A_r)]^0\\ \Pi_{jr}\\ C\end{array}}{C}\,0 \qquad \begin{array}{c}\Pi_{i'r}\\ \theta^{i'_r}(A_r)\end{array} \qquad \begin{array}{c}[\theta^{j'_r}(A_r)]^1\\ \Pi_{j'r}\\ C\end{array}}{C}\,1
$$

As the rules for $\theta^{l}(\bigstar(A_1,\ldots,A_k))$ are defined by the partitions of $\{0,1,\ldots,s\}$, we have that one of the following cases hold:

Case 1: there exists a formula $\theta^{u_v}(A_v)$, $u_v \in I_r$, of the same form of a formula discharged by a rule of type 1. Then Π can be transformed into a derivation of the form $\begin{array}{c}\Pi_{uv}\\ (\theta^{u_v}(A_v))\\ \Pi_{j'r}\\ C\end{array}$.

Case 2: there exists a formula $\theta^{u_v}(A_v)$, $u_v \in I'_r$, of the same form of a formula discharged by a rule of type 0. Then Π can be transformed into a derivation of the form $\begin{array}{c}\Pi_{uv}\\ (\theta^{u_v}(A_v))\\ \Pi_{jr}\\ C\end{array}$.

■

Note that if a simplified derivation is transformed into a derivation Π in U-form, Π remains a simplified derivation.

Now we are going to prove the lemma known as *Critical Lemma*, which is used to prove the Normalization Theorem.

Lemma 2.4.3 *(Critical Lemma) If Π is a critical simplified derivation in U-form of C from Γ, then Π can be transformed into a derivation Π' such that $G(\Pi') < G(\Pi)$.*

Proof: Let Π be a simplified critical derivation in U-form of C from Γ. The proof is by induction on the pair $\langle \#G(\Pi), \ell(\Pi)\rangle$, where $\#G(\Pi)$ is the number of formulas in the segment of degree $G(\Pi)$ in Π and $\ell(\Pi)$ is the length of Π.

Suppose Π has one application of $(\langle I_r, J_r\rangle, 1)$ followed by $(\langle I'_r, J'_r\rangle, 0)$, where $\langle I_r, J_r\rangle$ and $\langle I'_r, J'_r\rangle$ are partitions of $\{0,1,\ldots,s\}$ with respect to the formula A_r. Π has the form:

$$
\dfrac{\dfrac{\begin{array}{c}[\theta^{l}(\bigstar(A_1,\ldots,A_k))]^1\\ \Pi_l\\ \theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))\end{array} \quad \begin{array}{c}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{array} \quad \begin{array}{c}[\theta^{j_r}(A_r)]^1\\ \Pi_{jr}\\ \theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))\end{array}}{\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))}\,1 \qquad \begin{array}{c}\Pi'_{i'r}\\ \theta^{i'_r}(B_r)\end{array} \qquad \begin{array}{c}[\theta^{j'_r}(B_r)]^0\\ \Pi'_{j'r}\\ C\end{array}}{C}\,0
$$

We have three cases to consider:

Case 1: There exists a $m_n \in I_r$ and $m'_{n'} \in J'_r$ such that $\theta^{m_n}(A_n) = \theta^{m'_{n'}}(B_{n'})$ in Π. Then Π can be transformed into a derivation Π' of the form

$$\begin{array}{c}\Pi_{mn}\\(\theta^{m_n}(A_n))\\\Pi'_{m'n'}\\C\end{array}$$

As $\theta^{m_n}(A_n)$ is a sub-formula of $\theta^l(\bigstar(A_1,\ldots,A_k))$, $G(\theta^{m_n}(A_n))$¡ $G(\theta^l(\bigstar(A_1,\ldots,A_k)))$ and $max\{G(\Pi_{mn}), G(\Pi_{m'n'})\} < G(\theta^l(\bigstar(A_1,\ldots,A_k)))$, for Π is a critical derivation. Thus, $G(\Pi') < G(\Pi)$.

