3
Natural Deduction for non-deterministic finite-valued logics

3.1
Introduction

This chapter extends the last chapter on finite-valued propositional logics to non-deterministic finite-valued propositional logics. In chapter 2, a framework for defining natural deduction systems for finite-valued propositional logics was defined based on the methods shown in (19) and (6) to reduce a many-valued semantics to a bivalent one. In a non-deterministic logic, a k-ary constant may have more than one truth-value assigned for a given vector $\langle x_1, \dots, x_k \rangle$. This kind of semantics has been recently used for formalizing some logics in a more concise and elegant way[1] (see (4, 2)).

In order to use the framework described in chapter 2, we must define a rule for evey possible interpretation a formula may have. Given a k-ary operator $\bigstar$, we are able to define rules for the introduction and elimination of $\bigstar$ based on its Nmatrix. The Natural Deduction system produced is shown to be complete and sound with respect to $\mathcal{L}$.

A different framework is described in (3) to obtain Sequent Calculus for this logics. One difference between our approach and those described in (3) is that we do not pass by n-sequents to achieve provability in ordinary sequent calculus. On the other hand one may say that we produce more rules than (3) when both frameworks are applied to the same logic.

3.2
Non-deterministic finite-valued basic system for single conclusion logic

Let $\mathcal{L}$ be a non-deterministic logic and $\mathcal{W}$ be the set of formulas in $\mathcal{L}$. We define now the system we are going to call **NBSS** which extends $\mathbf{N_{mv}}$.

Definition 1 (Nmatrix) *A* non-deterministic matrix *(*Nmatrix*) for $\mathcal{L}$ is a triple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where $\mathcal{T}$ is a finite non-empty set of truth-values, $\mathcal{D}$ is a*

[1] Some logics that do not admit finite matrices have sound and complete semantics for finite Nmatrices.

non-empty proper subset of $\mathcal{T}$ *(containig its designated values) and* $\mathcal{O}$ *includes a k-ary function* $\tilde{\diamond} : \mathcal{T}^k \rightarrow 2^{\mathcal{T}} - \{\emptyset\}$ *for every k-ary connective* $\diamond \in O_k$.

Definition 2 (Valuation) *Let* $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$ *be an Nmatrix. A* valuation *in* $\mathcal{M}$ *is a function* $\upsilon : \mathcal{W} \rightarrow T$ *such that*

$$\upsilon(\diamond(\psi_1, \ldots, \psi_k)) \in \tilde{\diamond}(\upsilon(\psi_1), \ldots, \upsilon(\psi_k))$$

for each k-ary connective $\diamond \in O_k$ *and for all* $\psi_1, \ldots, \psi_k \in \mathcal{W}$.

Definition 3 (Entry) *We will call an* entry *of a formula* $\bigstar(A_1, \ldots, A_k)$ *each possible truth-value* $\upsilon(\bigstar(A_1, \ldots, A_k))$, *for every* $A_1, \ldots, A_k \in \mathcal{W}$.

As with deterministic finite-valued propositional logics, the truth-values of non-deterministic finite-valued logics can be separated in two classes: the class of designated values ($\mathcal{D}$) and the class of undesignated values ($\mathcal{U}$). To define a Natural Deduction system for $\mathcal{L}$ we first have to define *separating formulas*. Consider an Nmatrix for $\mathcal{L}$ where there are two entries of the same class. For every pair of such entries, we have to define a unary formula $\theta(P)$, known as separating formula, that has only truth-values $\mathcal{D}$ in one entry and $\mathcal{U}$ in the other. Then we define a function t that takes designated values to $\{1\}$ and undesignated values to $\{0\}$.

Let $\bigstar$ be a k-place finite-value operator, $A_1, \ldots, A_k$ be formulas in $\mathcal{L}$ and $\theta^1(P)$, $\theta^2(P)$, $\ldots$, $\theta^s(P)$ be the separating formulas defined for $\mathcal{L}$. We write $\theta^0(P)$ to represent the proposition P.

