5
Correspondence between Natural Deduction and Sequent Calculus

In this chapter we examine how the correspondence shown in last chapter extends when we deal with non-normal and non-cut-free derivations. As in the previous chapter, we need the notion of pseudo-derivation and the translations between derivations in LJT and derivations in ND are extended to cope with derivations with cut and with non-normal derivations. We also define conversion steps for derivations in LJT and for derivations in ND so that there is a bijection between them.

The definitions of elimination sequence and of pseudo-derivation remain the same. The only difference is that Π does not need to be a cut-free/normal derivation. This means that the sequents in elimination sequences are allowed to be premisses of cut/substitution rules.

5.1
Definitions

Definition 23 (Pure elimination derivation in LJT) *A* pure elimination derivation *in LJT is a derivation according to definition 17 by replacing the second case by the item below:*

- $\Gamma; A_i \vdash B_i$, $1 < i \leq n$, *is the conclusion of the application of either the left rule or cut rule of which* $\Gamma; A_{i-1} \vdash B_{i-1}$ *is a premiss and,*

Lemma 11 *If there is a sequent of the form* $\Gamma; A \vdash B$*, in a derivation* Π*, which is conclusion of an application of cut, then, if we delete every sequent below* $\Gamma; A \vdash B$*, the resulting tree is a pseudo-derivation.*

Proof: Just note that when the conclusion of either a head-cut or a middle-cut has a head-formula, then one of the premisses has a head-formula. ■

Definition 24 (Pure elimination derivation in ND) *A* pure elimination derivation *in ND is a derivation according to definition 14 by considering the additional item below:*

– *a derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}\Pi' \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash B}\,\mathcal{S}$ *is a pure elimination derivation if* Π *is.*

We also define:

Definition 25 *The degree of a formula* P *is the number of connectives in* P. *The degree of a derivation* Π *(*$G(\Pi)$*) is the degree of the largest cut-formula of* Π, $\#G(\Pi)$ *is the number of formulas of* Π *with degree* $G(\Pi)$ *and* $\ell(\Pi)$ *is the length of* Π. *The definitions are analogous for pseudo-derivations.*

Definition 26 *The rank of a rule* r *in a (pseudo-)derivation* Ψ *is the number of rules applied in* Ψ *above* r *up to the top-formula.*

Proposition 1 *If for every derivation* Π *in LJT of* A *from* Γ *there exists a derivation* Π^B *of* A *from* $\Gamma \cup \{B\}$*, then, if* Σ *is a pseudo-derivation of* F *from* Γ *and* G *is a formula, then there exists a pseudo-derivation* Σ^G *of* F *from* $\Gamma \cup \{G\}$. *Moreover,* $\ell(\Sigma^G) = \ell(\Sigma)$.

Proof: Suppose that for every derivation Π of A from Γ there exists a derivation Π^B of A from $\Gamma \cup \{B\}$. The proof is by induction on the length of the pseudo-derivation. Let Σ be a pseudo-derivation of F from $\Gamma \cup \{H\}$ and let G be a formula.

Basic case: If $\Sigma = \Gamma; H \vdash F$, then $\Sigma^G = \Gamma, G; H \vdash F$.

Let r be the last rule applied in Σ. We have the following cases to consider:

$(r = \wedge\vdash)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1 \\ \Gamma; Q \vdash F\end{matrix}}{\Gamma; P \wedge Q \vdash F}\wedge\vdash$, then $\Sigma^G = \dfrac{\begin{matrix}\Sigma_1^G \\ \Gamma, G; Q \vdash F\end{matrix}}{\Gamma, G; P \wedge Q \vdash F}\wedge\vdash$

$(r = \wedge\vdash)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1 \\ \Gamma; P \vdash F\end{matrix}}{\Gamma; P \wedge Q \vdash F}\wedge\vdash$, then $\Sigma^G = \dfrac{\begin{matrix}\Sigma_1^G \\ \Gamma, G; P \vdash F\end{matrix}}{\Gamma, G; P \wedge Q \vdash F}\wedge\vdash$

$(r = \vee\vdash)$ If $\Sigma = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma, P; \vdash F\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, Q; \vdash F\end{matrix}}{\Gamma; P \vee Q \vdash F}\vee\vdash$, then

$$\Sigma^G = \dfrac{\begin{matrix}\Pi_1^G \\ \Gamma, G, P; \vdash F\end{matrix} \quad \begin{matrix}\Pi_2^G \\ \Gamma, G, Q; \vdash F\end{matrix}}{\Gamma G; P \vee Q \vdash F}\vee\vdash$$

$(r = \to\vdash)$ If $\Sigma = \dfrac{\overset{\Pi_1}{\Gamma; \vdash P} \quad \overset{\Sigma_2}{\Gamma; Q \vdash F}}{\Gamma; P \to Q \vdash F}{\scriptstyle \to\vdash}$, then

$$\Sigma^G = \dfrac{\overset{\Pi_1^G}{\Gamma, G; \vdash P} \quad \overset{\Sigma_2^G}{\Gamma, G; Q \vdash F}}{\Gamma G; P \to Q \vdash F}{\scriptstyle \to\vdash}$$

$(r = \perp\vdash)$ If $\Sigma = \dfrac{\overset{\Sigma_1}{\Gamma; P \vdash F}}{\Gamma; \perp \vdash F}{\scriptstyle \perp\vdash}$, then $\Sigma^G = \dfrac{\overset{\Sigma_1^G}{\Gamma, G; P \vdash F}}{\Gamma, G; \perp \vdash F}{\scriptstyle \perp\vdash}$

$(r = C_M)$ If $\Sigma = \dfrac{\overset{\Pi_1}{\Gamma; \vdash P} \quad \overset{\Sigma_2}{\Gamma, P; H \vdash F}}{\Gamma; H \vdash F}{\scriptstyle C_M}$, then

$$\Sigma^G = \dfrac{\overset{\Pi_1^G}{\Gamma, G; \vdash P} \quad \overset{\Sigma_2^G}{\Gamma, G, P; H \vdash F}}{\Gamma, G; H \vdash F}{\scriptstyle C_M}$$

$(r = C_H)$ If $\Sigma = \dfrac{\overset{\Pi_1}{\Gamma; H \vdash P} \quad \overset{\Sigma_2}{\Gamma; P \vdash F}}{\Gamma; H \vdash F}{\scriptstyle C_H}$, then

$$\Sigma^G = \dfrac{\overset{\Pi_1^G}{\Gamma, G; H \vdash P} \quad \overset{\Sigma_2^G}{\Gamma, G; P \vdash F}}{\Gamma, G; H \vdash F}{\scriptstyle C_H}$$

It is easy to see that Σ^G and Σ have the same length. ■

Proposition 2 *If Π is a derivation in LJT of A from Γ and B is a formula, then there exists a derivation Π^B of A from $\Gamma \cup \{B\}$. Moreover, $\ell(\Pi) = \ell(\Pi^B)$.*

Proof: The proof is by induction on the length of the derivation. Let Π be a derivation in LJT and let B be a formula.

Basic case: If $\Pi = \dfrac{\overline{\Gamma; A \vdash A}\,{\scriptstyle Ax}}{\Gamma; \vdash A}{\scriptstyle \mathcal{D}}$, then $\Pi^G = \dfrac{\overline{\Gamma, G; A \vdash A}\,{\scriptstyle Ax}}{\Gamma, G; \vdash A}{\scriptstyle \mathcal{D}}$.

Let r be the last rule applied in Π. We have the following cases to consider:

$(r = \vdash\wedge)$ If $\Pi = \dfrac{\overset{\Pi_1}{\Gamma; \vdash P} \quad \overset{\Pi_2}{\Gamma; \vdash Q}}{\Gamma; \vdash P \wedge Q}{\scriptstyle \vdash\wedge}$, then $\Pi^B = \dfrac{\overset{\Pi_1^B}{\Gamma, B; \vdash P} \quad \overset{\Pi_2^B}{\Gamma, B; \vdash Q}}{\Gamma, B; \vdash P \wedge Q}{\scriptstyle \vdash\wedge}$

$(r = \vdash\vee)$ If $\Pi = \dfrac{\overset{\Pi_1}{\Gamma; \vdash P}}{\Gamma; \vdash P \vee Q}{\scriptstyle \vdash\vee}$, then $\Pi^B = \dfrac{\overset{\Pi_1^B}{\Gamma, B; \vdash P}}{\Gamma, B; \vdash P \vee Q}{\scriptstyle \vdash\vee}$

$(r = \vdash\vee)$ If $\Pi = \dfrac{\overset{\Pi_1}{\Gamma; \vdash Q}}{\Gamma; \vdash P \vee Q}{\scriptstyle \vdash\vee}$, then $\Pi^B = \dfrac{\overset{\Pi_1^B}{\Gamma, B; \vdash Q}}{\Gamma, B; \vdash P \vee Q}{\scriptstyle \vdash\vee}$

$(r = \vdash\rightarrow)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1\\ \Gamma, P; \vdash Q\end{matrix}}{\Gamma; \vdash P \rightarrow Q}\vdash\rightarrow$, then $\Pi^B = \dfrac{\begin{matrix}\Pi_1^B\\ \Gamma, B, P; \vdash Q\end{matrix}}{\Gamma, B; \vdash P \rightarrow Q}\vdash\rightarrow$

$(r = \vdash \mathcal{D})$ If $\Pi = \dfrac{\begin{matrix}\Pi_1\\ \Gamma; \vdash A\end{matrix}}{\Gamma; P \vdash A}\mathcal{D}$, then $\Pi^B = \dfrac{\begin{matrix}\Pi_1^B\\ \Gamma, B; \vdash A\end{matrix}}{\Gamma, B; P \vdash A}\mathcal{D}$

$(r = C_M)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1\\ \Gamma; \vdash P\end{matrix} \quad \begin{matrix}\Pi_2\\ \Gamma, P; \vdash A\end{matrix}}{\Gamma; \vdash A}C_M$, then

$$\Pi^B = \dfrac{\begin{matrix}\Pi_1^B\\ \Gamma, B; \vdash P\end{matrix} \quad \begin{matrix}\Pi_2^B\\ \Gamma, B, P; \vdash A\end{matrix}}{\Gamma, B; \vdash A}C_M$$

$(r = C_H)$ If $\Sigma = \dfrac{\begin{matrix}\Pi_1\\ \Gamma; \vdash P\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma; P \vdash A\end{matrix}}{\Gamma; \vdash A}C_H$, then

$$\Sigma^B = \dfrac{\begin{matrix}\Pi_1^B\\ \Gamma, B; \vdash P\end{matrix} \quad \begin{matrix}\Sigma_2^B\\ \Gamma, B; P \vdash A\end{matrix}}{\Gamma, B; \vdash A}C_H$$

It is easy to see that Π^B and Π have the same length. ■

Proposition 3 *If Σ is a pseudo-derivation of F from Γ and G is a formula, then there exists a pseudo-derivation Σ^G of F from $\Gamma \cup \{G\}$. Moreover, $\ell(\Sigma^G) = \ell(\Sigma)$.*

Proof: Let Σ be a pseudo-derivation of A from Γ and let B be a formula. The proof is by induction on the length of Σ.

$(r = E_\wedge)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1\\ \Gamma \vdash P \wedge Q\end{matrix}}{\Gamma \vdash P}E_\wedge$, then $\Sigma^B = \dfrac{\begin{matrix}\Sigma_1^B\\ \Gamma, B \vdash P \wedge Q\end{matrix}}{\Gamma, B \vdash P}E_\wedge$

$(r = E_\wedge)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1\\ \Gamma \vdash P \wedge Q\end{matrix}}{\Gamma \vdash Q}E_\wedge$, then $\Sigma^B = \dfrac{\begin{matrix}\Sigma_1^B\\ \Gamma, B \vdash P \wedge Q\end{matrix}}{\Gamma, B \vdash Q}E_\wedge$

$(r = E_\vee)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1\\ \Gamma \vdash P \vee Q\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma, P \vdash A\end{matrix} \quad \begin{matrix}\Sigma_3\\ \Gamma, Q \vdash A\end{matrix}}{\Gamma \vdash A}E_\vee$, then

$$\Sigma^B = \dfrac{\begin{matrix}\Sigma_1\\ \Gamma, B \vdash P \vee Q\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma, B, P \vdash A\end{matrix} \quad \begin{matrix}\Sigma_3\\ \Gamma, B, Q \vdash A\end{matrix}}{\Gamma, B \vdash A}E_\vee$$

$(r = E_\rightarrow)$ If $\Sigma = \dfrac{\begin{matrix}\Sigma_1\\ \Gamma \vdash P \rightarrow Q\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma \vdash P\end{matrix}}{\Gamma \vdash Q}E_\rightarrow$, then

$$\Sigma^B = \dfrac{\begin{array}{cc}\Sigma_1^B & \Sigma_2^B \\ \Gamma, B \vdash P \to Q & \Gamma, B \vdash P\end{array}}{\Gamma, B \vdash Q} E_{\to}$$

$(r = E_\perp)$ If $\Sigma = \dfrac{\begin{array}{c}\Sigma_1 \\ \Gamma \vdash \perp\end{array}}{\Gamma \vdash P} E_\perp$, then $\Sigma^B = \dfrac{\begin{array}{c}\Sigma_1^B \\ \Gamma, B \vdash \perp\end{array}}{\Gamma, B \vdash P} E_\perp$

$(r = S)$ If $\Sigma = \dfrac{\begin{array}{cc}\Sigma_1 & \Sigma_2 \\ \Gamma \vdash P & \Gamma, P \vdash A\end{array}}{\Gamma \vdash A} S$, then $\Sigma^B = \dfrac{\begin{array}{cc}\Sigma_1^B & \Sigma_2^B \\ \Gamma, B \vdash P & \Gamma, B, P \vdash A\end{array}}{\Gamma, B \vdash A} S$

It is easy to see that Σ^B and Σ have the same length. ∎

Proposition 4 *If Π is a derivation in ND of A from Γ and B is a formula, then there exists a derivation Π^B of A from $\Gamma \cup \{B\}$. Moreover, $\ell(\Pi) = \ell(\Pi^B)$.*

Proof: The proof is by induction on the length of the derivation. Let Π be a derivation of A from Γ and let B be a formula.

