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Introduction

The theory developed by Kolmogorov, Arnold and Moser to show the persistence, under C^k perturbations, of integrable Hamiltonians on the torus and, under C^k perturbations, of integrable exact twist maps of the annulus, for k large enough, is one of the landmarks of classical mechanics and mathematical physics. Hermann ([14]) showed that actually C^3 perturbations of exact twist maps preserve the existence of invariant curves with prescribed diophantine rotation number. Moreover, Hermann shows that the C^3 class is sharp: given any rotation number, there exist $C^{2+\beta}$ perturbations with $\beta \in (0,1)$ of exact twist maps which destroy invariant curves with this rotation number. The diophantine condition proves to be necessary for the persistence problem since J. Mather ([18]) showed that Lioville invariant curves of exact twist maps can be eliminated by C^∞ perturbations. The question that arises naturally from these results is the following: what is the lowest $\alpha > 0$ such that $C^{2,\alpha}$ perturbations of an integrable Hamiltonian destroys any Lagrangian invariant graph?

Many interesting answers to these questions appeared in the literature in the last 40 years. Takens ([24]) showed that C^1 close to any exact twist map there exists one with no invariant curves in the interior of the annulus. Hermann ([15]) considered the nonexistence of C^1 invariant Lagrangian graphs of Tonelli Hamiltonians, and showed that for any $\beta \in (0,1)$ and $C^{d+1-\beta}$ close to any Hamiltonian in the d-torus there exists one without C^1 Lagrangian invariant graphs. Notice however that Lagrangian graphs are just Lipschitz in general, so Hermann's result does not imply the destruction of Lagrangian invariant graphs by $C^{d+1-\beta}$ perturbations. Regarding specific families of Hamiltonians like geodesic flows, Bangert ([3]) came out with a simple and beautiful idea to show that there exists C^1 perturbations of a flat metric in T^2 without Lagrangian invariant graphs. Since C^1 perturbations of the metric are just C^0 perturbations of the geodesic flow the result might seem unsatisfactory from the point of view of twist maps. It is true that the geodesic flow of a flat torus admits local cross sections where the Poincaré return map is a twist map, but this is not possible globally. Moreover, a perturbation of such Poincaré return map as a twist map might not be the Poincaré return map of another geodesic flow, this is very hard to determine in general. The previous two remarks show the relevance of Bangert's result in the context of integrable geodesic flows. MacKay ([17]) and Ruggiero ([22]) showed that C^2 perturbations of a Riemannian metric create regions in the phase space of the geodesic flow without Lagrangian invariant graphs. The technique is linked to the creation of conjugate points, and it works only locally.

So the best idea so far to obtain perturbations of geodesic flows in T^n without Lagrangian invariant graphs is Bangert's idea, which is based on smoothing singular metrics having cones and proving by direct calculus of variations that no geodesic through the vertex of the cone is a minimizer. This idea was extended and applied by Ruggiero ([21]) to show the C^1-density in the family of mechanical Lagrangians in T^2 of the nonexistence of Lagrangian invariant graphs in all supercritical energy levels. The goal of this paper to consider the gap between C^1 and C^2 perturbations of Finsler metrics. The main result is the following:

Theorem 1.0.1 (Main theorem)**.** *Let F be a C^∞ reversible Finsler metric on the the $2-$torus T^2. Given $\epsilon > 0$, there exist $\beta \in (0,1)$ and a C^∞ function $\sigma_\beta : T^2 \to \mathbb{R}$ satisfying the following properties:*

i) $||F - F_\beta||_1 < \epsilon$, *where* $F_\beta = e^{\sigma_\beta} F$;

ii) $||\sigma_\beta||_{1,\beta} < \epsilon$;

iii) The geodesic flow of F_β admits no invariant continuous graphs.

In the theorem above, the $C^{1,\beta}$ norm is defined as follows: equip M with a Riemannian metric g and let G be the Sasaki metric on TM associated with g. Denote by $|| \cdot ||^G$ the norm given the Sasaki metric G. For $L : TM \to \mathbb{R}$, let $||L||_0 = \sup_{\theta \in TM} |L(\theta)|$ and $||L||_1 = \max\{||L||_0, \sup_{\theta \in T^1M} ||\nabla L(\theta)||\}$, where ∇L is the gradient of L. Finally, define

$$||L||_{1,\beta} = \max\left\{||L||_1, \sup_{\theta,\eta \in TM, \theta \neq \eta} \frac{||\nabla L(\theta) - \nabla L(\eta)||^G}{d_{TM}(\theta,\eta)^\beta}\right\},$$

where d_{TM} is the distance in TM given by G.