For each $1 \leq r \leq k$, let $< I_r, J_r >$ be a partition of $\{0, 1, \ldots, s\}$. Note that not all partitions are used as each of the n truth-values of $\mathcal{L}$ is represented by only one of those partitions and there may exist more than n partitions of $\{0, 1, \ldots, s\}$.

Each entry of $\theta^l(\bigstar(A_1, \ldots, A_k))$, $l \in \{0, 1, \ldots, s\}$, can be classified in two types with respect to the partitions. For all $1 \leq r \leq k$, and $i_r \in I_r$ and $j_r \in J_r$:

type 0 $\theta^l(\bigstar(A_1, \ldots, A_k)), \{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}$

type 1 $\{\theta^{i_r}(A_r)\} \vDash \{\theta^{j_r}(A_r)\}, \theta^l(\bigstar(A_1, \ldots, A_k))$

According to this classification, we define the following rules:

If $\theta^l(\bigstar(A_1, \ldots, A_k))$, $1 \leq l \leq s$, is of type 0 with respect to the partitions $< I_r, J_r >$, then,

$$\frac{\theta^l(\bigstar(A_1,\ldots,A_k)) \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{matrix}[\theta^{j_r}(A_r)]\\ \vdots \\ C(\text{all } j_r \in J_r)\end{matrix}}{C}\ (\langle I_r, J_r\rangle,\ 0)$$

where $1 \leq r \leq k$.

If $\theta^l(\bigstar(A_1,\ldots,A_k))$, $1 \leq l \leq s$, is of type 1 with respect to the partitions $< I_r, J_r >$,

$$\frac{\begin{matrix}[\theta^l(\bigstar(A_1,\ldots,A_k))]\\ \vdots \\ C\end{matrix} \quad \theta^{i_r}(A_r)(\text{all } i_r \in I_r) \quad \begin{matrix}[\theta^{j_r}(A_r)]\\ \vdots \\ C(\text{all } j_r \in J_r)\end{matrix}}{C}\ (\langle I_r, J_r\rangle,\ 1)$$

where $1 \leq r \leq k$

Besides these rules, we have rules defined by the partitions that do not define any trut-value. Let $\langle I, J\rangle$ be partitions of $\{0, 1, \ldots, s\}$ that do not represent any truth-value and P a formula in $\mathcal{L}$. According to those partitions, we define the following rules, that we call U-rules:

$$\frac{\theta^i(P)(\text{all } i \in I) \quad \begin{matrix}[\theta^j(P)]\\ \vdots \\ C(\text{all } j \in J)\end{matrix}}{C}\ (\langle I, J\rangle,\ \text{U})$$

for every $i \in I$ and $j \in J$.

Definition 4 (Derived matrix) *A* derived matrix *N of $\mathcal{N}$ is a matrix achieved from $\mathcal{N}$ by the method just presented.*

Definition 5 (Logical consequence) *Let $\mathcal{N}$ be a Nmatrix of a logic $\mathcal{L}$. $\vDash_{\mathcal{N}} \alpha$ iff there exists a derived matrix N of $\mathcal{N}$ and $\vDash_N \alpha$.*

Definition 6 (Derivability) *Let $\mathcal{N}$ be a Nmatrix of a logic $\mathcal{L}$. $\vdash_{\mathcal{N}} \alpha$ iff there exists a derived matrix N of $\mathcal{N}$ and $\vdash_N \alpha$.*

The soundness and completeness proofs follow the scheme of the proofs of soundness and completeness presented in the last chapter even taking into account that the statements consider Nmatrices.

3.3 Soundness and completeness

Let Γ be a set of formulas. We define the *degree of a formula* A, $G(A)$, to be the number of occurrences in A of logical symbols.

The formulas $\theta^i(A_r)$, $0 \leq i \leq s$ and $1 \leq r \leq k$, and $\theta^l(\bigstar(A_1, \ldots, A_k))$, $l \in \{0, 1, \ldots, s\}$, are *sub-formulas* of $\theta^l(\bigstar(A_1, \ldots, A_k))$. Consequently, $G(\theta^i(A_r)) < G(\theta^l(\bigstar(A_1, \ldots, A_k)))$.