Basic case: If $\Pi = \dfrac{}{\Gamma, A \vdash A} Ax$, then $\Pi^B = \dfrac{}{\Gamma, B, A \vdash A} Ax$.

Let r be the last rule applied in Π. We have the following cases to consider:

$(r = I_\wedge)$ If $\Pi = \dfrac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ \Gamma \vdash P & \Gamma \vdash Q\end{array}}{\Gamma \vdash P \wedge Q} I_\wedge$, then $\Pi^B = \dfrac{\begin{array}{cc}\Pi_1^B & \Pi_2^B \\ \Gamma, B \vdash P & \Gamma, B \vdash Q\end{array}}{\Gamma, B \vdash P \wedge Q} \vdash \wedge$

$(r = I_\vee)$ If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash P\end{array}}{\Gamma \vdash P \vee Q} I_\vee$, then $\Pi^B = \dfrac{\begin{array}{c}\Pi_1^B \\ \Gamma, B \vdash P\end{array}}{\Gamma, B \vdash P \vee Q} I_\vee$

$(r = I_\vee)$ If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash Q\end{array}}{\Gamma \vdash P \vee Q} I_\vee$, then $\Pi^B = \dfrac{\begin{array}{c}\Pi_1^B \\ \Gamma, B \vdash Q\end{array}}{\Gamma, B \vdash P \vee Q} I_\vee$

$(r = I_\to)$ If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma, P \vdash Q\end{array}}{\Gamma \vdash P \to Q} I_\to$, then $\Pi^B = \dfrac{\begin{array}{c}\Pi_1^B \\ \Gamma, B, P \vdash Q\end{array}}{\Gamma, B \vdash P \to Q} I_\to$

$(r = E_\wedge)$ If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash P \wedge Q\end{array}}{\Gamma \vdash P} E_\wedge$, then $\Pi^B = \dfrac{\begin{array}{c}\Pi_1^B \\ \Gamma, B \vdash P \wedge Q\end{array}}{\Gamma, B \vdash P} E_\wedge$

$(r = E_\wedge)$ If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash P \wedge Q\end{array}}{\Gamma \vdash Q} E_\wedge$, then $\Pi^B = \dfrac{\begin{array}{c}\Pi_1^B \\ \Gamma, B \vdash P \wedge Q\end{array}}{\Gamma, B \vdash Q} E_\wedge$

$(r = E_\vee)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash P \vee Q\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, P \vdash A\end{matrix} \quad \begin{matrix}\Pi_3 \\ \Gamma, Q \vdash A\end{matrix}}{\Gamma \vdash A}\, E_\vee$, then

$$\Pi^B = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma, B \vdash P \vee Q\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, B, P \vdash A\end{matrix} \quad \begin{matrix}\Pi_3 \\ \Gamma, B, Q \vdash A\end{matrix}}{\Gamma, B \vdash A}\, E_\vee$$

$(r = E_\rightarrow)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash P \rightarrow Q\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma \vdash P\end{matrix}}{\Gamma \vdash Q}\, E_\rightarrow$, then

$$\Pi^B = \dfrac{\begin{matrix}\Pi_1^B \\ \Gamma, B \vdash P \rightarrow Q\end{matrix} \quad \begin{matrix}\Pi_2^B \\ \Gamma, B \vdash P\end{matrix}}{\Gamma, B \vdash Q}\, E_\rightarrow$$

$(r = E_\perp)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash \perp\end{matrix}}{\Gamma \vdash P}\, E_\perp$, then $\Pi^B = \dfrac{\begin{matrix}\Pi_1^B \\ \Gamma, B \vdash \perp\end{matrix}}{\Gamma, B \vdash P}\, E_\perp$

$(r = S)$ If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash P\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, P \vdash A\end{matrix}}{\Gamma \vdash A}\, S$, then $\Pi^B = \dfrac{\begin{matrix}\Pi_1^B \\ \Gamma, B \vdash P\end{matrix} \quad \begin{matrix}\Pi_2^B \\ \Gamma, B, P \vdash A\end{matrix}}{\Gamma, B \vdash A}\, S$

It is easy to see that Π^B and Π have the same length. ■

5.2 Conversions in LJT

Let c be a cut rule (either head or middle-cut) application in Π. Conversions on Π can be separated into three kinds: *trivial*, *permutative* and *reductive*. Trivial conversions are those in which one of the premisses of c is initial, permutative conversions are those in which $\#G(\Pi)$ remains the same when the conversion is applied and reductive conversions are those in which $\#G(\Pi)$ diminishes when the conversion is applied.

Definition 27 *Let Σ be a pseudo-derivation with at least one occurrence of cut. We have the following conversion cases for Σ:*

1. *Conversion for head-cut:*

   Let Σ be a pseudo-derivation of the form

$$\dfrac{\dfrac{\Sigma_1}{\Gamma; D \vdash A}\, r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\, r_2}{\begin{matrix}\Gamma; D \vdash B \\ \Psi_3\end{matrix}}\, c$$

   (a) *Trivial conversions (either r_1 or r_2 is Ax):*

$$i.\quad \begin{array}{c}\dfrac{\dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; A \vdash B}\,c \\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2 \\ \Psi_3\end{array}$$

$$ii.\quad \begin{array}{c}\dfrac{\dfrac{\Sigma_1}{\Gamma; D \vdash A}\,r_1 \quad \dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax}}{\Gamma; D \vdash A}\,{\scriptstyle c} \\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\dfrac{\Sigma_1}{\Gamma; D \vdash A}\,r_1 \\ \Psi_3\end{array}$$

(b) *Permutative conversions (r_1 is a left rule):*

i. *$r_1 = \wedge\vdash$ and $D = P \wedge Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \wedge Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \wedge Q \vdash B}\,c \\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \wedge Q \vdash B}\,r_1 \\ \Psi_3\end{array}$$

The case where $D = Q \wedge P$ is analogous.

ii. *$r_1 = \vee\vdash$ and $D = P \vee Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_{11}\\ \Gamma, P; \vdash A\end{array} \quad \begin{array}{c}\Pi_{12}\\ \Gamma, Q; \vdash A\end{array}}{\Gamma; P \vee Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vee Q \vdash B}\,c \\ \Psi_3\end{array} \;\rhd$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_{11}\\ \Gamma, P; \vdash A\end{array} \quad \dfrac{\Sigma_2^P}{\Gamma, P; A \vdash B}\,r_2}{\Gamma, P; \vdash B}\,{\scriptstyle C_H} \quad \dfrac{\begin{array}{c}\Pi_{12}\\ \Gamma, Q; \vdash A\end{array} \quad \dfrac{\Sigma_2^Q}{\Gamma, Q; A \vdash B}\,r_2}{\Gamma, Q; \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \vee Q \vdash B}\,r_1 \\ \Psi_3\end{array}$$

Note that we applied propositions 1 and 2 in the rewriting of the pseudo-derivation.

iii. *$r_1 = \to\vdash$ and $D = P \to Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_{11}\\ \Gamma; \vdash P\end{array} \quad \begin{array}{c}\Sigma_{12}\\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \to Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \to Q \vdash B}\,c \\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\dfrac{\begin{array}{c}\Pi_{11}\\ \Gamma; \vdash P\end{array} \quad \dfrac{\begin{array}{c}\Sigma_{12}\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \to Q \vdash B}\,r_1 \\ \Psi_3\end{array}$$

iv. *$r_1 = \bot\vdash$ and $D = \bot$:*

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; P \vdash A\end{array}}{\Gamma; \bot \vdash A}\,{\scriptstyle \bot\vdash} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; \bot \vdash B}\,c \\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; P \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vdash B}\,{\scriptstyle C_H}}{\Gamma; \bot \vdash B}\,{\scriptstyle \bot\vdash} \\ \Psi_3\end{array}$$

2. *Conversion for middle-cut*

Let Σ be a pseudo-derivation of the form

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma, A; D \vdash B}\,r_2}{\Gamma; D \vdash B}\,c \\ \Psi_3\end{array}$$

(a) *Trivial conversions:*

r_1 cannot be Ax (for the stoup is empty). If r_2 is Ax, then $D = B$ and

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{}{\Gamma, A; B \vdash B}\,Ax}{\Gamma; B \vdash B}\,c \\ \Psi_3\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{}{\Gamma; B \vdash B}\,Ax \\ \Psi_3\end{array}$$

(b) *Permutative conversions (r_2 is a left rule):*

i. *$r_2 = \wedge\vdash$ and $D = P \wedge Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{array}{c}\Sigma_2\\ \Gamma, A; Q \vdash B\end{array}}{\Gamma, A; P \wedge Q \vdash B}\,\wedge\vdash}{\Gamma; P \wedge Q \vdash B}\,c \\ \Psi_3\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{array}{c}\Sigma_2\\ \Gamma, A; Q \vdash B\end{array}}{\Gamma; Q \vdash B}\,c_M}{\Gamma; P \wedge Q \vdash B}\,\wedge\vdash \\ \Psi_3\end{array}$$

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{array}{c}\Sigma_2\\ \Gamma, A; P \vdash B\end{array}}{\Gamma, A; P \wedge Q \vdash B}\,\wedge\vdash}{\Gamma; P \wedge Q \vdash B}\,c \\ \Psi_3\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{array}{c}\Sigma_2\\ \Gamma, A; P \vdash B\end{array}}{\Gamma; P \vdash B}\,c_M}{\Gamma; P \wedge Q \vdash B}\,\wedge\vdash \\ \Psi_3\end{array}$$

ii. *$r_2 = \rightarrow\vdash$ and $D = P \rightarrow Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{array}{c}\Pi_{21}\\ \Gamma, A;\vdash P\end{array} \quad \begin{array}{c}\Sigma_{22}\\ \Gamma, A; Q \vdash B\end{array}}{\Gamma, A; P \rightarrow Q \vdash B}\,\rightarrow\vdash}{\Gamma; P \rightarrow Q \vdash B}\,c \\ \Psi_3\end{array}$$

$$\rhd \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{array}{c}\Pi_{21}\\ \Gamma, A;\vdash P\end{array}}{\Gamma;\vdash P}\,c_M \quad \dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{array}{c}\Sigma_{22}\\ \Gamma, A; Q \vdash B\end{array}}{\Gamma; Q \vdash B}\,c_M}{\Gamma; P \rightarrow Q \vdash B}\,\rightarrow\vdash \\ \Psi_3\end{array}$$

iii. *$r_2 = \vee\vdash$ and $D = P \vee Q$:*

$$\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{array}{c}\Pi_{21}\\ \Gamma, A, P;\vdash B\end{array} \quad \begin{array}{c}\Pi_{22}\\ \Gamma, A, Q;\vdash B\end{array}}{\Gamma, A; P \vee Q \vdash B}\,\vee\vdash}{\Gamma; P \vee Q \vdash B}\,c \\ \Psi_3\end{array}$$

$$\rhd \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\Pi_1^P}{\Gamma, P;\vdash A}\,r_1 \quad \begin{array}{c}\Pi_{21}\\ \Gamma, A, P;\vdash B\end{array}}{\Gamma, P;\vdash B}\,c_M \quad \dfrac{\dfrac{\Pi_1^Q}{\Gamma, Q;\vdash A}\,r_1 \quad \begin{array}{c}\Pi_{22}\\ \Gamma, A, Q;\vdash B\end{array}}{\Gamma, Q;\vdash B}\,c_M}{\Gamma; P \vee Q \vdash B}\,\vee\vdash \\ \Psi_3\end{array}$$

iv. *$r_2 = \bot\vdash$ and $D = \bot$:*

$$\cfrac{\cfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \cfrac{\begin{matrix}\Sigma_2\\ \Gamma,A;C\vdash B\end{matrix}}{\Gamma,A;\bot\vdash B}\,\bot\vdash}{\begin{matrix}\Gamma;\bot\vdash B\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \cfrac{\cfrac{\cfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Sigma_2\\ \Gamma,A;C\vdash B\end{matrix}}{\Gamma;C\vdash B}\,C_M}{\begin{matrix}\Gamma;\bot\vdash B\\ \Psi_3\end{matrix}}\,\bot\vdash$$