The main contribution of this work is a study of the second variation of certain smoothed singular perturbations of metrics which generalize Bangert's idea of cone-type singular metrics. We construct explicitly $C^{1,\beta}$ metrics in T^2 with just one singular point p and small $C^{1,\beta}$ norm. This singular point plays the role of the vertex of a cone in Bangert's construction. Then we look at the Jacobi equation of the geodesics passing through p. The Jacobi equation will have smooth solutions which extend continuously to the singularity and we show that there exist conjugate points along every singular geodesic. Then we smoothen the metric and show that the smoothed Jacobi equation still has conjugate points along each geodesic passing through a point in a neighborhood of p. This implies in the case of the torus that the smoothed metric has no Lagrangian invariant graphs since the canonical projection of such a graph in the torus would give a continuous flow of minimizers.

The proof of Theorem 1.0.1 combines calculus of variations techniques and Riemann-Finsler geometry, which gives a Riemannian flavor to the arguments. Although we can apply variational theory in the general context of Lagrangian or Hamiltonian systems, the Riemman-Finsler point of view allows us to get a better insight of Bangert's idea of cone-type Riemannian metrics in the Finsler category. The second variation study of geodesics in a Finsler metric and its perturbations involves the Jacobi equation, and the application of Riemann-Finsler theory will simplify in many steps of the argument the expression relating these Jacobi equations.

We begin the exposition with an extension of an idea of Ruggiero ([21]): we push forward Bangert's argument for Riemannian surfaces using a geometrical cone-type perturbation. We show that by gluing spherical cones to a regular Riemannian surface, we can approach the surface in the C^1 topology by another one without continuous invariant graphs whose $C^{1,\frac{1}{3}}$ norm is finite. This result is the main motivation to study our problem.

The preliminaries about Finsler geometry contain a brief introduction to Riemann-Finsler geometry and Lagrangian dynamics of Finsler geodesic flows. This includes an account of some variational and geometrical basic results of the theory of minimizers. We define the Chern-Rund connection and the Flag curvature, concepts that are the Finslerian counterparts of the Levi-Civita connection and the sectional curvature of Riemannian geometry. The chapter ends with a brief introduction to Lagrangian graphs, in a way that is well suited for our study.

In Chapter 5 we study perturbations of the Finsler Jacobi equation by conformal changes of the metric, which are equivalent to the so-called Mañé's perturbations or pertubations of a Lagrangian by adding potentials. Since there is not a conformal theory for the Chern-Rund connection and, hence, we cannot appeal to what is done by Ruggiero ([23], [21]), we use the Lagrangian formalism. There are advantages in the use of conformal changes in the metric. For instance, the fundamental tensor of the conformal metric also changes by a scalar function so, some geometrical aspects, such as perpendicularity, are preserved. Another advantage is that it is possible to simplify the Jacobi equation by making use of the Fermi-Finsler coordinates. These coordinates are similar to those of Riemannian geometry but, since they are not easily found in the literature, we give a description here. We finish the chapter with one of the main technical results of our study, a simple system of equations for the conformal Jacobi equation.

Chapter 6 is devoted to construct the conformal perturbation. Here it becomes clear why our methods do not extend to the C^2 case: we create conjugate points in every geodesic through a given point in a small neighborhood of it, while by C^2 perturbations we just create conjugate points in long subsets of geodesics. The conformal perturbations are based on the perturbations considered in the previous Chapter, and to show the existence of conjugate points we combine the results in Chapter 5 with the Sturm comparison theorem applied on the Jacobi equation. The construction of the new Finsler metric grants that its $C^{1,\beta}$ norm is small and that it is C^1 close to the initial Finsler metric.

In the appendix, we present the Hamiltonian version of the theorem and its proof. We suppose that the Hamiltonian is $2-$homogeneous and reversible. In this case, the perturbation is made by a radial potential that must preserve the radial orbits of the Hamiltonian flow. We simplify the Hamiltonian Jacobi equations using special coordinates found in [11]. No knowledge of Finsler geometry is needed.