4
Some ODE results

In this section we will state and proof some results about a special type of linear second order ODE. First of all consider a smooth function $K : [-a, a] \to \mathbb{R}$ for some $a > 0$. Given $\rho > 0$, define $K^\rho(t) = \frac{a^2}{\rho^2} K(\frac{a}{\rho} t)$ where $t \in [-\rho, \rho]$. Our main object of study will be solutions of the equation

$$x''(t) + K^\rho(t) x(t) = 0 \qquad (4.0.1)$$

where, by definition, K^ρ is defined on the interval $[-\rho, \rho]$. This is a family of differential equations parametrized by $\rho > 0$.

Remark 4.0.6. *The equation* (4.0.1) *is defined on the interval* $[-\rho, \rho]$. *So, for different values of* ρ, *the interval of definition of the differential equation* (4.0.1) *changes.*

The definition of conjugate point was done in the context of the Jacobi equation. Now we will restate it for second order ODE, which is the object of this section.

Definition 4.0.7. *We say that equation* (4.0.1) *has conjugate points if there exist* $s, t \in [-\rho, \rho]$ *with* $s < t$ *and a non-trivial solution* x *of* (4.0.1) *such that*

$$x(s) = x(t) = 0.$$

Lemma 4.0.8. *Consider the family of equations* (4.0.1) *parametrized by* ρ. *If this equation does not have conjugate points for* $\rho = a$ *then it also does not have conjugate points for every* $\rho \geq a$.

Proof. Given $\rho \geq a$, consider $x : [-\rho, \rho] \to \mathbb{R}$ a solution of

$$x'' + K^\rho x = 0$$

such that $x(t_1) = x(t_2) = 0$ are two consecutive zeros and $[t_1, t_2] \subset [-\rho, \rho]$. Define $y(u) = x(t)$ where $u = \frac{a}{\rho} t$. We have that y is a solution of

$$\ddot{y}(u) + K(u) y(u) = 0$$

where the dot corresponds to the derivative with respect to u. Since the above equation does not have conjugate points and y also has to consecutive zeros u_1 and u_2 such that $[u_1, u_2] \subset [-a, a]$ then it has to be identically zero which implies that x is also identically zero. □

Lemma 4.0.9. *If equation* (4.0.1) *does not have conjugate points then two given different solutions x_1 and x_2 cannot cross twice, that is, if $x_1(t_0) = x_2(t_0)$ then $x_1(t) \neq x_2(t)$ for all $t \neq t_0$.*

Proof. Suppose that $\exists \bar{t} \neq t_0$ such that $x_1(\bar{t}) = x_2(\bar{t})$. Define the new solution $x(t) = x_1(t) - x_2(t)$. This solution has two zeros at t_0 and $\bar{t}$ and therefore $x \equiv 0$, a contradiction with the fact that equation (4.0.1) does not have conjugate points. □

Lemma 4.0.10. *Suppose that equation* (4.0.1) *does not have conjugate poits. Consider a solution x of equation* (4.0.1) *with initial conditions $x(-b) = 0$ and $x'(-b) > 0$ for $b \in (0, \rho]$. If $y(t) = x(-t)$ is also a solution then $x'(0) > 0$.*

Proof. Suppose that $x'(0) \leq 0$. Since $y(t) = x(-t)$ then the two solutions cross at $t = 0$. In the case that $x'(0) = 0$, we have that $y'(t) = -x'(-t) \Rightarrow y'(0) = 0$ and from Picard's theorem $x \equiv y$, a contradiction. If $x'(0) < 0$ then $y(t) < x(t)$ for $t \in [-\rho, 0)$. But lemma 4.0.9 says that they can't cross again, therefore $y(\bar{t}) = 0$ for some $\bar{t} \in (-\rho, 0)$, a contradiction. □

Consider $\epsilon, \alpha > 0$ and $\beta \in (0, 1)$. On the interval $[-\rho, \rho]$ let $K^{\rho}_{\epsilon,\alpha,\beta}$ be a smooth non-negative function such that

$$K^{\rho}_{\epsilon,\alpha,\beta}(t) = K^{\rho}(t) + \frac{\epsilon}{|t|^{1-\beta}}$$

in $[-\rho, -\alpha] \cup [\alpha, \rho]$. These functions will be called of $(\epsilon, \alpha, \beta)-$type.