Case 2: There exists a $m_n \in J_r$ and $m'_{n'} \in I'_r$ such that $\theta^{m_n}(A_n) = \theta^{m'_{n'}}(B_{n'})$ in Π. Then Π can be transformed into a simplified derivation in U-form $\Pi^\sharp$ of the form

$$\dfrac{\begin{array}{c}\Pi_{m'n'}\\(\theta^{m'_{n'}}(A_{n'}))\\\Pi_{mn}\\\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))\end{array}\quad\begin{array}{c}\Pi_{i'r}\\\theta^{i'_r}(B_r)\end{array}\quad\begin{array}{c}[\theta^{j'_r}(B_r)]^0\\\Pi_{j'r}\\C\end{array}}{C}\,0$$

If $\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))$ is a maximal formula in $\Pi^\sharp$, then $\#G(\Pi^\sharp) < \#G(\Pi)$ and, by induction hypothesis, $\Pi^\sharp$ can be transformed into a derivation Π' such that $G(\Pi') < G(\Pi^\sharp) = G(\Pi)$. If $\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))$ is not a maximal formula in $\Pi^\sharp$, then $\Pi' = \Pi^\sharp$ and $G(\Pi') < G(\Pi)$.

Case 3: Neither the conditions of cases 1 and 2 hold. Then Π can be transformed into a derivation $\Pi^\sharp$ of the form

$$\dfrac{\begin{array}{c}[\theta^l(\bigstar(A_1,\ldots,A_k))]^1\\\Psi_1\\C\end{array}\quad\begin{array}{c}\Pi_{ir}\\\theta^{i_r}(A_r)\end{array}\quad\begin{array}{c}[\theta^{j_r}(A_r)]^1\\\Psi_{jr}\\C\end{array}}{C}\,1$$

where $\Psi_1 = \dfrac{\begin{array}{c}\theta^l(\bigstar(A_1,\ldots,A_k))\\\Pi_l\\\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))\end{array}\quad\begin{array}{c}\Pi_{i'r}\\\theta^{i'_r}(B_r)\end{array}\quad\begin{array}{c}[\theta^{j'_r}(B_r)]^0\\\Pi_{j'r}\\C\end{array}}{C}\,0$ and $\Psi_{jr} =$

$$\dfrac{\begin{array}{c}\theta^{j_r}(A_r)\\\Pi_{jr}\\\theta^{l'}(\bigstar'(B_1,\ldots,B_{k'}))\end{array}\quad\begin{array}{c}\Pi_{i'r}\\\theta^{i'_r}(B_r)\end{array}\quad\begin{array}{c}[\theta^{j'_r}(B_r)]^0\\\Pi_{j'r}\\C\end{array}}{C}\,0$$

, for all $j_r \in J_r$.

It is easy to see that no new maximal formulas are created by this transformation and that the sub-derivations Ψ_1 and Ψ_{jr}, $j_r \in J_r$, that are not normal, are critical, simplified and in U-form and have degree $G(\Pi)$. As these critical derivations are smaller than Π, by induction hypothesis they can be transformed into derivations Λ_1 and Λ_{jr}, respectively, such that $G(\Lambda_1) < G(\Psi_1)$ and $G(\Lambda_{jr}) < G(\Psi_{jr})$. By substituting each occurrence of the critical sub-derivations Ψ_1 and Ψ_{jr} in $\Pi^\sharp$ by Λ_1 and Λ_{jr}, respectively, we achieve a derivation Π'. If Π' is a simplified derivation in U-form, then Π' is the wanted derivation. If not, we just need to apply lemmas 2.4.1 and 2.4.2 to achieve the wanted derivation.

■

Theorem 1 *(Normalization theorem) Every derivation of C from Γ can be reduced to a normal derivation of C from Γ.*

Proof: Let Π be a derivation of C from Γ. By lemma 2.4.2, Π can be transformed into a simplified derivation Π'. By lemma 2.4.1, Π' can be transformed into a simplified derivation Π_0 in U-form. The proof of this Normalization theorem is by induction on the pair $\langle \ell(\Pi), \#G(\Pi)\rangle$. Choose a critical sub-derivation Σ such that $G(\Sigma) = G(\Pi)$. By the critical lemma (lemma 2.4.3), Σ can be transformed into a derivation Σ_0 such that $G(\Sigma_0) < G(\Sigma)$. Let Π_1 be the result of substituting Σ' for Σ in Π_0 and let A be the end-formula of Σ. Note that it does not have an effect on maximal segments of degree $G(\Pi)$ except, possibly, for segments containing the formula A. We have that, either A is not a maximal premise, or it is, in which case it was already a maximal premiss in Π_0 and $G(A) < G(\Pi_0)$. Hence, $G(\Pi_1) < G(\Pi_0)$ and, by induction hypothesis, Π_1 can be transformed into a normal derivation of C from Γ. ■

2.5 Some Further Results

As consequence of the Normalization Theorem, we have the usual results.