Definition 28 *[Conversion for derivations] Most of the cases are analogous to the cases shown in definition 27. The other cases are as follows:*

1. *Conversion for head-cut:*

   Let Π *be a derivation of the form*

$$\cfrac{\cfrac{\Psi_1}{\Gamma;\Delta\vdash A}\,r_1 \quad \cfrac{\Pi_2}{\Gamma;A\vdash B}\,r_2}{\begin{matrix}\Gamma;\Delta\vdash B\\ \Psi_3\end{matrix}}\,c$$

   (a) *Trivial conversion:*

$$\cfrac{\cfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \cfrac{}{\Gamma;A\vdash A}\,Ax}{\begin{matrix}\Gamma;\vdash A\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \cfrac{\Pi_1}{\begin{matrix}\Gamma;\vdash A\\ \Psi_3\end{matrix}}\,r_1$$

   (b) *Permutative conversion:*

$$\cfrac{\cfrac{\begin{matrix}\Pi_1\\ \Gamma;P\vdash A\end{matrix}}{\Gamma;\vdash A}\,\mathcal{D} \quad \cfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\Gamma;\vdash B}\,c \quad \rhd \quad \cfrac{\cfrac{\begin{matrix}\Pi_1\\ \Gamma;P\vdash A\end{matrix} \quad \cfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\Gamma;P\vdash B}\,C_H}{\Gamma;\vdash B}\,\mathcal{D}$$

   (c) *Reductive conversions:*

   If r_1 *is a right rule, then* Δ *must be empty. As* A *is the active formula in the major premiss, then, if* A *is of the form* $P\odot Q$*, where* $\odot$ *is either* $\to$, $\wedge$ *or* $\vee$*, then* r_1 *is* $\vdash\odot$ *and* r_2 *is* $\odot\vdash$ *(the case in which* r_2 *is* Ax *was already shown as a trivial conversion). We have the following cases to consider:*

   i. A *is* $P\wedge Q$*:*

$$\cfrac{\cfrac{\begin{matrix}\Pi_{11}\\ \Gamma;\vdash P\end{matrix} \quad \begin{matrix}\Pi_{12}\\ \Gamma;\vdash Q\end{matrix}}{\Gamma;\vdash P\wedge Q}\,\vdash\wedge \quad \cfrac{\begin{matrix}\Sigma_2\\ \Gamma;Q\vdash B\end{matrix}}{\Gamma;P\wedge Q\vdash B}\,\wedge\vdash}{\Gamma;\vdash B}\,c \quad \rhd \quad \cfrac{\begin{matrix}\Pi_{12}\\ \Gamma;\vdash Q\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma;Q\vdash B\end{matrix}}{\Gamma;\vdash B}\,C_H$$

$$\cfrac{\cfrac{\begin{matrix}\Pi_{11}\\ \Gamma;\vdash P\end{matrix} \quad \begin{matrix}\Pi_{12}\\ \Gamma;\vdash Q\end{matrix}}{\Gamma;\vdash P\wedge Q}\,\vdash\wedge \quad \cfrac{\begin{matrix}\Sigma_2\\ \Gamma;P\vdash B\end{matrix}}{\Gamma;P\wedge Q\vdash B}\,\wedge\vdash}{\Gamma;\vdash B}\,c \quad \rhd \quad \cfrac{\begin{matrix}\Pi_{11}\\ \Gamma;\vdash P\end{matrix} \quad \begin{matrix}\Sigma_2\\ \Gamma;P\vdash B\end{matrix}}{\Gamma;\vdash B}\,C_H$$

   ii. A *is* $P\vee Q$*:*

$$\cfrac{\cfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash P\end{matrix}}{\Gamma;\vdash P\vee Q}\,\vdash\vee \quad \cfrac{\begin{matrix}\Pi_{21}\\ \Gamma,P;\vdash B\end{matrix} \quad \begin{matrix}\Pi_{22}\\ \Gamma,Q;\vdash B\end{matrix}}{\Gamma;P\vee Q\vdash B}\,\vee\vdash}{\Gamma;\vdash B}\,c \quad \rhd \quad \cfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash P\end{matrix} \quad \begin{matrix}\Pi_{21}\\ \Gamma,P;\vdash B\end{matrix}}{\Gamma;\vdash B}\,C_M$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash Q\end{matrix}}{\Gamma;\vdash P\vee Q}{\scriptstyle\vdash\vee} \quad \dfrac{\begin{matrix}\Pi_{21}\\ \Gamma,P;\vdash B\end{matrix}\quad\begin{matrix}\Pi_{22}\\ \Gamma,Q;\vdash B\end{matrix}}{\Gamma;P\vee Q\vdash B}{\scriptstyle\vee\vdash}}{\Gamma;\vdash B}\,c \quad\rhd\quad \dfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash Q\end{matrix}\quad\begin{matrix}\Pi_{21}\\ \Gamma,Q;\vdash B\end{matrix}}{\Gamma;\vdash B}\,c_M$$

iii. A is $P\to Q$*:*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma,P;\vdash Q\end{matrix}}{\Gamma;\vdash P\to Q}{\scriptstyle\vdash\to} \quad \dfrac{\begin{matrix}\Pi_{21}\\ \Gamma;\vdash P\end{matrix}\quad\begin{matrix}\Sigma_{22}\\ \Gamma;Q\vdash B\end{matrix}}{\Gamma;P\to Q\vdash B}{\scriptstyle\to\vdash}}{\Gamma;\vdash B}\,c \quad\rhd\quad \dfrac{\dfrac{\begin{matrix}\Pi_{21}\\ \Gamma;\vdash P\end{matrix}\quad\begin{matrix}\Pi_1\\ \Gamma,P;\vdash Q\end{matrix}}{\Gamma;\vdash Q}\,c_M \quad \begin{matrix}\Sigma_{22}\\ \Gamma;Q\vdash B\end{matrix}}{\Gamma;\vdash B}\,c_H$$

Conversion for middle-cut:

Let Π *be a derivation of the form*

$$\begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\Psi_2}{\Gamma,A;\Delta\vdash B}\,r_2}{\Gamma;\Delta\vdash B}\,c\\ \Psi_3\end{matrix}$$

1. *Permutative conversion:*

(a)
$$\begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Sigma_2\\ \Gamma,A;A\vdash B\end{matrix}}{\Gamma,A;\vdash B}\,\mathcal{D}}{\Gamma;\vdash B}\,c_M\\ \Psi_3\end{matrix} \quad\rhd\quad \begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Sigma_2\\ \Gamma,A;A\vdash B\end{matrix}}{\Gamma;A\vdash B}\,c_M}{\Gamma;\vdash B}\,c_H\\ \Psi_3\end{matrix}$$

$$\begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Sigma_2\\ \Gamma,A;C\vdash B\end{matrix}}{\Gamma,A;\vdash B}\,\mathcal{D}}{\Gamma;\vdash B}\,c\\ \Psi_3\end{matrix} \quad\rhd\quad \begin{matrix}\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Sigma_2\\ \Gamma,A;C\vdash B\end{matrix}}{\Gamma;C\vdash B}\,c_M}{\Gamma;\vdash B}\,\mathcal{D}\\ \Psi_3\end{matrix}$$

(b) $r_2 = \vdash\wedge$ *and* $\Delta = P\wedge Q$*:*

$$\begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Pi_{21}\\ \Gamma,A;\vdash P\end{matrix}\quad\begin{matrix}\Pi_{22}\\ \Gamma,A;\vdash Q\end{matrix}}{\Gamma,A;\vdash P\wedge Q}\vdash\wedge}{\Gamma;\vdash P\wedge Q}\,c\\ \Psi_3\end{matrix}$$

$$\rhd\quad \begin{matrix}\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Pi_{21}\\ \Gamma,A;\vdash P\end{matrix}}{\Gamma;\vdash P}\,c_M \quad \dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Pi_{22}\\ \Gamma,A;\vdash Q\end{matrix}}{\Gamma;\vdash Q}\,c_M}{\Gamma;\vdash P\wedge Q}\vdash\wedge\\ \Psi_3\end{matrix}$$

(c) $r_2 = \vdash\to$ *and* $\Delta = P\to Q$*:*

$$\begin{matrix}\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Pi'_2\\ \Gamma,A,P;\vdash Q\end{matrix}}{\Gamma,A;\vdash P\to Q}\vdash\to}{\Gamma;\vdash P\to Q}\,c\\ \Psi_3\end{matrix} \quad\rhd\quad \begin{matrix}\dfrac{\dfrac{\dfrac{\Pi_1^P}{\Gamma,P;\vdash A}\,r_1 \quad \begin{matrix}\Pi_2\\ \Gamma,A,P;\vdash Q\end{matrix}}{\Gamma,P;\vdash Q}\,c_M}{\Gamma;\vdash P\to Q}\vdash\to\\ \Psi_3\end{matrix}$$

(d) $r_2 = \vdash\vee$ *and* $\Delta = P\vee Q$*:*

$$\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Pi_2'\\ \Gamma,A;\vdash P\end{matrix}}{\Gamma,A;\vdash P\vee Q}\vdash\vee}{\begin{matrix}\Gamma;\vdash P\vee Q\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Pi_2'\\ \Gamma,A;\vdash P\end{matrix}}{\Gamma;\vdash P}\,C_M}{\begin{matrix}\Gamma;\vdash P\vee Q\\ \Psi_3\end{matrix}}\vdash\vee$$

$$\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \dfrac{\begin{matrix}\Pi_2'\\ \Gamma,A;\vdash Q\end{matrix}}{\Gamma,A;\vdash P\vee Q}\vdash\vee}{\begin{matrix}\Gamma;\vdash P\vee Q\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}\,r_1 \quad \begin{matrix}\Pi_2'\\ \Gamma,A;\vdash Q\end{matrix}}{\Gamma;\vdash Q}\,C_M}{\begin{matrix}\Gamma;\vdash P\vee Q\\ \Psi_3\end{matrix}}\vdash\vee$$

Definition 29 *We say that two derivations* Π *and* Π' *are* equivalent *(*$\Pi \approx \Pi'$*) if* Π *and* Π' *reduce to a same cut-free derivation.*

5.2.1 Normalization Theorem

To define translations between derivations in ND and derivations in LJT, we need the normalization property. To prove normalization, we take into account the uppermost application of cut in LJT and the uppermost detour in ND.

Lemma 12 *If every derivation in LJT can be transformed into a cut-free derivation, then every pseudo-derivation* Σ *of a derivation in LJT can be transformed into a cut-free pseudo-derivation.*

Proof: Suppose Σ is a pseudo-derivation of degree $G(\Sigma)$, $\#G(\Sigma)$ is the number of formulas of Σ with degree $G(\Sigma)$ and $\ell(\Sigma)$ is the lenght of Σ. The proof is by induction on the pair $\langle \#G(\Sigma), \ell(\Sigma)\rangle$.

Let c be an application of cut such that the degree of the cut-formula is the degree of Σ. We only show three cases, the others are analogous to cases already seen.

1. $$\Sigma = \dfrac{\dfrac{}{\Gamma;A\vdash A}\,Ax \quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\begin{matrix}\Gamma;A\vdash B\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \dfrac{\Sigma_2}{\begin{matrix}\Gamma;A\vdash B\\ \Psi_3\end{matrix}}\,r_2 = \Sigma'$$

   As $\#G(\Sigma') < \#G(\Sigma)$, by IH Σ' can be reduced to a cut-free pseudo-derivation.