The main result in this section is the proposition bellow.

Proposition 4.0.11. *Let $\epsilon > 0$. Then there exist $\beta \in (0, 1)$, $\rho = \rho(\beta, \epsilon)$ and $\alpha(\rho) > 0$ such that the equation*

$$x''(t) + K^{\rho}_{\epsilon,\alpha,\beta}(t)x(t) = 0, \tag{4.0.2}$$

defined on $[-\rho, \rho]$, has conjugate points.

Although basic, the Sturm comparison theorem plays a important role in the arguments along this work. The theorem is stated bellow. For a proof, see [6].

Theorem 4.0.12 (Sturm comparison theorem)**.** *Let K_i be piecewise continuous on $[a,b]$, and let*

$$K_2(t) \leq K_1(t)$$

on $[a,b]$. Let $f_1(t)$ be a solution of

$$f_1''(t) + K_1(t)\, f_1(t) = 0$$

and $f_2(t)$ a solution of

$$f_1''(t) + K_2(t)\, f_2(t) = 0$$

such that $f_1(a) = f_2(a)$ and $f_1'(a) = f_2'(a)$. Then

$$f_2(t) \geq f_1(t) \qquad (a \leq t \leq b). \tag{4.0.3}$$

Moreover, if $K_1 > K_2$ on (a,b), then

$$f_2(t) > f_1(t) \qquad (a < t \leq b). \tag{4.0.4}$$

Before proceeding to the proof of the proposition, we will need a few facts about the Riccati equation.

Definition 4.0.13. *If x is a non-trivial solution of a equation of type* (4.0.1) *then define $u(t) = \frac{x'(t)}{x(t)}$ for t such that $x(t) \neq 0$. Where it is defined, the function u satisfies the non-linear first order equation*

$$u'(t) + u^2(t) + K^{\rho}(t) = 0 \tag{4.0.5}$$

called the Riccati equation.

From now on, suppose that equation (4.0.1) does not have conjugate points. Consider any function $K_{\epsilon,\alpha,\beta}$ of $(\epsilon,\alpha,\beta)-$type and (4.0.2) the corresponding equation. Let x_1 be a solution of (4.0.1) and x_2 be a solution of (4.0.2) with initial conditions $x_1(-\rho) = x_2(-\rho) = 0$ and $x_1'(-\rho) = x_2'(-\rho) = \lambda > 0$. Moreover, let u_1 and u_2 the solutions of the Riccati equation corresponding to x_1 and x_2. Although $u_1, u_2 \to \infty$ when $t \mapsto -\rho^+$ their difference converges:

Lemma 4.0.14. *The solutions of the Riccati equations u_1, u_2 satisfy*

$$\lim_{t \to -\rho^+} u_1(t) - u_2(t) = 0.$$

Proof. Since $x_1(\rho) = x_2(\rho) = 0$, by the L'Hospital rule,

$$\begin{aligned}\lim_{t\to-\rho^+} u_1(t) - u_2(t) &= \lim_{t\to-\rho^+} \frac{x_1'(t)x_2(t) - x_1(t)x_2'(t)}{x_1(t)x_2(t)} \\ &= \lim_{t\to-\rho^+} \frac{(K_{\epsilon,\alpha,\beta}(t) - K(t))x_1(t)x_2(t)}{x_1'(t)x_2(t) + x_1(t)x_2'(t)} \\ &= \lim_{t\to-\rho^+} \frac{\epsilon}{|t|^{1-\beta}} \frac{1}{u_1(t)+u_2(t)}.\end{aligned}$$

The fact that $u_i(t) \to \infty$ as $t \to -\rho^+$ for $i = 1, 2$ completes the proof. □