Theorem 1 *(Soundness) If $\Gamma \vdash_{\mathcal{N}} A$, then $\Gamma \vDash_{\mathcal{N}} A$.*

Proof: Assume that $\Gamma \vdash_{\mathcal{N}} A$. We want to prove that

$$\Gamma \vDash_{\mathcal{N}} A$$

From our assunption and the definition 5, there exists a derived matrix N of $\mathcal{N}$. Hence, we just need to show that $\Gamma \vDash_N A$.

Let v be any truth-value assignment for the set of boolean atoms such that $\bar{v}(C) \in \mathcal{D}$ for all $C \in \Gamma$. The proof is by induction on the complexity of derivations. Let Π be a derivation of A from Γ and suppose the theorem holds for every derivation less complex than Π.

If A is the conclusion of an application of a rule of type 0 with partitions $\langle I_r, J_r\rangle$, then, by induction hypotheses, there are sets of formulas Γ', $\Gamma_{ir}, \Gamma_{jr} \subset \Gamma$ such that

1. $\Gamma' \vDash_N \theta^l(\bigstar(A_1, \ldots, A_k))$
2. $\Gamma_{ir} \vDash_N \theta^{i_r}(A_r)$, for all $i_r \in I_r$
3. $\Gamma_{jr}, \theta^{j_r}(A_r) \vDash_N A$, for all $j_r \in J_r$

By 1. and 2., we have that

1'. $\bar{v}(\theta^l(\bigstar(A_1, \ldots, A_k))) \in \mathcal{D}$

2'. $\bar{v}(\theta^{i_r}(A_r)) \in \mathcal{D}$, for all $i_r \in I_r$

Suppose that $\bar{v}(A) \in \mathcal{U}$. Then, by 3.

3'. $\bar{v}(\theta^{j_r}(A_r)) \in \mathcal{U}$, for all $j_r \in J_r$

It follows from 1'. and 2'. that $\theta^l(\bigstar(A_1, \ldots, A_k))$ is of type 1 with respect to $\langle I_r, J_r\rangle$. Consequently, $\bar{v}(A) \in \mathcal{D}$, i.e., $\Gamma \vDash_N A$.

The case where A is the conclusion of an application of a rule of type 1 is analogous.

Suppose A is the conclusion of an application of $(\langle I, J\rangle, U)$. Then, the partition $\langle I, J\rangle$ does not define any truth-value of $\mathcal{L}$. By induction hypotheses, there exist $\Gamma_i, \Gamma_j \subseteq \Gamma$ such that

1. $\Gamma_i \vDash_N \theta^i(B)$, for all $i \in I$

2. $\Gamma_j, \theta^j(B) \vDash_N A$, for all $j \in J$

Then,

1'. $\bar{v}(\theta^i(B)) \in \mathcal{D}$, for all $i \in I$

Suppose that $\bar{v}(A) \in \mathcal{U}$. By 2.,

2'. $\bar{v}(\theta^j(B)) \in \mathcal{U}$, for all $j \in J$.

The information given by 1'. and 2'. is unobtainable from the initial truth-values of $\mathcal{L}$. A tuple so defined does not represent any truth-value in $\mathcal{L}$.

Hence, $\bar{v}(A) \in \mathcal{D}$ and $\Gamma \vDash_N A$. ■

Theorem 2 *(Completeness) If* $\Gamma \vDash_{\mathcal{N}} A$*, then* $\Gamma \vdash_{\mathcal{N}} A$*.*

Proof: Suppose $\Gamma \nvdash_{\mathcal{N}} A$ and let N be a derived matrix of $\mathcal{N}$. Hence, $\Gamma \nvdash_N A$ and there exists Γ^* such that $\Gamma \subseteq \Gamma^*, \Gamma^* \nvdash_N A$ and $\Gamma^* \cup \{B\} \vdash A$, $B \notin \Gamma^*$. Define a particular truth-value assignment v for the set of atoms: $v(p) \in \mathcal{D}$ iff $p \in \Gamma^*$. The theorem now is reduced to the claim

$$\bar{v}(B) \in \mathcal{D} \text{ iff } B \in \Gamma^*.$$

With this condition we show that $\Gamma^* \nvDash_N A$, for $A \notin \Gamma^*$ ($\Gamma^* \nvdash_N A$ by assumption).