2. $$\Sigma = \dfrac{\dfrac{\begin{matrix}\Sigma_1'\\ \Gamma;Q\vdash A\end{matrix}}{\Gamma;P\wedge Q\vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\begin{matrix}\Gamma;P\wedge Q\vdash B\\ \Psi_3\end{matrix}}\,c \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Sigma_1'\\ \Gamma;Q\vdash A\end{matrix} \quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\Gamma;Q\vdash B}\,C_H}{\begin{matrix}\Gamma;P\wedge Q\vdash B\\ \Psi_3\end{matrix}}\,r_1$$

By IH the sub-pseudo-derivation $\dfrac{\begin{array}{c}\Sigma'_1\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2^P}{\Gamma; A \vdash B}\, r_2}{\Gamma; Q \vdash B}\, C_H$ reduces to a cut-free pseudo-derivation $\Sigma^{\#}$. If $\Sigma' = \begin{array}{c}\Sigma^{\#}\\ \Gamma; P \wedge Q \vdash B\\ \Psi_3\end{array}$, as $\#G(\Sigma') < \#G(\Sigma)$, by IH Σ' can be reduced to a cut-free pseudo-derivation.

3. $$\Sigma = \begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \dfrac{\begin{array}{c}\Pi_{21}\\ \Gamma, A, P; \vdash B\end{array} \quad \begin{array}{c}\Pi_{22}\\ \Gamma, A, Q; \vdash B\end{array}}{\Gamma, A; P \vee Q \vdash B}\, \vee\vdash}{\Gamma; P \vee Q \vdash B}\, c\\ \Psi_3\end{array}$$

$$\rhd \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\Pi_1^P}{\Gamma, P; \vdash A}\, r_1 \quad \begin{array}{c}\Pi_{21}\\ \Gamma, A, P; \vdash B\end{array}}{\Gamma, P; \vdash B}\, C_M \quad \dfrac{\dfrac{\Pi_1^Q}{\Gamma, Q; \vdash A}\, r_1 \quad \begin{array}{c}\Pi_{22}\\ \Gamma, A, Q; \vdash B\end{array}}{\Gamma, Q; \vdash B}\, C_M}{\Gamma; P \vee Q \vdash B}\, \vee\vdash\\ \Psi_3\end{array}$$

By hypothesis both $\dfrac{\dfrac{\Pi_1^P}{\Gamma, P; \vdash A}\, r_1 \quad \begin{array}{c}\Pi_{21}\\ \Gamma, A, P; \vdash B\end{array}}{\Gamma, P; \vdash B}\, C_M$ and $\dfrac{\dfrac{\Pi_1^Q}{\Gamma, Q; \vdash A}\, r_1 \quad \begin{array}{c}\Pi_{22}\\ \Gamma, A, Q; \vdash B\end{array}}{\Gamma, Q; \vdash B}\, C_M$ reduce to cut-free derivations $\begin{array}{c}\Pi'_1\\ \Gamma, P; \vdash B\end{array}$ and $\begin{array}{c}\Pi'_2\\ \Gamma, Q; \vdash B\end{array}$, respectively. Then, $\Sigma' = \begin{array}{c}\dfrac{\begin{array}{c}\Pi'_1\\ \Gamma, P; \vdash B\end{array} \quad \begin{array}{c}\Pi'_2\\ \Gamma, Q; \vdash B\end{array}}{\Gamma; P \vee Q \vdash B}\, \vee\vdash\\ \Psi_3\end{array}$ and, as $\#G(\Sigma') < \#G(\Sigma)$, by IH Σ' can be reduced to a pseudo-derivation.

■

Theorem 7 *Every derivation* Π *in LJT can be transformed into a cut-free derivation.*

Proof: Suppose Π is a derivation of degree $G(\Pi)$, $\#G(\Pi)$ is the number of formulas of Π with degree $G(\Pi)$ and $\ell(\Pi)$ is the lenght of Π. The proof is by induction on the pair $\langle \#G(\Pi), \ell(\Pi) \rangle$.

Let c be an application of cut such that the degree of the cut-formula of c is the degree of Π. We only show two cases, the others are analogous to cases already seen.

1. $$\Pi = \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma'_1\\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \wedge Q \vdash A}\, r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\, r_2}{\Gamma; P \wedge Q \vdash B}\, c\\ \Psi_3\end{array} \quad \rhd \quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma'_1\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\, r_2}{\Gamma; Q \vdash B}\, C_H}{\Gamma; P \wedge Q \vdash B}\, r_1\\ \Psi_3\end{array}$$

By induction hypothesis, any subderivation of Σ'_1 and Σ_2 reduces to a cut-free derivation. Hence, by lemma 12, the pseudo-derivation $\dfrac{\overset{\Sigma'_1}{\Gamma; Q \vdash A} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\, r_2}{\Gamma; Q \vdash B}\, c_H$ reduces to a cut-free pseudo-derivation Σ'. The resulting derivation $\Pi' = \begin{array}{c}\Sigma' \\ \Gamma; P \wedge Q \vdash B \\ \Psi_3\end{array}$ is such that $\#G(\Pi') < \#G(\Pi)$.

2. $$\Pi = \begin{array}{c}\dfrac{\dfrac{\overset{\Pi'_1}{\Gamma, P; \vdash Q}}{\Gamma; \vdash P \to Q}\,\vdash\to \quad \dfrac{\overset{\Pi_{21}}{\Gamma; \vdash P} \quad \overset{\Sigma_{22}}{\Gamma; Q \vdash B}}{\Gamma; P \to Q \vdash B}\,\to\vdash}{\Gamma; \vdash B}\, c \\ \Psi_3\end{array} \rhd$$

$$\begin{array}{c}\dfrac{\dfrac{\overset{\Pi_{21}}{\Gamma; \vdash P} \quad \overset{\Pi'_1}{\Gamma, P; \vdash Q}}{\Gamma; \vdash Q}\, c_M \quad \overset{\Sigma_{22}}{\Gamma; Q \vdash B}}{\Gamma; \vdash B}\, c_H \\ \Psi_3\end{array}$$

By IH, the sub-derivation $\dfrac{\overset{\Pi_{21}}{\Gamma; \vdash P} \quad \overset{\Pi'_1}{\Gamma, P; \vdash Q}}{\Gamma; \vdash Q}\, c_M$ reduces to a cut-free derivation $\Pi^{\#}$. The resulting derivation $\Pi' = \begin{array}{c}\dfrac{\overset{\Pi^{\#}}{\Gamma; \vdash Q} \quad \overset{\Sigma_{22}}{\Gamma; Q \vdash B}}{\Gamma; \vdash B} \\ \Psi_3\end{array}$ is such that $\#G(\Pi') < \#G(\Pi)$.

3. $$\Pi = \begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \dfrac{\overset{\Sigma_2}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}\, \mathcal{D}}{\Gamma; \vdash B}\, c_M \\ \Psi_3\end{array} \rhd \begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \overset{\Sigma_2}{\Gamma, A; A \vdash B}}{\Gamma; A \vdash B}\, c_M}{\Gamma; \vdash B}\, c_H \\ \Psi_3\end{array}$$

By lemma 12, $\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \overset{\Sigma_2}{\Gamma, A; A \vdash B}}{\Gamma; A \vdash B}\, c_M$ reduces to a cut-free pseudo-derivation Σ'. That the resulting derivation $\begin{array}{c}\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}\, r_1 \quad \overset{\Sigma'}{\Gamma; A \vdash B}}{\Gamma; \vdash B}\, c_H \\ \Psi_3\end{array}$ reduces to a cut-free derivation follows from the other cases.

■

5.3 Conversions in ND

Before defining the conversions, we need to define the notion of simplified derivation.

Definition 30 (Simplified (pseudo-)derivation) *A (pseudo-)derivation Ψ is* simplified *if no major premiss in Ψ is conclusion of an elimination rule.*

Lemma 13 *Every (pseudo-)derivation can be transformed into a simplified (pseudo-)derivation.*

Proof:

– If Ψ is of the form

$$\dfrac{\dfrac{\overset{\Psi_1}{\Gamma\vdash A\vee B}\quad \overset{\Psi_2}{\Gamma,A\vdash C\vee D}\quad \overset{\Psi_3}{\Gamma,B\vdash C\vee D}}{\Gamma\vdash C\vee D}E_\vee \qquad \overset{\Psi_4}{\Gamma,C\vdash E}\quad \overset{\Psi_5}{\Gamma,D\vdash E}}{\underset{\Psi_6}{\Gamma\vdash E}}E_\vee$$

then Ψ can be transformed in $\dfrac{\overset{\Psi_1}{\Gamma\vdash A\vee B}\quad \Psi_7\quad \Psi_8}{\underset{\Psi_6}{\Gamma\vdash E}}E_\vee$, where

$$\Psi_7 = \dfrac{\overset{\Psi_2}{\Gamma,A\vdash C\vee D}\quad \overset{\Psi_4^A}{\Gamma,A,C\vdash E}\quad \overset{\Psi_5^A}{\Gamma,A,D\vdash E}}{\Gamma,A\vdash E}E_\vee \quad \text{and}$$

$$\Psi_8 = \dfrac{\overset{\Psi_3}{\Gamma,B\vdash C\vee D}\quad \overset{\Psi_4^B}{\Gamma,B,C\vdash E}\quad \overset{\Psi_5^B}{\Gamma,B,D\vdash E}}{\Gamma,B\vdash E}E_\vee \ .$$

– If Ψ is of the form $\dfrac{\dfrac{\overset{\Psi_1}{\Gamma\vdash A\vee B}\quad \overset{\Psi_2}{\Gamma,A\vdash C}\quad \overset{\Psi_3}{\Gamma,B\vdash C}}{\Gamma\vdash C}E_\vee \qquad \overset{\Psi_4}{\Gamma\vdash D}}{\underset{\Psi_5}{\Gamma\vdash E}}E_\odot$,

then Ψ can be transformed into

$$\dfrac{\overset{\Psi_1}{\Gamma\vdash A\vee B}\qquad \dfrac{\overset{\Psi_2}{\Gamma,A\vdash C}\quad \overset{\Psi_4^A}{\Gamma,A\vdash D}}{\Gamma,A\vdash E}E_\odot \quad \dfrac{\overset{\Psi_3}{\Gamma,B\vdash C}\quad \overset{\Psi_4^B}{\Gamma,B\vdash D}}{\Gamma,B\vdash E}E_\odot}{\underset{\Psi_5}{\Gamma\vdash E}}E_\vee$$

where $\odot\in\{\wedge,\rightarrow\}$.

■

As in conversions for LJT, conversions in ND can be separated into trivial, reductive and permutative.

Definition 31 *Let Ψ be a non-normal pseudo-derivation of a derivation in ND. By definition, a pseudo-derivation cannot have a maximal sequent, hence all the cases shown here are cases where Ψ has at least one occurrence of a substitution rule.*

Let Ψ be a pseudo-derivation of the form

$$\begin{array}{c}\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \dfrac{\Psi_2}{\Gamma, A\vdash B}\,r_2}{\Gamma\vdash B}\,S\\ \Psi_3\end{array}$$

1. *Trivial conversions:*

$$\begin{array}{c}\dfrac{\dfrac{}{\Gamma, A\vdash A}\,Ax\quad \begin{array}{c}\Psi_2\\ \Gamma, A\vdash B\end{array}}{\Gamma, A\vdash B}\,S\\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\Psi_2\\ \Gamma, A\vdash B\\ \Psi_3\end{array}$$

$$\begin{array}{c}\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \dfrac{}{\Gamma, A\vdash A}\,Ax}{\Gamma\vdash A}\,S\\ \Psi_3\end{array} \;\rhd\; \begin{array}{c}\Psi_1\\ \Gamma\vdash A\\ \Psi_3\end{array}$$

2. *Reductive conversions:*

   (a) *If $r_2 \in \{E_{\rightarrow}, E_{\wedge}, E_{\perp}\}$, then*
$$\begin{array}{c}\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \dfrac{\begin{array}{c}\Psi_{12}\\ \Gamma, A\vdash B_1\end{array}\quad\ldots\quad\begin{array}{c}\Psi_{n2}\\ \Gamma, A\vdash B_n\end{array}}{\Gamma, A\vdash B}\,r_2}{\Gamma\vdash B}\,S\\ \Psi_3\end{array}$$
$$\rhd$$
$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \begin{array}{c}\Psi_{12}\\ \Gamma, A\vdash B_1\end{array}}{\Gamma\vdash B_1}\,S\quad\ldots\quad \dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \begin{array}{c}\Psi_{n2}\\ \Gamma, A\vdash B_n\end{array}}{\Gamma\vdash B_n}\,S}{\Gamma\vdash B}\,r_2\\ \Psi_3\end{array}$$

   where $1\leq n\leq 2$.