The first result states that, in each branch of a derivation, it cannot happen that a rule of type 0 occurs after a rule of type 1.

Corollary 2.5.1 *Let Π be a normal derivation in $\mathbf{N_{mv}}$ and let $\beta = A_1, \ldots, A_n$ be a sequence of formulas in Π. Then, there exist formula occurrences $A_i, \ldots A_j$ in β, $1 \leq i, j \leq n$, called the* minimum segment *which separates β into the following (possibly empty) parts:*

- *a part called the* 0-part *of β, where each A_r, $r < i$, is a major premiss of a rule of type 0 and contains A_{r+1} as a sub-formula;*
- *A_i, if $i \neq n$, is a premiss of a rule of type 1 or of a U-rule;*
- *a part called the* U-part *of β, where each A_s, $i < s < j$, is a major premiss of a U-rule and has the same form that A_{s+1};*
- *A_j, if $j \neq n$, is a premiss of a rule of type 1;*
- *and a part called the* 1-part *of β, where each A_t, $j < t < n$, is a premiss of a rule of type 1 and is a sub-formula of A_{t+1}.*

The sub-formula principle states that every formula that occurs in a derivation has the same form of either a sub-formula of a top-formula or of the conclusion. As with natural deduction rule for the elimination of disjunction, this sub-formula structure of normal derivations can be lost with rules of type 0 and in a similar way those derivations can be transformed with the following permutation:

$$\dfrac{\dfrac{\begin{matrix}\Pi_l\\ \theta^l(\bigstar(A_1,\ldots,A_k))\end{matrix} \quad \begin{matrix}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{matrix} \quad \begin{matrix}[\theta^{j_r}(A_r)]^0\\ \Pi_{jr}\\ \theta^{l'}(\bigstar'(B_1,\ldots,B'_k))\end{matrix}}{\theta^{l'}(\bigstar'(B_1,\ldots,B'_k))}\,0 \quad \begin{matrix}\Pi_{i'r}\\ \theta^{i'_r}(B_r)\end{matrix} \quad \begin{matrix}[\theta^{j'_r}(B_r)]^{0'}\\ \Pi_{j'r}\\ C\end{matrix}}{C}\,0'$$

$\triangleright$

$$\dfrac{\begin{matrix}\Pi_l\\ \theta^l(\bigstar(A_1,\ldots,A_k))\end{matrix} \quad \begin{matrix}\Pi_{ir}\\ \theta^{i_r}(A_r)\end{matrix} \quad \dfrac{\begin{matrix}[\theta^{j_t}(A_t)]^0\\ \Pi_{jt}\\ \theta^{l'}(\bigstar'(B_1,\ldots,B'_k))\end{matrix} \quad \begin{matrix}\Pi_{i'r}\\ \theta^{i'_r}(B_r)\end{matrix} \quad \begin{matrix}[\theta^{j'_r}(B_r)]^{0'}\\ \Pi_{j'r}\\ C\end{matrix}}{C}\,0'}{C}\,0$$

for each $j_t \in J_r$.

Corollary 2.5.2 *(Sub-formula Principle) Every formula occurrence in a normal derivation of C from Γ is a sub-formula of a formula in $\Gamma \cup \{C\}$.*

Proof: Let Π be a normal derivation of C from Γ and $\beta = A_1, \ldots, A_n$ be a branch in Π of order p. From corollary 2.5.1, we have that every formula in β is a sub-formula of either A_1 or A_n. Hence, we need to prove that A_1 and A_n are sub-formulas of $\Gamma \cup \{C\}$.

We have that either A_1 belongs to Γ or A_1 was discharged by an application r of

– a rule of type 0, in which case A_1 is a sub-formula of the major premiss M of r. We have that either $M \in \Gamma$ or M is a minor discharged premise, in which case M is a sub-formula of a formula that belongs to a branch of order less than p.