Lemma 4.0.15. *On* $[-\rho, \rho]$,

$$u_1(t) \geq u_2(t). \tag{4.0.6}$$

Proof. Define $u(t) = u_1(t) - u_2(t)$ and observe that u is a solution of

$$u'(t) + (u_1(t) + u_2(t))\, u(t) + K(t) = 0 \tag{4.0.7}$$

where $K(t) = K^\rho - K^\rho_{\epsilon,\alpha,\beta}$. Since $x_1(-\rho) = x_2(-\rho) = 0$ and $x_1'(-\rho) = x_2'(-\rho) > 0$ there is a $\delta > 0$ such that $x_1(t), x_2(t) > 0$ and $x_1'(t), x_2'(t) > 0$ on $I_\delta = (-\rho, -\rho + \delta)$. For convenience, suppose that $-\rho + \delta < -\alpha$.

Suppose that, on I_δ, $u < 0$. Since on this interval both u_1 and u_2 are bigger than zero and $K = -\frac{\epsilon}{|t|^{1-\beta}}$, from the equation (4.0.7) we conclude that $u'(t) > 0$. But this contradicts lemma 4.0.14 for u wouldn't converge to zero as t goes to $-\rho$.

Therefore, $u(t) \geq 0$ on I_δ. This proves the lemma because if $u(t_0) = 0$ for some t_0 then equation (4.0.7) implies that $u'(t) \geq 0$ for every $t > t_0$ so $u(t) \geq 0$ for every $t > t_0$. □

4.1 Proof of proposition 4.0.11

Consider the auxiliary equation

$$y''(t) + K_0 y(t) = 0 \tag{4.1.1}$$

where $K_0 = \inf\{K(t) | t \in [-a, a]\}$, and suppose that $K_0 \leq 0$. Observe that, since for $\rho = a$ equation (4.0.1) does not have conjugate points, equation (4.1.1) does not have conjugate points on the interval $[-a, a]$. In this way, let y_a be a solution of (4.1.1) with initial conditions $y_a(-a) = 0$ and $y_a'(-a) = 1$.

Definition 4.1.1. *For $\beta \in (0,1)$ and $\epsilon > 0$, define*

$$\rho(\beta,\epsilon) = \max\left\{ \left(\frac{(y_a'(0))^2(\beta+1)(\beta+2)}{2\epsilon} \right)^{\frac{1}{\beta+2}}, \right. \left. \left(\frac{a^2|K_0|}{\epsilon} \right)^{\frac{1}{\beta+2}}, a, 1 \right\}. \tag{4.1.2}$$

Remark 4.1.2. *Observe that ρ is at most of order $\epsilon^{-\frac{1}{\beta+2}}$. Therefore,*

$$\epsilon\, \rho(\beta,\epsilon)^{1+\beta} \approx \epsilon^{\frac{1+\beta}{2+\beta}} \to 0$$

when $\epsilon \to 0^+$.

Let $K_0^\rho = \frac{a^2}{\rho^2} K_0$. We will consider a singular $(\epsilon, \alpha, \beta)-$type perturbation of K_0^ρ defined on interval $I_\rho = [-\rho, \rho]$ such that the new equation will have conjugate points on this interval.

Consider y_0 a solution of

$$y''(t) + K_0^\rho y(t) = 0 \tag{4.1.3}$$

with initial conditions $y_0(-\rho) = 0$ and $y_0'(-\rho) = 1$. Let u_0 be the solution of the corresponding Riccati equation associated with y_0.

Lemma 4.1.3. *For ρ defined in 4.1.2 we have that*

$$y_a'(0) \geq y_0'(0).$$

Remark 4.1.4. *Lemma 4.1.3 implies that*

$$\rho \geq \left(\frac{(y_0'(0))^2(\beta+1)(\beta+2)}{2\epsilon} \right)^{\frac{1}{\beta+2}}.$$

For ρ chosen as in (4.1.2), consider $K^\rho_{0,(\epsilon,\beta)}(t) = K_0^\rho + \frac{\epsilon}{|t|^{1-\beta}}$ defined on the interval I_ρ. The equation obtained is

$$y''(t) + K^\rho_{0,(\epsilon,\beta)} y(t) = 0. \tag{4.1.4}$$

This equation has a singularity at $t = 0$ and some observations concerning the regularity of solutions are needed.