   (b) *If $r_2 = E_{\vee}$, then*
$$\begin{array}{c}\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \dfrac{\begin{array}{c}\Psi_{12}\\ \Gamma, A\vdash C\vee D\end{array}\quad \begin{array}{c}\Psi_{22}\\ \Gamma, A, C\vdash B\end{array}\quad \begin{array}{c}\Psi_{32}\\ \Gamma, A, D\vdash B\end{array}}{\Gamma, A\vdash B}\,E_{\vee}}{\Gamma\vdash B}\,S\\ \Psi_3\end{array} \;\rhd$$
$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Psi_1\\ \Gamma\vdash A\end{array}\quad \begin{array}{c}\Psi_{12}\\ \Gamma, A\vdash C\vee D\end{array}}{\Gamma\vdash C\vee D}\,S\quad \dfrac{\begin{array}{c}\Psi_1^C\\ \Gamma, C\vdash A\end{array}\quad \begin{array}{c}\Psi_{22}\\ \Gamma, A, C\vdash B\end{array}}{\Gamma, C\vdash B}\,S\quad \dfrac{\begin{array}{c}\Psi_1^D\\ \Gamma, D\vdash A\end{array}\quad \begin{array}{c}\Psi_{32}\\ \Gamma, A, D\vdash B\end{array}}{\Gamma, D\vdash B}\,S}{\Gamma\vdash B}\,E_{\vee}\\ \Psi_3\end{array}$$

Definition 32 (Conversion for derivations) *Most of the cases are analogous to the cases shown in definition 31. The other cases are as follows:*

1. *Σ has a maximal sequent of the form $\Gamma\vdash F$:*

$$F = A \land B: \quad \begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_2}{\Gamma \vdash B}}{\Gamma \vdash A \land B}\,{\scriptstyle I_\land}}{\Gamma \vdash A}\,{\scriptstyle E_\land} \\ \Psi_3 \end{array} \quad \rhd \quad \begin{array}{c} \Psi_1 \\ \Gamma \vdash A \\ \Psi_3 \end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_2}{\Gamma \vdash B}}{\Gamma \vdash A \land B}\,{\scriptstyle I_\land}}{\Gamma \vdash B}\,{\scriptstyle E_\land} \\ \Psi_3 \end{array} \quad \rhd \quad \begin{array}{c} \Psi_2 \\ \Gamma \vdash B \\ \Psi_3 \end{array}$$

$$F = A \to B: \quad \begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \to B}\,{\scriptstyle I_\to} \quad \overset{\Psi_2}{\Gamma \vdash A}}{\Gamma \vdash B}\,{\scriptstyle E_\to} \\ \Psi_3 \end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\overset{\Psi_2}{\Gamma \vdash A} \quad \overset{\Psi_1}{\Gamma, A \vdash B}}{\Gamma \vdash B}\,{\scriptstyle S} \\ \Psi_3 \end{array}$$

$$F = A \lor B: \quad \begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash A}}{\Gamma \vdash A \lor B}\,{\scriptstyle I_\lor} \quad \overset{\Psi_2}{\Gamma, A \vdash C} \quad \overset{\Psi_3}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,{\scriptstyle E_\lor} \\ \Psi_4 \end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_2}{\Gamma, A \vdash C}}{\Gamma \vdash C}\,{\scriptstyle S} \\ \Psi_4 \end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash B}}{\Gamma \vdash A \lor B}\,{\scriptstyle I_\lor} \quad \overset{\Psi_2}{\overset{\Psi_1}{\Gamma, A \vdash C}} \quad \overset{\Psi_3}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,{\scriptstyle E_\lor} \\ \Psi_4 \end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\overset{\Psi_1}{\Gamma \vdash B} \quad \overset{\Psi_3}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,{\scriptstyle S} \\ \Psi_4 \end{array}$$

2. *If* $r_2 \in \{I_\land, I_\lor,\}$, *then*

$$\begin{array}{c} \dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\Psi_{12}}{\Gamma, A \vdash B_1} \quad \ldots \quad \overset{\Psi_{n2}}{\Gamma, A \vdash B_n}}{\Gamma, A \vdash B}\,{\scriptstyle r_2}}{\Gamma \vdash B}\,{\scriptstyle S} \\ \Psi_3 \end{array} \quad \rhd$$

$$\begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_{12}}{\Gamma, A \vdash B_1}}{\Gamma \vdash B_1}\,{\scriptstyle S} \quad \ldots \quad \dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_{n2}}{\Gamma, A \vdash B_n}}{\Gamma \vdash B_n}\,{\scriptstyle S}}{\Gamma \vdash B}\,{\scriptstyle r_2} \\ \Psi_3 \end{array}$$

where $1 \leq n \leq 2$.

3. *If* $r_2 = I_\to$, *then*

$$\begin{array}{c} \dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\Psi'_2}{\Gamma, A, C \vdash D}}{\Gamma, A \vdash C \to D}\,{\scriptstyle I_\to}}{\Gamma \vdash C \to D}\,{\scriptstyle S} \\ \Psi_3 \end{array} \quad \rhd \quad \begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1^C}{\Gamma, C \vdash A} \quad \overset{\Psi'_2}{\Gamma, A, C \vdash D}}{\Gamma, C \vdash D}\,{\scriptstyle S}}{\Gamma \vdash C \to D}\,{\scriptstyle I_\to} \\ \Psi_3 \end{array}$$

Definition 33 *We say that two derivations* Π *and* Π' *are* equivalent *(*$\Pi \approx \Pi'$*) if* Π *and* Π' *reduce to a same normal derivation.*

Lemma 14 *If every derivation in ND can be transformed into a normal derivation, then every pseudo-derivation of a derivation in ND can be transformed into a normal pseudo-derivation.*

Proof: Suppose Σ is a pseudo-derivation of degree $G(\Sigma)$, $\#G(\Sigma)$ is the number of formulas of Σ with degree $G(\Sigma)$ and $\ell(\Sigma)$ is the lenght of Σ. The proof is by induction on the pair $\langle \#G(\Sigma), \ell(\Sigma)\rangle$.

Let S be an application of substitution rule such that the degree of the cut-formula is the degree of Σ. We only show one case, the others are analogous to cases already seen.

If $\Sigma = \begin{array}{c} \dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \dfrac{\overset{\Pi_{21}}{\Gamma, A \vdash B} \quad \overset{\Sigma_{22}}{\Gamma, A \vdash C \to B}}{\Gamma, A \vdash B}{\scriptstyle E_\to}}{\Gamma \vdash B}{\scriptstyle S} \\ \Sigma_3 \end{array}$, then

$$\Sigma \rhd \begin{array}{c} \dfrac{\dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \overset{\Pi_{21}}{\Gamma, A \vdash C}}{\Gamma \vdash C}{\scriptstyle S} \quad \dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \overset{\Sigma_{22}}{\Gamma, A \vdash C \to B}}{\Gamma \vdash C \to B}{\scriptstyle S}}{\Gamma \vdash B}{\scriptstyle E_\to} \\ \Sigma_3 \end{array}.$$

By hypothesis, $\Pi^\star = \dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \overset{\Pi_{21}}{\Gamma, A \vdash C}}{\Gamma \vdash C}{\scriptstyle S}$ reduces to a normal derivation $\Pi^\#$ and by IH $\Sigma^\star = \dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \overset{\Sigma_{22}}{\Gamma, A \vdash C \to B}}{\Gamma \vdash B}{\scriptstyle S}$ reduces to a normal pseudo-derivation $\Sigma^\#$. The resulting pseudo-derivation $\Sigma' = \begin{array}{c} \dfrac{\overset{\Pi^\#}{\Gamma \vdash C} \quad \overset{\Sigma^\#}{\Gamma \vdash C \to B}}{\Gamma \vdash B}{\scriptstyle E_\to} \\ \Sigma_3 \end{array}$ is such that $\#G(\Sigma') < \#G(\Sigma)$. ■

Theorem 8 *Every derivation in ND can be transformed into a normal derivation.*

Proof: Suppose Π is a derivation of degree $G(\Pi)$, $\#G(\Pi)$ is the number of formulas of Π with degree $G(\Pi)$ and $\ell(\Pi)$ is the lenght of Π. The proof is by induction on the pair $\langle \#G(\Pi), \ell(\Pi)\rangle$.

We only show two cases, the others are analogous to cases already seen. Let c be an application of $\mathcal{S}$ such that the degree of the cut-formula of c is the degree of Π.

1. $\Sigma = \begin{array}{c} \dfrac{\dfrac{\overset{\Psi_1}{\Gamma \vdash A} \quad \overset{\Psi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A} \\ \Sigma_3 \end{array} \rhd \begin{array}{c} \Psi_1 \\ \Gamma \vdash A \\ \Sigma_3 \end{array} = \Sigma'$

   It is easy to see that $\ell(\Sigma') < \ell(\Sigma)$.

2. If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash A\end{matrix} \quad \dfrac{\begin{matrix}\Pi'_2 \\ \Gamma, A, C \vdash D\end{matrix}}{\Gamma, A \vdash C \to D} I_\to}{\begin{matrix}\Gamma \vdash C \to D \\ \Psi_3\end{matrix}} S$, then $\Pi \rhd \dfrac{\dfrac{\begin{matrix}\Pi_1^C \\ \Gamma, C \vdash A\end{matrix} \quad \begin{matrix}\Pi'_2 \\ \Gamma, A, C \vdash D\end{matrix}}{\Gamma, C \vdash D} S}{\begin{matrix}\Gamma \vdash C \to D \\ \Psi_3\end{matrix}} I_\to$

   As $\dfrac{\begin{matrix}\Pi_1^C \\ \Gamma, C \vdash A\end{matrix} \quad \begin{matrix}\Pi'_2 \\ \Gamma, A, C \vdash D\end{matrix}}{\Gamma, C \vdash D} S$ is smaller than Π, by IH it reduces to a normal derivation $\Pi^{\#}$. The resulting derivation $\Pi' = \begin{matrix}\Pi^{\#} \\ \Gamma \vdash C \to D \\ \Psi_3\end{matrix}$ is such that $\#G(\Pi') < \#G(\Pi)$.

■

5.4 Translation

In this section we define transformations f' from pseudo-derivations of derivations in LJT to pseudo-derivations of derivations in ND, g' from derivations in LJT to derivations in ND, s' from pseudo-derivations of derivations in ND to pseudo-derivations of derivations in LJT and t' from derivations in ND to derivations in LJT and prove that these transformations hold. Finally, we show that the transformations form a bijection that preserves equivalence induced by normal form as defined in sections 5.2 and 5.3.

Lemmas 5 and 6 from section 4 can be extended to deal with cut and substitution rules:

Lemma 15 *If* $\begin{matrix}\Psi_i \\ \Gamma; \Delta_i \vdash A_i\end{matrix}$, $i = 1, 2$, *are (pseudo-)derivations in LJT, then*

1. $\dfrac{\begin{matrix}\Psi_1 \\ \Gamma; \vdash B\end{matrix} \quad \begin{matrix}\Psi_2 \\ \Gamma; B \vdash A\end{matrix}}{\Gamma; \vdash A} C_H$ *and* $\dfrac{\begin{matrix}\Psi_1 \\ \Gamma; \vdash B\end{matrix} \quad \begin{matrix}\Psi_2 \\ \Gamma, B; \vdash A\end{matrix}}{\Gamma; \vdash A} C_M$ *are derivations in LJT.*

2. $\dfrac{\begin{matrix}\Psi_1 \\ \Gamma; C \vdash B\end{matrix} \quad \begin{matrix}\Psi_2 \\ \Gamma; B \vdash A\end{matrix}}{\Gamma; C \vdash A} C_H$ *and* $\dfrac{\begin{matrix}\Psi_1 \\ \Gamma; \vdash B\end{matrix} \quad \begin{matrix}\Psi_2 \\ \Gamma, B; C \vdash A\end{matrix}}{\Gamma; C \vdash A} C_M$ *are pseudo-derivations of derivations in LJT.*

Proof: The proof is straight from the definition of derivation and of pseudo-derivation. ■

Lemma 16 *Let* $\dfrac{\Gamma \vdash A_1 \quad \Gamma, A_1 \vdash A}{\Gamma \vdash A}$ *be a substitution rule in ND.*

1. *If* $\begin{array}{c}\Pi_1\\ \Gamma\vdash A_1\end{array}$ *and* $\begin{array}{c}\Pi_2\\ \Gamma, A_2\vdash A\end{array}$ *are derivations in ND and* $\begin{array}{c}\Gamma\vdash A\\ \Sigma\end{array}$ *is a pseudo-derivation of a derivation in ND, then* $\begin{array}{c}\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma\vdash A_1 & \Gamma, A_1\vdash A\end{array}}{\Gamma\vdash A}\\ \Sigma\end{array}$ *is a derivation in ND.*

2. *If* $\begin{array}{c}\Pi_1\\ \Gamma\vdash A_1\end{array}$ *is a derivation in ND and* $\begin{array}{c}\Gamma\vdash A\\ \Sigma\end{array}$ *is a pseudo-derivation of a derivation in ND, then* $\begin{array}{c}\dfrac{\begin{array}{cc}\Pi_1 & \\ \Gamma\vdash A_1 & \Gamma, A_1\vdash A\end{array}}{\Gamma\vdash A}\\ \Sigma\end{array}$ *is a pseudo-derivation of a derivation in ND.*

Proof: The proof follows from the definition of pseudo-derivation and of derivation in ND. ■

Definition 34 **(f')** *Let Σ be a pseudo-derivation of a derivation Π in LJT. If g' is a translation from derivations in LJT to derivations in ND then the translation f' from pseudo-derivations of derivations in LJT to pseudo-derivations of derivations in ND is defined recursively as follows:*

Let c be the bottommost rule applied in Σ.