– a U-rule, in which case A_1 is a sub-formula of the minor premises of r, which belongs to a branch of order less than p,

– a rule of type 1, in which case A_1 is either a minor premiss of a rule of type 0 or of a U-rule, or A_1 belongs to the 1-part of the derivation and is a sub-formula of A_n.

Concerning A_n, we have that either A_n is the conclusion of Π or A_n is a minor premiss of an application r of a rule of type 0, in which case A_n is a sub-formula of the major premiss of r, which belongs to a branch of order $p - 1$.

■

A deduction system S is *consistent* if there exists a formula β such that $\nvdash_S \beta$. The next corollary states that the system here presented is consistent.

Corollary 2.5.3 $\mathbf{N_{mv}}$ *is consistent.*

Proof: Suppose that $\mathbf{N_{mv}}$ is not consistent and let ϕ be a formula of $\mathcal{L}$. Hence, there exists a normal proof Π of ϕ. As it holds for any ϕ, it holds for $\phi = B$, B an atomic formula. By 2.5.1, we have that B is either the conclusion of a rule of type 0 or of a U-rule. In either case, it is easy to see that Π has at least one hypothesis that was not discharged. Therefore, Π is not a proof. ■

2.6 Related Works

In the literature of deduction systems for non-classical logics and extensions of classical logics, we can find two main classes of deductive systems: internal or external based. The first one deals only with formulas as they are defined by the logic itself. The second type involves the manipulation of non-logical elements, such as signed or labeled formulas and constraints marks and controls. The use of Kripke worlds indexing in tableaux for modal logics is an example of externally based deductive system, while the sequent calculus for the same modal logic can be an example of an internally based one. We believe that the fewer external signs and controls a deductive system has, the better it is from the proof-theoretical point of view. Of course, when implementing a theorem prover, many more controls and marks are needed to take care of proving strategies and efficiency. We should not mix these implementation mechanisms with marks and labels needed to ensure completeness and soundness of the 'pure' deductive system.

Our approach is internally based, since the natural deduction system obtained is defined using only the original language of the logic. In the sequel we comment other approaches to obtain deductive systems for finite-valued logics.

In (1) it is reported how to obtain complete and sound sequent calculi for the finite-valued Łukasiewicz logics. The sequent calculus deals with multi-sequents reminding a hypersequent approach. Considering the Łukasiewicz logic $\mathcal{L}_{n+1}$ of $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, $n > 1$, truth-values, a sequent for $\mathcal{L}_{n+1}$ is of the form

$$\Gamma_1 \vdash \Delta_1 | \Gamma_2 \vdash \Delta_2 | \ldots | \Gamma_n \vdash \Delta_n$$

where each $\vdash$ is related to the respective truth-value in $\mathcal{L}_n$.

This form of the sequents works because of the well-known fact that states that 'the truth of many-valued formulas in $\mathcal{L}_{n+1}$ can be reduced to the truth of n classical logic formulas'. That is, for each formula ϕ of $\mathcal{L}_{n+1}$, there is a n-tuple $\phi^1, \ldots, \phi^n$ of bi-valued classical formulas stating that each ϕ^i takes $\frac{i}{n}$ as truth-value. This is called *boolean decomposition* of $\mathcal{L}_{n+1}$. In fact, for each n, there are n mapping functions B^i_n, $i = 1, n$, that maps each ϕ to its corresponding ϕ^i.

The mapping from a usual sequent $\Gamma \vdash \Delta$ to the form above can be obtained by means of the B^i_n. The sequent rules are, in this way, localized by the respective truth value. For example, when applying a $\neg$-right rule to a premiss having $\Gamma_i, A \vdash \Delta_i$ as its i-th component, we obtain $\Gamma_{n-i} \vdash \Delta_{n-i}, \neg A$.

In this sequent calculus, a sequent is valid, iff, for every interpretation (valuation) v for $\mathcal{L}_n$, there is a component $\Gamma_i \vdash \Delta_i$, such that $v(\bigwedge_{\gamma \in \Gamma_i} B^i_n(\gamma)) \leq v(\bigvee_{\delta \in \Delta_i} B^i_n(\delta))$ holds. There is a cut-rule for each pair i, j with $i \leq j$. The elimination of the cut is not proved in the article. However, the article shows that the cut rule is not needed by proving completeness without taking the cut rule into account. Comparing with our approach, it does not provide a natural deduction system for the Łukasiewicz logics and the approach is restricted to logics satisfying boolean decompositions like Łukasiewicz logics, that is, there must be formulas B^i_n, $i = 1, n$ and a way to express $i \leq j$ in boolean terms. Finally, the similarity to hypersequents does not allow a simple way to obtain a natural deduction system from the sequent calculus.