Lemma 4.1.5. *Let $\bar{y}$ be a solution of (4.1.4) defined on $[-\rho, 0)$ such that $\bar{y}(-\rho) = 0$ and $\bar{y}(-\rho) = 1$. If $\bar{y} \geq 0$ on $[-\rho, 0)$ then we have that the limit*

$$\lim_{t \to 0^-} \bar{y}(t) \text{ and } \lim_{t \to 0^-} \bar{y}'(t)$$

exists.

Proof. We will show the second limit by proving that $\bar{y}'$ is a non-increasing bounded function. Simple integration gives the first.

Let y_0 satisfy equation (4.1.3) such that $y_0(0) = 0$ and $y_0'(0) = 1$. From the Sturm comparison theorem, we have that $y_0(t) \geq \bar{y}(t) \geq 0$ for $t \in [-\rho, 0)$. Integrating the differential equation (4.1.4), we obtain

$$\begin{aligned} \bar{y}'(t) &= 1 - \int_{-\rho}^{t} K^{\rho}_{0,(\epsilon,\beta)} \bar{y}(s)\, ds \\ &\geq 1 - \int_{-\rho}^{t} K^{\rho}_{0,(\epsilon,\beta)} y_0(s)\, ds \end{aligned}$$

The the choice of K_0^ρ and the positiveness of $\bar{y}$ implies that $\bar{y}'(t)$ and

$$g(t) : t \mapsto 1 - \int_{-\rho}^{t} K^{\rho}_{0,(\epsilon,\beta)} y_0(s)\, ds$$

are non-crescent functions. But, since y_0 is a well defined solution for all $t \in [-\rho, \rho]$, the function $\bar{y}'$ has a lower bound, namely, the infimum of $g(t)$. Define $\bar{y}(0)$ and $\bar{y}'(0)$ as the limits in the statement of the lemma.

□

Lemma 4.1.6. *Let $\bar{y}$ be a solution of* (4.1.4) *such that $\bar{y}(-\rho) = 0$ and $\bar{y}'(-\rho) = 1$. There exists $\beta \in (0, 1)$ such that if $\bar{y}(t) > 0$ for all $t \in (-\rho(\beta), 0]$ then*

$$\lim_{t \to 0^-} \bar{y}'(t) < 0.$$

Proof. First of all, choose $\beta \in (0, 1)$ such that

$$\beta\, \rho \leq 1.$$

Let $v(t) = \frac{\bar{y}'(t)}{\bar{y}(t)}$ and define $\phi(t) = v(t) - u_0(t)$. We have that ϕ is a solution of

$$\phi'(t) + \psi(t)\phi(t) + \frac{\epsilon}{|t|^{1-\beta}} = 0. \tag{4.1.5}$$

where $\psi(t) = v(t) + u_0(t)$. Using variation of parameters to determine ϕ, for $\bar{t} \in (-\rho, 0)$, we have

$$\begin{aligned} \phi(t) &= \left(\phi(\bar{t}) - \int_{\bar{t}}^{t} \frac{\epsilon}{|s|^{1-\beta}} \exp\left(\left(\int_{\bar{t}}^{s} \psi(l)\, dl \right) ds \right) \exp\left(- \int_{\bar{t}}^{t} \psi(l)\, ds \right) \right. \\ &= \left(\phi(\bar{t}) - \int_{\bar{t}}^{t} \frac{\epsilon}{|s|^{1-\beta}} \frac{y_0(s)\bar{y}(s)}{y_0(\bar{t})\bar{y}(\bar{t})}\, ds \right) \frac{y_0(\bar{t})\bar{y}(\bar{t})}{y_0(t)\bar{y}(t)} \\ &= \frac{1}{y_0(t)\bar{y}(t)} \left(\phi(\bar{t}) - \int_{\bar{t}}^{t} \frac{\epsilon}{|s|^{1-\beta}}\, y_0(s)\bar{y}(s)\, ds \right). \end{aligned}$$