1. *If Σ is cut-free, then $f'(\Sigma) = f(\Sigma)$, where f is the function defined last chapter (definition 19).*

2. *If c is a left rule, then the definition of f' is analogous to the definition of f (by the definition of pseudo-derivation, c cannot be a right rule):*

   (a) *If* $\Sigma = \dfrac{\begin{array}{cc}\Pi' & \Sigma'\\ \Gamma;\vdash A & \Gamma; B\vdash C\end{array}}{\Gamma; A\to B\vdash C}\ {\to}\vdash$ *, then* $f'(\Sigma) =$
   $$\begin{array}{c}\dfrac{\begin{array}{cc}g'(\Pi') & \\ \Gamma\vdash A & \Gamma\vdash A\to B\end{array}}{\Gamma\vdash B}\ E_{\to}\\ f'(\Sigma')\end{array}$$

   (b) *If* $\Sigma = \dfrac{\begin{array}{c}\Sigma'\\ \Gamma, A; B\vdash C\end{array}}{\Gamma; A\wedge B\vdash C}\ \wedge\vdash$ *, then* $f'(\Sigma) = \begin{array}{c}\dfrac{\Gamma\vdash A\wedge B}{\Gamma\vdash B}\ E_{\wedge}\\ f'(\Sigma')\end{array}$

   (c) *If* $\Sigma = \dfrac{\begin{array}{c}\Sigma'\\ \Gamma, B; A\vdash C\end{array}}{\Gamma; A\wedge B\vdash C}\ \wedge\vdash$ *, then* $f'(\Sigma) = \begin{array}{c}\dfrac{\Gamma\vdash A\wedge B}{\Gamma\vdash A}\ E_{\wedge}\\ f'(\Sigma')\end{array}$

   (d) *If* $\Sigma = \dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma, A;\vdash C & \Gamma, B;\vdash C\end{array}}{\Gamma; A\vee B\vdash C}\ \vee\vdash$ *,*

   then $f'(\Sigma) = \dfrac{\begin{array}{ccc} & g'(\Pi_1) & g'(\Pi_2)\\ \Gamma\vdash A\vee B & \Gamma, A\vdash C & \Gamma, B\vdash C\end{array}}{\Gamma\vdash C}\ E_{\vee}$

$$\text{(e) If } \Sigma = \begin{array}{c}\Sigma' \\ \dfrac{\Gamma; A \vdash C}{\Gamma; \bot \vdash C}\,{\scriptstyle \bot\vdash}\end{array}, \text{ then } f'(\Sigma) = \begin{array}{c}\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}\,{\scriptstyle E_\bot} \\ f'(\Sigma')\end{array}$$

3. If c is a head-cut, then

$$\text{(a)}\quad f'\left(\dfrac{\dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; A \vdash B}\,c\right) = f'\left(\begin{array}{c}\Sigma_2 \\ \Gamma; A \vdash B\end{array}\right)$$

$$\text{(b)}\quad f'\left(\dfrac{\dfrac{\Sigma_1}{\Gamma; D \vdash A}\,r_1 \quad \dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax}}{\Gamma; D \vdash A}\,c\right) = f'\left(\begin{array}{c}\Sigma_1 \\ \Gamma; D \vdash A\end{array}\right)$$

$$\text{(c)}\quad f'\left(\dfrac{\dfrac{\begin{array}{c}\Sigma_1' \\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \wedge Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \wedge Q \vdash B}\,c\right) =$$

$$f'\left(\dfrac{\dfrac{\begin{array}{c}\Sigma_1' \\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \wedge Q \vdash B}\,r_1\right)$$

$$\text{(d)}\quad f'\left(\dfrac{\dfrac{\begin{array}{c}\Sigma_1' \\ \Gamma; P \vdash A\end{array}}{\Gamma; P \wedge Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \wedge Q \vdash B}\,c\right) =$$

$$f'\left(\dfrac{\dfrac{\begin{array}{c}\Sigma_1' \\ \Gamma, Q; P \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \wedge Q \vdash B}\,r_1\right)$$

$$\text{(e)}\quad f'\left(\dfrac{\dfrac{\begin{array}{c}\Pi_{11} \\ \Gamma, P; \vdash A\end{array} \quad \begin{array}{c}\Pi_{12} \\ \Gamma, Q; \vdash A\end{array}}{\Gamma; P \vee Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vee Q \vdash B}\,c\right) =$$

$$f'\left(\dfrac{\dfrac{\begin{array}{c}\Pi_{11} \\ \Gamma, P; \vdash A\end{array} \quad \dfrac{\Sigma_2^P}{\Gamma, P; A \vdash B}\,r_2}{\Gamma, P; \vdash B}\,{\scriptstyle C_H} \quad \dfrac{\begin{array}{c}\Pi_{12} \\ \Gamma, Q; \vdash A\end{array} \quad \dfrac{\Sigma_2^Q}{\Gamma, Q; A \vdash B}\,r_2}{\Gamma, Q; \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \vee Q \vdash B}\,r_1\right)$$

$$\text{(f)}\quad f'\left(\dfrac{\dfrac{\begin{array}{c}\Pi_{11} \\ \Gamma; \vdash P\end{array} \quad \begin{array}{c}\Sigma_{12} \\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \to Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \to Q \vdash B}\,c\right) =$$

$$f'\left(\dfrac{\begin{array}{c}\Pi_{11} \\ \Gamma; \vdash P\end{array} \quad \dfrac{\begin{array}{c}\Sigma_{12} \\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,{\scriptstyle C_H}}{\Gamma; P \to Q \vdash B}\,r_1\right)$$

$$\textit{(g)}\ \ f'\left(\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ \Gamma;P\vdash A\end{matrix}}{\Gamma;\bot\vdash A}{\scriptstyle\bot\vdash}\quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}{\scriptstyle r_2}}{\Gamma;\bot\vdash B}{\scriptstyle c}\right)=f'\left(\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ \Gamma;P\vdash A\end{matrix}\quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}{\scriptstyle r_2}}{\Gamma;P\vdash B}{\scriptstyle C_H}}{\Gamma;\bot\vdash B}{\scriptstyle\bot\vdash}\right)$$

4. *If c is a middle-cut, then*

$$\textit{(a)}\ \ f'\left(\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}{\scriptstyle r_1}\quad \dfrac{}{\Gamma,A;B\vdash B}{\scriptstyle Ax}}{\Gamma;B\vdash B}{\scriptstyle c}\right)=f'\left(\dfrac{}{\Gamma;B\vdash B}{\scriptstyle Ax}\right)$$

$$\textit{(b)}\ \ f'\left(\dfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash A\end{matrix}\quad \dfrac{\Sigma_2}{\Gamma,A;C\vdash B}{\scriptstyle\odot\vdash}}{\Gamma;C\vdash B}{\scriptstyle c}\right)\quad=\quad \begin{matrix}\dfrac{\dfrac{\begin{matrix}g'(\Pi_1)\\ \Gamma\vdash A\end{matrix}\quad \Gamma,A\vdash C}{\Gamma\vdash C}{\scriptstyle S}}{f'(\Sigma_2)}{\scriptstyle E_\odot}\end{matrix},\quad \odot\in$$

$\{\wedge,\vee,\rightarrow,\bot\}$

Definition 35 **(g')** *Let Π be a derivation in LJT. The transformation g' from derivations in LJT to derivations in ND is defined recursively as follows:*

Let c be the bottommost rule applied in Π.

1. *If Π is cut-free, then $g'(\Pi)=g(\Pi)$, where g is the function defined last chapter (definition* **??***).*

2. *If c is either a $\mathcal{D}$ rule or a right rule, then the definition of g' is analogous to the definition of g:*

$$(\vdash\rightarrow)\ \ \textit{If}\ \Pi=\dfrac{\begin{matrix}\Pi'\\ \Gamma,A;\vdash B\end{matrix}}{\Gamma;\vdash A\rightarrow B}{\scriptstyle\vdash\rightarrow},\ \textit{then}\ g'(\Pi)=\dfrac{\begin{matrix}g'(\Pi')\\ \Gamma,A\vdash B\end{matrix}}{\Gamma\vdash A\rightarrow B}{\scriptstyle I_\rightarrow}$$

$$(\vdash\wedge)\ \ \textit{If}\ \Pi=\dfrac{\begin{matrix}\Pi_1\\ \Gamma;\vdash A\end{matrix}\quad\begin{matrix}\Pi_2\\ \Gamma;\vdash B\end{matrix}}{\Gamma;\vdash A\wedge B}{\scriptstyle\vdash\wedge},\ \textit{then}\ g'(\Pi)=\dfrac{\begin{matrix}g'(\Pi_1)\\ \Gamma\vdash A\end{matrix}\quad\begin{matrix}g'(\Pi_2)\\ \Gamma\vdash B\end{matrix}}{\Gamma\vdash A\wedge B}{\scriptstyle I_\wedge}$$

$$(\vdash\vee)\ \ \textit{If}\ \Pi=\dfrac{\begin{matrix}\Pi'\\ \Gamma;\vdash A\end{matrix}}{\Gamma;\vdash A\vee B}{\scriptstyle\vdash\vee},\ \textit{then}\ g'(\Pi)=\dfrac{\begin{matrix}g'(\Pi')\\ \Gamma\vdash A\end{matrix}}{\Gamma\vdash A\vee B}{\scriptstyle I_\vee}$$

$$\textit{If}\ \Pi=\dfrac{\begin{matrix}\Pi'\\ \Gamma;\vdash B\end{matrix}}{\Gamma;\vdash A\vee B}{\scriptstyle\vdash\vee},\ \textit{then}\ g'(\Pi)=\dfrac{\begin{matrix}g'(\Pi')\\ \Gamma\vdash B\end{matrix}}{\Gamma\vdash A\vee B}{\scriptstyle I_\vee}$$

$$(\mathcal{D})\ \ \textit{If}\ \Pi=\dfrac{\begin{matrix}\Sigma'\\ \Gamma;A\vdash B\end{matrix}}{\Gamma;\vdash B}{\scriptstyle\mathcal{D}},\ \textit{then}\ g'(\Pi)=\begin{matrix}\dfrac{}{\Gamma\vdash A}{\scriptstyle Ax}\\ f'(\Sigma')\end{matrix}$$

3. *If c is a head-cut, then*

$$\textit{(a)}\ \ g'\left(\dfrac{\dfrac{\Pi_1}{\Gamma;\vdash A}{\scriptstyle r_1}\quad \dfrac{}{\Gamma;A\vdash A}{\scriptstyle Ax}}{\Gamma;\vdash A}{\scriptstyle c}\right)=g'\left(\begin{matrix}\Pi_1\\ \Gamma;\vdash A\end{matrix}\right)$$

$$\textit{(b)}\ g'\left(\dfrac{\dfrac{\stackrel{\Sigma_1'}{\Gamma; P \vdash A}}{\Gamma; \vdash A}\mathcal{D} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}r_2}{\Gamma; \vdash B}c\right) = g'\left(\dfrac{\dfrac{\stackrel{\Sigma_1'}{\Gamma; P \vdash A} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}r_2}{\Gamma; P \vdash B}C_H}{\Gamma; \vdash B}\mathcal{D}\right)$$

$$\textit{(c)}\ g'\left(\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash C}\vdash\odot \quad \dfrac{\Sigma_2}{\Gamma; C \vdash B}\odot\vdash}{\Gamma; \vdash B}c\right) = \begin{array}{c}\dfrac{g'(\Pi_1)}{\Gamma \vdash C}I_\odot \\ \hline f'(\Sigma_2)\end{array}E_\odot$$

4. If c is a middle-cut, then

$$\textit{(a)}\ g'\left(\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}r_1 \quad \dfrac{\stackrel{\Sigma_2'}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}\mathcal{D}}{\Gamma; \vdash B}C_M\right) =$$

$$g'\left(\dfrac{\dfrac{\dfrac{\Pi_1^A}{\Gamma, A; \vdash A}r_1 \quad \stackrel{\Sigma_2'}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}C_H \quad \dfrac{\Pi_1}{\Gamma; \vdash A}r_1}{\Gamma; \vdash B}C_M\right)$$

$$\textit{(b)}\ g'\left(\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}r_1 \quad \dfrac{\stackrel{\Sigma_2'}{\Gamma, A; C \vdash B}}{\Gamma, A; \vdash B}\mathcal{D}}{\Gamma; \vdash B}c\right) = g'\left(\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma; \vdash A}r_1 \quad \stackrel{\Sigma_2'}{\Gamma, A; C \vdash B}}{\Gamma; C \vdash B}C_M}{\Gamma; \vdash B}\mathcal{D}\right)$$

$$\textit{(c)}\ g'\left(\dfrac{\stackrel{\Pi_1}{\Gamma; \vdash A} \quad \stackrel{\Pi_2}{\Gamma, A; \vdash B}}{\Gamma; \vdash B}c\right) = \dfrac{\stackrel{g'(\Pi_1)}{\Gamma \vdash A} \quad \stackrel{g'(\Pi_2)}{\Gamma, A \vdash B}}{\Gamma \vdash B}\mathcal{S}$$

Lemma 17 *If g' is a translation from derivations in LJT to derivations in ND, then, if Σ is a pseudo-derivation of a derivation in LJT, then $f'(\Sigma)$ is a pseudo-derivation of a derivation in ND.*

Proof: Let Π be a derivation of B from Γ and let Σ be a pseudo-derivation of Π. The proof is by induction on the length of Σ and on the rank of the cut-formulas of Σ (if any) and it is straight from definition 34. We show two cases.