The proof is by induction on the degree of B. The basic step of the induction is taken care of by the definition of v. Suppose B is of the form $\bigstar(A_1, \ldots, A_k)$ for some k-ary operator $\bigstar$ and formulas $A_1, \ldots, A_k$. For each $1 \leq r \leq k$, put $I_r = \{i_r : \theta^{i_r}(A_r) \in \Gamma^*\}$ and $J_r = \{j_r : \theta^{j_r}(A_r) \notin \Gamma^*\}$. Then, $\langle I_r, J_r \rangle$ are partitions of $\{0, 1, \ldots, s\}$. By assumption,

1. $\theta^{i_r}(A_r) \in \Gamma^*$, that is, $\Gamma^* \vdash_N \theta^{i_r}(A_r)$, for each $i_r \in I_r$

2. $\theta^{j_r}(A_r) \notin \Gamma^*$, that is, $\Gamma^* \cup \{\theta^{j_r}(A_r)\} \vdash_N A$, for each $j_r \in J_r$

If one of those partitions, say $\langle I_t, J_t \rangle$, does not define a truth-value of $\mathcal{L}$, then by applying $(\langle I_t, J_t \rangle, U)$, by 1. and 2., we get that $\Gamma^* \vdash_N A$, what goes against our initial supposition. Hence, all partitions as defined above define truth-values of $\mathcal{L}$.

By the induction hypothesis, $\bar{v}(\theta^{i_r}(A_r)) \in \mathcal{D}$, for all $i_r \in I_r$ and $\bar{v}(\theta^{j_r}(A_r)) \in \mathcal{U}$, for all $j_r \in J_r$. Consequently, $\bar{v}(\bigstar(A_1, \ldots, A_k)) \in \mathcal{D}$ iff $\bigstar(A_1, \ldots, A_k)$ is of type 1 with respect to the partitions. Hence, we need to prove that

$\bar{v}(\bigstar(A_1, \ldots, A_k)) \in \mathcal{D}$ iff $\bigstar(A_1, \ldots, A_k)$ is of type 1 with respect to the partitions $\langle I_r, J_r \rangle$, for all $1 \leq r \leq k$.

Suppose that $\bigstar(A_1, \ldots, A_k) \in \Gamma^*$. Then

3. $\Gamma^* \vdash_N \bigstar(A_1, \ldots, A_k)$

If $\bigstar(A_1, \ldots, A_k)$ were of type 0 with respect to the partitions, then $\Gamma^* \vdash_N A$ (by 1., 2. and 3.) what is against our assumption. Consequently, $\bigstar(A_1, \ldots, A_k)$ is of type 1 with respect to the partitions.

Suppose that $\bigstar(A_1, \ldots, A_k)) \notin \Gamma^*$. Then

3'. $\Gamma^* \cup \{\bigstar(A_1, \ldots, A_k)\} \vdash_N A$

If $\bigstar(A_1, \ldots, A_k)$ were of type 1 with respect to the partitions, then $\Gamma^* \vdash_N A$ (by 1., 2. and 3'.), what is against our assumption. Consequently, $\bigstar(A_1, \ldots, A_k)$ is of type 0 with respect to the partitions. ■

3.4
Conclusion

We are now able to deal with finitely non-deterministic valued logics using the extension of our previous work on deterministic finite valued semantics. This abstract reports the first result that was obtained on this basis. We can compare the systems obtained by our framework with other more mature ones. As a first comparison we noted that, despite the use of formulas to distinguish truth-values, which are present in all frameworks, the systems produced according to our schema is more traditional in the sense that it only involves usual deductive concepts, not appealing to any semantic feature in the calculus. We have to investigate whether this is an advantage or not.