In (5) Baaz, Fermüller and Zach propose a systematic way to obtain natural deduction systems for arbitrary finite-valued first order logics. The main idea behind their strategy is as follows:

1. Take the truth-tables for the many-valued logic under consideration.

2. Extract from these truth-tables a sequent calculus.

3. Show that the sequent calculus is sound and complete with respect to the intended semantics.

4. Cut-elimination follows from the completeness proof (method of reduction trees used by Schütte).

5. Extract from the sequent calculus rules introduction and elimination rules that define a natural deduction system for the same logic.

6. Soundness and completeness of the natural deduction system is a natural consequence of soundness and completeness for the corresponding sequent calculus.

7. The normal form theorem is obtained through cut-elimination for the sequent calculus: take a derivation Π of $\Gamma \vdash \Delta$ in the natural deduction system. By soundness, there is a cut-free sequent calculus proof Π' of $\Gamma \Rightarrow \Delta$. The proof Π' can now be translated into a normal derivation $\Pi^\star$ of $\Gamma \vdash \Delta$ in the natural deduction system.

The main differences between this result and ours are:

– We do not use sequent calculus as a step to produce natural deduction systems for many-valued logics.

– Soundness and completeness are proved directly for the natural deduction systems.

– The natural deduction systems defined in (5) are multiple conclusion systems, while our natural deduction systems are single conclusion systems.

– We prove the normalization theorem directly for our systems and not just a normal form theorem based on cut-free proofs in sequent calculus.

The bivalent representation used here and presented in (20, 6), was used to define tableaux systems for arbitrary finite-valued logics. After reducing the many-valued semantics of a truth-table to a bivalent one with the help of the separating formulas, the idea to define the rules is very simple: the rules can be "read" straight from the truth-table. For instance, in (6), one of the rules to handle $Ł_3$ implication is

$$
\begin{array}{ccc}
 & F(A \to B) & \\
\diagup & | & \diagdown \\
T(A) & T(A) & F(A) \\
| & | & | \\
T(\theta(A)) & T(\theta(A)) & T(\theta(A)) \\
| & | & | \\
F(B) & F(B) & F(B) \\
| & | & | \\
T(\theta(B)) & F(\theta(B)) & F(\theta(B))
\end{array}
$$

The reader just needs to compare this rule with the three lines in which the truth-value of $A \to B$ is 0 (which generates three elimination rules in $\mathbf{N_{mv}}$) on the bivalent truth-table of implication presented in subsection 2.3.1 to understand how it was defined. Besides the rules for introduction and elimination of the operators, $\mathbf{N_{mv}}$ has rules (the U-rules) defined by tuples that do not define any of the truth-values in question. In (6), these tuples define closure rules for the tableaux.

2.7 Conclusion

The main results of this chapter are:

- A systematic way to extract natural deduction systems for finitely-many-valued logics from truth-tables;
- A proof that the systems so defined are sound and complete with respect to the intended semantics;
- A proof that the systems so defined satisfies the normalization theorem.

Our approach is based on a combination of techniques introduced by Segerberg in (19) and the method of internal decoding of binary prints defined in (6). It's worth mentioning again that the proof-theoretical analysis is carried out in a very systematic way directly for natural deduction without any resource to auxiliary systems.

A future work we would like to mention is the improvement of the complexity of the natural deduction systems. For example, in the (6, 20) we find a method to reduce both the number of rules and the number of premises in each rule (when possible). We believe these methods can be used with our approach. For example, according to the method presented in (6), the rules for elimination of the Ł$_3$ implication presented in subsection 2.3.1 (rules 1, 2 and 3) can be replaced by the following two rules:

$$\dfrac{P \to Q \quad \theta(P) \quad \begin{matrix}[Q] \\ \vdots \\ C\end{matrix}}{C} \qquad \dfrac{P \to Q \quad \theta(P) \quad \begin{matrix}[\theta(Q)] \\ \vdots \\ C\end{matrix}}{C}$$

In the next chapter we extend the result shown here to non-deterministic finite-valued logics.