1. If $\Sigma = \dfrac{\stackrel{\Pi'}{\Gamma; \vdash P} \quad \stackrel{\Sigma'}{\Gamma; Q \vdash B}}{\Gamma; P \to Q \vdash B}{\to\vdash}$, then, by definition 34, $f'(\Sigma) =$

$$\begin{array}{c}\dfrac{\stackrel{g'(\Pi')}{\Gamma \vdash P} \quad \Gamma \vdash P \to Q}{\Gamma \vdash Q}E_\to \\ f'(\Sigma')\end{array}$$

By hypothesis, $g'(\Pi')$ is a derivation in ND and, as $\ell(\Sigma') < \ell(\Sigma)$, by IH, $f'(\Sigma')$ is a pseudo-derivation of a derivation in ND. Hence, $f'(\Sigma)$ is a pseudo-derivation of a derivation in ND.

2. If $\Sigma = \dfrac{\dfrac{\begin{matrix}\Pi_{11} \\ \Gamma;\vdash P\end{matrix} \quad \begin{matrix}\Sigma_{12} \\ \Gamma;Q\vdash A\end{matrix}}{\Gamma;P\to Q\vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\Gamma;P\to Q\vdash B}\,c$ then, by definition 34,

$$f'(\Sigma) = f'(\Sigma'),\ \Sigma' = \dfrac{\begin{matrix}\Pi_{11} \\ \Gamma;\vdash P\end{matrix} \quad \dfrac{\begin{matrix}\Sigma_{12} \\ \Gamma;Q\vdash A\end{matrix} \quad \dfrac{\Sigma_2}{\Gamma;A\vdash B}\,r_2}{\Gamma;Q\vdash B}\,c_H}{\Gamma;P\to Q\vdash B}\,r_1$$

It is easy to see that the rank of the lowest cut-rule applied in Σ' is smaller than the one in Σ. Hence, by IH, $f'(\Sigma)$ is a pseudo-derivation of a derivation of in ND.

■

Theorem 9 *If* Π *is a derivation in LJT, then* $g'(\Pi)$ *is a derivation in ND.*

Proof: The proof is by induction on the length of Π and it is straight from definition 35. We show two cases.

Let Π be a derivation in LJT of B from Γ.

1. If $\Pi = \dfrac{\begin{matrix}\Sigma' \\ \Gamma, A;\vdash B\end{matrix}}{\Gamma;\vdash B}\,\mathcal{D}$, then, by definition 35, $g'(\Pi) = \begin{matrix}\dfrac{}{\Gamma\vdash A}\,Ax \\ f'(\Sigma')\end{matrix}$

   By lemma **??**, $f'(\Sigma')$ is a pseudo-derivation of a derivation in ND. Hence, $g'(\Pi)$ is a derivation of B from Γ in ND.

2. If $\Pi = \dfrac{\dfrac{\Pi_1}{\Gamma;\vdash B}\,r_1 \quad \dfrac{}{\Gamma;B\vdash B}\,Ax}{\Gamma;\vdash B}\,c$ then, by definition 35, $g'(\Pi) = g'\left(\begin{matrix}\Pi_1 \\ \Gamma;\vdash B\end{matrix}\right)$.

   As $\begin{matrix}\Pi_1 \\ \Gamma;\vdash B\end{matrix}$ is smaller than Π, by IH, $g'(\Pi)$ is a derivation of B from Γ in ND.

■

Definition 36 (s') *Let* Σ *be a pseudo-derivation of a derivation* Π *in ND. If* t' *is a translation from derivations in ND to derivations in LJT then the translation* s' *from pseudo-derivations of derivations in ND to pseudo-derivations of derivation in LJT is defined recursively as follows:*

Let c *be the uppermost rule applied in* Σ.

1. *If* Σ *is normal, then* $s'(\Sigma) = s(\Sigma)$*, where* s *is the function defined last chapter (definition 21).*

2. *If* c *is an elimination rule, then the definition of* s' *is analogous to the definition of* s*:*

(a) *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}{\scriptstyle E_\wedge} \\ \Sigma'\end{array}$ *, then* $s'(\Sigma) = \begin{array}{c} s'(\Sigma') \\ \dfrac{\Gamma, B; A \vdash C}{\Gamma; A \wedge B \vdash C}{\scriptstyle \wedge\vdash}\end{array}$

(b) *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash B \wedge A}{\Gamma \vdash A}{\scriptstyle E_\wedge} \\ \Sigma'\end{array}$ *, then* $s'(\Sigma) = \begin{array}{c} s'(\Sigma') \\ \dfrac{\Gamma, B; A \vdash C}{\Gamma; B \wedge A \vdash C}{\scriptstyle \wedge\vdash}\end{array}$

(c) *If* $\Sigma = \begin{array}{c}\dfrac{\begin{array}{c}\Pi' \\ \Gamma \vdash A\end{array} \quad \Gamma \vdash A \to B}{\Gamma \vdash B}{\scriptstyle E_\to} \\ \Sigma'\end{array}$ *, then* $s'(\Sigma) = \dfrac{\begin{array}{c}t'(\Pi') \\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s'(\Sigma') \\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}{\scriptstyle \to\vdash}$

(d) *If* $\Sigma = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}\Pi_1 \\ \Gamma, A \vdash D\end{array} \quad \begin{array}{c}\Pi_2 \\ \Gamma, B \vdash D\end{array}}{\Gamma \vdash D}{\scriptstyle E_\vee}$,

then $s'(\Sigma) = \dfrac{\begin{array}{c}t'(\Pi_1) \\ \Gamma, A; \vdash D\end{array} \quad \begin{array}{c}t'(\Pi_2) \\ \Gamma, B; \vdash D\end{array}}{\Gamma, A \vee B; \vdash D}{\scriptstyle \vee\vdash}$

(e) *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}{\scriptstyle E_\bot} \\ \Sigma'\end{array}$ *, then* $t'(\Sigma) = \begin{array}{c} s'(\Sigma') \\ \dfrac{\Gamma; A \vdash C}{\Gamma; \bot \vdash C}{\scriptstyle \bot\vdash}\end{array}$

3. *If* c *is a substitution rule, then*

$$s'\left(\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A\end{array} \quad \Gamma, A \vdash C}{\Gamma \vdash C}{\scriptstyle s} \\ \Sigma_3\end{array}\right) = \dfrac{\begin{array}{c}t'(\Pi_1) \\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s'(\Sigma_3^A) \\ \Gamma, A; C \vdash B\end{array}}{\Gamma; C \vdash B}{\scriptstyle C_M}$$

, where B *is a c-formula. Remember that, from the definition of pseudo-derivation, both* $\Gamma, A \vdash C$ *and* $\Gamma \vdash C$ *are major premisses.*

Definition 37 **(t')** *Let* Π *be a derivation in ND. The transformation* t' *from derivations in LJT to derivations in ND is defined recursively as follows:*

Let c *be the bottommost rule applied in* Π.

1. *If* Π *is normal, then* $t'(\Pi) = t(\Pi)$.

2. *If* c *is an introduction rule, then* $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma_1 \vdash C_1\end{array} \quad \dots \quad \begin{array}{c}\Pi_n \\ \Gamma_n \vdash C_n\end{array}}{\Gamma \vdash C}{\scriptstyle I_\odot}$,

and $t(\Pi) = \dfrac{\begin{array}{c}t(\Pi_1) \\ \Gamma_1; \vdash C_1\end{array} \quad \dots \quad \begin{array}{c}t(\Pi_n) \\ \Gamma_n; \vdash C_n\end{array}}{\Gamma; \vdash C}{\scriptstyle \odot\vdash}$.

3. *If* c *is an elimination rule, then we have two cases to consider:*

   (a) *there is no occurrence of an introduction rule in* Π *and then the definition of* s' *is analogous to the definition of* s.

(b) *there is an occurrence of an introduction rule in* Π *and then, if it is the case, we first simplify* Π. *The resulting derivation has a maximal sequent* $\Gamma \vdash C$ *and is of the form* $\Pi' = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash C\end{array}}{\Sigma_2}\begin{array}{l}{\scriptstyle I_\odot}\\ {\scriptstyle E_\odot}\end{array}$ *and* $t'(\Pi') =$

$$\dfrac{\dfrac{t'(\Pi_1)}{\Gamma; \vdash C}\vdash\odot \quad \dfrac{s'(\Sigma_2)}{\Gamma; C \vdash E}\odot\vdash}{\Gamma; \vdash E} C_H .$$

4. *If c is an* $\mathcal{S}$ *rule, then* $t'\left(\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma \vdash A & \Gamma, A \vdash B\end{array}}{\Gamma \vdash B}\mathcal{S}\right) = \dfrac{\begin{array}{cc}t'(\Pi_1) & t'(\Pi_2)\\ \Gamma; \vdash A & \Gamma, A; \vdash B\end{array}}{\Gamma; \vdash B} C_M$

Lemma 18 *If* t' *is a translation from derivations in ND to derivations in LJT, then, if* Σ *is a pseudo-derivation of a derivation in ND, then* $s'(\Sigma)$ *is a pseudo-derivation of a derivation in LJT.*

Proof: Let Σ be a pseudo-derivation of a derivation in ND. The proof is by induction on the length of Σ and it is straight from definition 36. We show one case:

If $\Sigma = \dfrac{\begin{array}{cc}\Pi_1 & \\ \Gamma \vdash A & \Gamma, A \vdash C\end{array}}{\begin{array}{c}\Gamma \vdash C\\ \Sigma_3\end{array}}S$, then, by definition 36, $s'(\Sigma) =$ $\dfrac{\begin{array}{cc}t'(\Pi_1) & s'(\Sigma_3^A)\\ \Gamma; \vdash A & \Gamma, A; C \vdash B\end{array}}{\Gamma; C \vdash B} C_M$

By hypothesis, $t'(\Pi_1)$ is a derivation in LJT and, by IH, $s'(\Sigma_3^A)$ is a pseudo-derivation of a derivation in LJT. Hence, by lemma 15, $s'(\Sigma)$ is a pseudo-derivation of a derivation in LJT.

■

Theorem 10 *If* Π *is a derivation in ND, then* $t'(\Pi)$ *is a derivation in LJT.*

Proof: Let Π be a derivation in ND. The proof is by induction on the length of Π and it is straight from definition 37. We show one case:

If $\Pi' = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash C\end{array}}{\Sigma_2}\begin{array}{l}{\scriptstyle I_\odot}\\ {\scriptstyle E_\odot}\end{array}$, $\odot \in \{\wedge, \vee, \rightarrow\}$, then, by definition 37, $t'(\Pi') =$ $\dfrac{\dfrac{t'(\Pi_1)}{\Gamma; \vdash C}\vdash\odot \quad \dfrac{s'(\Sigma_2)}{\Gamma; C \vdash E}\odot\vdash}{\Gamma; \vdash E} C_H .$

By IH, $t'(\Pi_1)$ is a derivation in LJT and by lemma 18, $s'(\Sigma_2)$ is a pseudo-derivation of a derivation in LJT. Hence, by lemma 15, $t'(\Pi')$ is a derivation in LJT. ■

Lemma 19 *If $t'(g'(\Pi)) \approx \Pi$ for every derivation Π in LJT then, for every pseudo-derivation Σ of Π,*

$$s'(f'(\Sigma)) \approx \Sigma.$$

Proof: The proof is by induction on the length of Σ. Suppose that $t'(g'(\Pi)) \approx \Pi$ for every derivation Π and let Σ be a pseudo-derivation of Π. We will make three cases, the others are analogous to cases already seen.