From lemma 4.0.14, we have that

$$\phi(\bar{t}) \to 0$$

when $\bar{t} \to -\rho^+$. Therefore we obtain the following formula for ϕ :

$$\phi(t) = -\frac{\epsilon}{y_0(t)\bar{y}(t)} \int_{-\rho}^{t} \frac{y_0(s)\bar{y}(s)}{|s|^{1-\beta}}\, ds. \tag{4.1.6}$$

Recall that the solutions y_0 and $\bar{y}$ satisfy $y_0(-\rho) = \bar{y}(-\rho) = 0$ and $y_0'(-\rho) = \bar{y}'(-\rho) = 1$. So, from the Sturm comparison theorem, we have $y_0(t) \geq \bar{y}(t) > 0$ for every $t \in (-\rho, 0]$. This and the choice of $K_0^\rho \geq 0$ implies that $\bar{y}'' < 0$ for $t \in (-\rho, 0]$. So, $\bar{y}$ is bigger than the affine function connecting its endpoints at

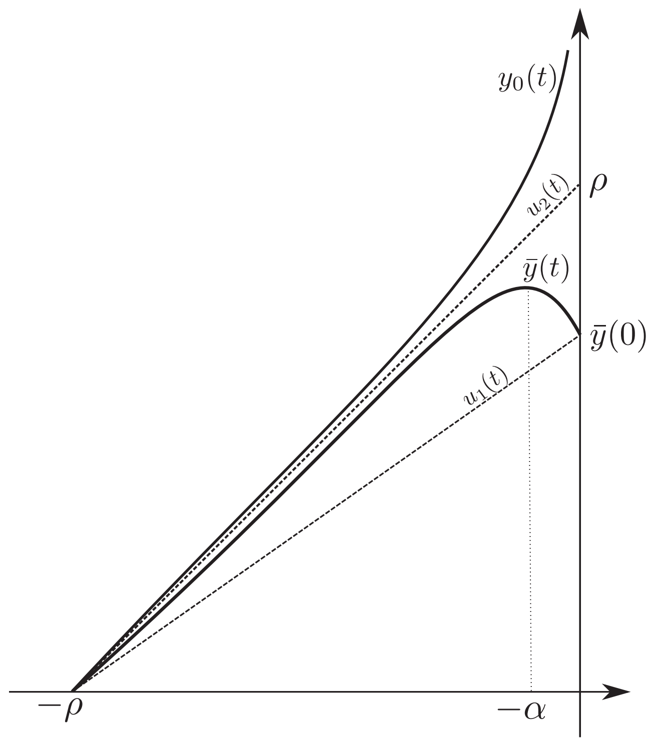

Figure 4.1: $u_1(t) = \frac{\bar{y}(0)}{\rho}(t+\rho)$ and $u_2(t) = t + \rho$.

$t = -\rho$ and at $t = 0$:

$$\bar{y}(t) \geq \frac{\bar{y}(0)}{\rho}(t+\rho),$$

where $\bar{y}(0) = \lim_{t\to 0^-} \bar{y}(t) > 0$ exists, as was seen in lemma 4.1.5. In a similar way, since $K_0^\rho \leq 0$ and $y_0 \geq 0$ on $[-\rho, 0)$, we have that $y_0'' > 0$ and then

$$y_0(t) \geq (t+\rho)$$

on $(-\rho, 0]$. Equation (4.1.3) and the inequalities above implies that, as $t \to 0^-$,

$$\begin{aligned}\phi(0) &\leq -\frac{\epsilon}{y_0(0)\bar{y}(0)} \int_{-\rho}^{0} \frac{\bar{y}(0)}{\rho} \frac{(s+\rho)^2}{|s|^{1-\beta}}\, ds \\ &< -\epsilon \frac{2\rho^{\beta+2}}{y_0(0)\beta(1+\beta)(2+\beta)\rho}.\end{aligned}$$