1. If $\Sigma = \dfrac{\dfrac{\Sigma_1}{\Gamma; D \vdash A}\,r_1 \quad \dfrac{}{\Gamma; A \vdash A}\,Ax}{\Gamma; D \vdash A}\,c$, then $s'f'(\Sigma) = s'f'\left(\begin{array}{c}\Sigma_1\\ \Gamma; D \vdash A\end{array}\right)$.

   As $\Sigma' = \begin{array}{c}\Sigma_1\\ \Gamma; D \vdash A\end{array}$ is smaller than Σ, by IH $s'f'(\Sigma') \approx \Sigma'$. As $\Sigma \approx \Sigma'$, $s'f'(\Sigma) \approx \Sigma$.

2. If $\Sigma = \dfrac{\dfrac{\begin{array}{c}\Sigma_1'\\ \Gamma; Q \vdash A\end{array}}{\Gamma; P \wedge Q \vdash A}\,r_1 \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \wedge Q \vdash B}\,c$, then $s'f'(\Sigma) =$
   $s'f'\left(\dfrac{\dfrac{\begin{array}{c}\Sigma_1'\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,c_H}{\Gamma; P \wedge Q \vdash B}\,r_1\right)$.

   As $\Sigma' = \dfrac{\begin{array}{c}\Sigma_1'\\ \Gamma; Q \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; Q \vdash B}\,c_H$ is smaller than Σ, we have that $s'f'(\Sigma') \approx \Sigma'$. As the last rule applied in $\begin{array}{c}\Sigma'\\ \Gamma; P \wedge Q \vdash B\end{array}$ is an elimination rule, we have that
   $s'f'\left(\begin{array}{c}\Sigma'\\ \Gamma; P \wedge Q \vdash B\end{array}\right) = s'\left(\begin{array}{c}\Gamma \vdash P \wedge Q\\ f'(\Sigma')\end{array}\right) = \begin{array}{c}s'f'(\Sigma')\\ \Gamma; P \wedge Q \vdash B\end{array} \approx \begin{array}{c}\Sigma'\\ \Gamma; P \wedge Q \vdash B\end{array}$.

   As $\Sigma \approx \begin{array}{c}\Sigma'\\ \Gamma; P \wedge Q \vdash B\end{array}$, $s'f'(\Sigma) \approx \Sigma$.

3. If $\Sigma = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma; \vdash A\end{array} \quad \dfrac{\Sigma_2}{\Gamma, A; C \vdash B}\,\odot\vdash}{\Gamma; C \vdash B}\,c$, then $s'f'(\Sigma) =$
   $s'\left(\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}g'(\Pi_1)\\ \Gamma \vdash A\end{array} \quad \Gamma, A \vdash C}{\Gamma \vdash C}\,S}{f'(\Sigma_2)}\,E_\odot\end{array}\right) = \dfrac{\begin{array}{c}t'g'(\Pi_1)\\ \Gamma; \vdash A\end{array} \quad \dfrac{s'f'(\Sigma_2)}{\Gamma, A; C \vdash B}\,\odot\vdash}{\Gamma; C \vdash B}\,c$.

   By hypothesis, $t'g'(\Pi_1) \approx \Pi_1$ and by IH, $s'f'(\Sigma_2) = \Sigma_2$. Hence,
   $\dfrac{\begin{array}{c}t'g'(\Pi_1)\\ \Gamma; \vdash A\end{array} \quad \dfrac{s'f'(\Sigma_2)}{\Gamma, A; C \vdash B}\,\odot\vdash}{\Gamma; C \vdash B}\,c \approx \Sigma$

■

Theorem 11 *For every derivation* Π *in LJT*

$$t'(g'(\Pi)) \approx \Pi.$$

Proof: The proof is by induction on the length of the derivation. Let Π be a derivation. We will make three cases, the others are analogous to cases already seen.

1. If $\Pi = \dfrac{\dfrac{\overset{\Sigma_1'}{\Gamma; P \vdash A}}{\Gamma; \vdash A}\,\mathcal{D} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; \vdash B}\,c$, then

   $$t'g'(\Pi) = t'g'\left(\dfrac{\dfrac{\overset{\Sigma_1'}{\Gamma; P \vdash A} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vdash B}\,c_H}{\Gamma; \vdash B}\,\mathcal{D}\right).$$

   Consider $\Sigma' = \dfrac{\overset{\Sigma_1'}{\Gamma; P \vdash A} \quad \dfrac{\Sigma_2}{\Gamma; A \vdash B}\,r_2}{\Gamma; P \vdash B}\,c_H$. By lemma 19 and by IH, we have that $s'f'(\Sigma') \approx \Sigma'$. As the last rule applied in $\overset{\Sigma'}{\Gamma; \vdash B}$ is the $\mathcal{D}$ rule having $\Gamma; P \vdash B$ as hypothesis, we have that

   $$t'g'\left(\overset{\Sigma'}{\Gamma; \vdash B}\right) = t'\left(\begin{matrix}\dfrac{}{\Gamma \vdash P}\,Ax \\ f'(\Sigma')\end{matrix}\right) = \overset{s'f'(\Sigma')}{\Gamma; \vdash B} \approx \overset{\Sigma'}{\Gamma; \vdash B}$$

   As $\Pi \approx \overset{\Sigma'}{\Gamma; \vdash B}$, $t'g'(\Pi) \approx \Pi$.

2. If $\Pi = \dfrac{\dfrac{\Pi_1}{\Gamma; \vdash C} \vdash\odot \quad \dfrac{\Sigma_2}{\Gamma; C \vdash B} \odot\vdash}{\Gamma; \vdash B}\,c$, then $t'g'(\Pi) = t'\left(\begin{matrix}\dfrac{g'(\Pi_1)}{\Gamma \vdash C}\,I_\odot \\ \overline{f'(\Sigma_2)}\,E_\odot\end{matrix}\right) =$ $\dfrac{\dfrac{t'g'(\Pi_1)}{\Gamma; \vdash C} \vdash\odot \quad \dfrac{s'f'(\Sigma_2)}{\Gamma; C \vdash B} \odot\vdash}{\Gamma; \vdash B}\,c$.

   By IH, $t'g'(\Pi_1) \approx \Pi_1$ and by lemma 19 and by IH, $s'f'(\Sigma_2) \approx \Sigma_2$. Hence, $t'g'(\Pi) \approx \Pi$.

3. If $\Pi = \dfrac{\overset{\Pi_1}{\Gamma; \vdash A} \quad \dfrac{\overset{\Sigma_2'}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}\,\mathcal{D}}{\Gamma; \vdash B}\,c_M$, then

   $$t'g'(\Pi) = t'g'\left(\dfrac{\overset{\Pi_1}{\Gamma; \vdash A} \quad \dfrac{\overset{\Pi_1^A}{\Gamma, A; \vdash A} \quad \overset{\Sigma_2'}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}\,c_H}{\Gamma; \vdash B}\,c_M\right).$$

As $\Pi' = \dfrac{\overset{\Pi_1^A}{\Gamma, A; \vdash A} \quad \overset{\Sigma_2'}{\Gamma, A; A \vdash B}}{\Gamma, A; \vdash B}\, C_H$ is smaller than Π, we have that $t'g'(\Pi') \approx \Pi'$. Hence, we have that

$$t'g'\left(\dfrac{\overset{\Pi_1}{\Gamma; \vdash A} \quad \overset{\Pi'}{\Gamma, A; \vdash B}}{\Gamma; \vdash B}\, C_M\right) = t'\left(\dfrac{\overset{g'(\Pi_1)}{\Gamma \vdash A} \quad \overset{g'(\Pi')}{\Gamma, A \vdash B}}{\Gamma \vdash B}\, \mathcal{S}\right) =$$

$$\dfrac{\overset{t'g'(\Pi_1)}{\Gamma; \vdash A} \quad \overset{t'g'(\Pi')}{\Gamma, A; \vdash B}}{\Gamma; \vdash B}\, C_M \approx \dfrac{\overset{\Pi_1}{\Gamma; \vdash A} \quad \overset{\Pi'}{\Gamma, A; \vdash B}}{\Gamma; \vdash B}\, C_M .$$

As $\Pi \approx \dfrac{\overset{\Pi_1}{\Gamma; \vdash A} \quad \overset{\Pi'}{\Gamma, A; \vdash B}}{\Gamma; \vdash B}\, C_M$, $t'g'(\Pi) \approx \Pi$.

■

Lemma 20 *If $g'(t'(\Pi)) \approx \Pi$ for every derivation Π in ND then, for every pseudo-derivation Σ of Π*

$$f'(s'(\Sigma)) \approx \Sigma.$$

Proof: The proof is by induction on the length of the pseudo-derivation. Suppose that $g'(t'(\Pi)) \approx \Pi$ for every derivation Π and let Σ be a pseudo-derivation. We will only show one case:

$$f's'\left(\begin{matrix}\dfrac{\overset{\Pi_1}{\Gamma \vdash A} \quad \Gamma, A \vdash C}{\Gamma \vdash C}\, \mathcal{S} \\ \Sigma_3\end{matrix}\right) = f'\left(\dfrac{\overset{t'(\Pi_1)}{\Gamma; \vdash A} \quad \overset{s'(\Sigma_3^A)}{\Gamma, A; C \vdash B}}{\Gamma; C \vdash B}\, C_M\right) =$$

$$\begin{matrix}\dfrac{\overset{g't'(\Pi_1)}{\Gamma \vdash A} \quad \Gamma, A \vdash C}{\Gamma \vdash C}\, \mathcal{S} \\ f's'(\Sigma_3)\end{matrix}$$

By hypothesis, $g't'(\Pi_1) \approx \Pi_1$ and by IH, $f's'(\Sigma_3) \approx \Sigma_3$. Hence, $f'(s'(\Sigma)) \approx \Sigma$. ■

Theorem 12 *For every derivation Π in ND*

$$g'(t'(\Pi)) \approx \Pi.$$

Proof: The proof is by induction on the length of the derivation. Let Π be a derivation. We will make two cases, the others are analogous to cases already seen.

1. If $\Pi = \dfrac{\dfrac{\Pi_1}{\Gamma \vdash C}\, I_\odot}{\Sigma_2}\, E_\odot$, then $g't'(\Pi) = g'\left(\dfrac{\dfrac{t'(\Pi_1)}{\Gamma; \vdash C} \vdash \odot \quad \dfrac{s'(\Sigma_2)}{\Gamma; C \vdash E} \odot \vdash}{\Gamma; \vdash E}\right) =$
$\dfrac{\dfrac{g't'(\Pi_1)}{\Gamma \vdash C}\, I_\odot}{f's'(\Sigma_2)}\, E_\odot$.
By IH, $g't'(\Pi_1) \approx \Pi_1$ and by lemma 20 , $f's'(\Sigma_2) \approx \Sigma_2$. Hence, $g't'(\Pi) \approx \Pi$.

2. If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash B}\, S$, then $g't'\,(\Pi) = g'\left(\dfrac{\begin{matrix}t'(\Pi_1) \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}t'(\Pi_2) \\ \Gamma, A; \vdash B\end{matrix}}{\Gamma; \vdash B}\, C_M\right) =$
$\dfrac{\begin{matrix}g't'(\Pi_1) \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}g't'(\Pi_2) \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash B}\, S$.
By IH, $g't'(\Pi_1) \approx \Pi_1$ and $g't'(\Pi_2) \approx \Pi_2$ and hence $g't'(\Pi) \approx \Pi$.

■

5.5 Conclusion

We have defined a direct mapping between Natural Deduction and Sequent Calculus. By 'direct', we mean we did not use an intermediate system to define the translations as is, for instance, the case of (13), where the author uses Linear Logic to define the translations. In chapter 4, we have achieved a one-to-one correspondence between LJT and ND so far as cut-free and normal derivations are concerned. When taking into account non-cut-free and non-normal derivations, we lose this one-to-one correspondence, but by defining an equivalence relation between derivations we have retrieved bijection.

Comparing our work with Herbelin's (11), the use of a substitution rule in ND is more natural as there is no need (apparently) of a formalization of explicit substitutions to achieve isomorphism. Comparing our work with Negri and von Plato's (15), we have achieved bijection between normal and cut-free derivations, although there is no gain (nor loss) when considering non-cut-free and non-normal derivations.

As future work, we suggest the development of the bijection by showing translation between the λ-calculus and a term notation for LJT (as, for example, Herbelin's syntax in (11)).