The inequality above is strict because both affine functions are equal to $\bar{y}$ and y_0 only on $t = -\rho$ and $t = 0$, since $\bar{y}$ and y_0 are strictly convex. From remark 4.1.4 and the choice of ρ we have that

$$\phi(0) < -u_0(0)\frac{y_0'(0)}{\beta\,\rho}.$$

Finally, since $\phi(0) = v(0) - u_0(0)$ then

$$v(0) = u_0(0)(1 - \frac{y_0'(0)}{\beta\,\rho}) < 0,$$

because $y_0'(0) \geq 1$ and $\beta\,\rho \leq 1$.

So,
$$\frac{\bar{y}'(0)}{\bar{y}(0)} = v(0) < 0 \text{ and } \bar{y}(0) > 0 \Longrightarrow \bar{y}'(0) < 0. \tag{4.1.7}$$

□

proof of proposition 4.0.11. Let $\epsilon > 0$. If (4.0.1) has conjugate points then the proposition if proved. So, from now on, suppose that (4.0.1) does not have conjugate points. From lemma 4.1.6 there exists $\alpha = \alpha(\rho) > 0$ such that $\bar{y}'(t) < 0$ for $t \in [-\alpha, 0)$, where $\bar{y}$ is the solution considered on that lemma. Let $\delta_\alpha : [-\rho, \rho] \to \mathbb{R}$ be a bump function such that

$$\delta_\alpha(t) = \begin{cases} 1, & t \in [-\frac{\alpha}{2}, \frac{\alpha}{2}]; \\ 0, & t \in [-\rho, \rho] \setminus (-\alpha, \alpha). \end{cases} \tag{4.1.8}$$

Define the $(\epsilon, \beta, \alpha)-$type function $K^\rho_{\epsilon,\beta,\alpha}(t) = K_\rho(t) + \kappa(t)$, where

$$\kappa(t) := \epsilon\,\frac{(1 - \delta_\alpha(t))}{|t|^{1-\beta}}. \tag{4.1.9}$$

Remains to show that for $K^\rho_{\epsilon,\beta,\alpha}$ equation (4.0.2) has conjugate points.

Claim 4.1.7. *If equation* (4.0.2) *has conjugate points for the function* $\widehat{K}(t) = K_0^\rho + \kappa(t)$ *then it also has for* $K^\rho_{\epsilon,\beta,\alpha}$.

Proof. Since, from the definition of K_0, $K_0^\rho \leq K^\rho(t)$ for all $t \in [-\rho, \rho]$, we have that $\widehat{K}(t) < K^\rho_{\epsilon,\beta,\alpha}(t)$. Let $\widehat{x}$ be a non trivial solution of equation (4.0.2) for the function $\widehat{K}$ with zeros at t_1, t_2 ($t_1 < t_2$) and such that $\widehat{x}|_{(t_1,t_2)} > 0$. Now, consider x a solution of equation (4.0.2) for the function $K^\rho_{\epsilon,\beta,\alpha}(t)$ with the same initial conditions of $\widehat{x}$ at $t = t_1$. Sturm comparison theorem implies that

$$x(t) \leq \widehat{x}(t)$$

for $t \in [t_1, t_2]$. Therefore $x(t_0) = 0$ for some $t_0 \in (t_1, t_2]$.

This claim also proves that the function K can be chosen non-positive.

□

The claim allows to prove the proposition for $\widehat{K}$, since this will imply we desired result. Recall $\bar{y}$ the solution defined in lemma 4.1.6 and let $\widehat{x}$ be a solution of (4.0.2) for $\widehat{K}$ with initial conditions $\widehat{x}(-\rho) = 0$ and $\widehat{x}'(-\rho) = 1$.

If $\widehat{x}(t_0) = 0$ for some $t_0 > -\rho$ then the lemma is proved. From now on, suppose that

$$\widehat{x}(t) > 0$$

for $t \in (-\rho, 0]$. Since $\widehat{K}(t) = K^{\rho}_{0,(\epsilon,\beta)}(t)$ on the interval $[-\rho, -\alpha]$ we have that $\widehat{x} = \bar{y}$ on this interval. Therefore, from lemma 4.1.6,

$$\widehat{x}'(-\alpha) < 0.$$

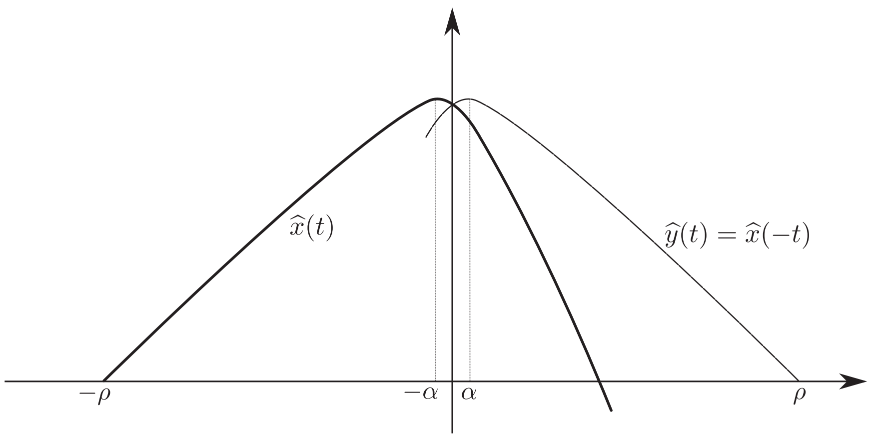

Figure 4.2: $\widehat{x}$ cannot intersect $\widehat{y}$ on $(0, \rho)$, so there is $t_0 \in (0, \rho)$ where $\widehat{x}(t_0) = 0$.

So,

$$\widehat{x}'(0) = \widehat{x}'(-\alpha) + \int_{-\alpha}^{0} \widehat{x}''(s)\, ds = \widehat{x}'(-\alpha) - \int_{-\alpha}^{0} \widehat{K}(s)\widehat{x}(s)\, ds < 0.$$

Since $\widehat{K}(t) = \widehat{K}(-t)$ we have that $\widehat{y}(t) = \widehat{x}(-t)$ is also a solution of equation (4.0.2) for $\widehat{K}$, then lemma 4.0.10 implies that $\widehat{x}$ has zero on $(-\rho, \rho]$. This and the claim proves the proposition. □

To end this chapter, we shall do some considerations about the function κ. First, recall the definition of the $\beta-$Hölder norm of a function.

Definition 4.1.8. *Let $\beta \in (0,1)$ and $f : [-\rho, \rho] \to \mathbb{R}$ be a positive function. Define*

$$||f||_\beta = \sup_{\substack{s,t \in [-\rho,\rho] \\ t \neq s}} \frac{|f(t) - f(s)|}{|t-s|^\beta}.$$

Lemma 4.1.9. *Given $\epsilon > 0$, there exist $\epsilon > \bar{\epsilon} > 0$, $\beta = \beta(\bar{\epsilon})$ and $\rho = \rho(\bar{\epsilon}, \beta)$ such that the function $f : [-\rho, \rho] \to \mathbb{R}$, defined by*

$$f(t) = \bar{\epsilon}\rho\,(1 - \delta_a(t))|t|^\beta$$

has $\beta-$Hölder norm less then ϵ.

Proof. The C^0 norm of this function is given by

$$||f||_0 \sup_{t \in [-\rho,\rho]} \bar{\epsilon}\,(1 - \delta_a(t))\,|t|^\beta \leq \bar{\epsilon}\rho^{1+\beta}.$$

The $\beta-$Hölder norm of the function is satisfied for $s = \rho$ and $t = -\rho$ because the function is differentiable. Therefore,

$$||f||_\beta \leq 2^\beta \bar{\epsilon}\rho.$$

By remark 4.1.2 we can choose $\bar{\epsilon} < \epsilon$ such that $\bar{\epsilon}\rho^{1+\beta} < \epsilon$. Then the result follows. □