5
Conformal Jacobi equation

On a compact smooth M let $F : TM \to \mathbb{R}$ be a smooth symmetric Finsler metric and $\sigma : M \to \mathbb{R}$ a smooth function. In this section we will be concerned with how the Jacobi equation changes when we consider the Finsler metric

$$F_\sigma := e^\sigma F.$$

We will actually deal with F_σ^2 since in this way we will have constant speed geodesics. Conformal changes of the Finsler metric are equivalent to the Mañé's perturbations, or perturbations by adding a potential to the original metric. We will not adopt this viewpoint because we intend to use the geometrical relations between the Finsler metric and its conformal counterpart.

Lemma 5.0.10. *Let $\gamma : [a, b] = I \to U$ be a unity speed geodesic of the Finsler metric F and suppose that*

$$\sigma_{x^i}(\gamma(t)) = f(t)\, g_{ik}(\gamma(t), \dot\gamma(t))\dot\gamma^k(t),$$

where $f : I \to \mathbb{R}$ is a smooth function. In this case, γ is also, up to reparametrization, a unity speed geodesic of the metric F_σ.

Proof. Define the change of coordinates $\alpha : I \to [0, b_0]$ by

$$\alpha(t) = \int_a^t \exp\left(\sigma(\gamma(u))\right) du. \tag{5.0.1}$$

Let $\beta = \alpha^{-1}$ and define the curve $\bar\gamma(s) = \gamma(\beta(s))$. If $\bar\gamma$ satisfy the Euler-Lagrange equation for F_σ^2 then the lemma is proved.

First of all, since F is $1-$homogeneous on the fiber coordinates,

$$(F_\sigma^2)_{y^i}\left(\bar\gamma(s), \bar\gamma'(s)\right) = \beta'(s) e^{2\sigma(\bar\gamma(s))}(F^2)_{y^i}(\gamma(\beta(s)), \dot\gamma(\beta(s))) \tag{5.0.2}$$

and

$$\begin{aligned}(F_\sigma^2)_{x^i}(\bar\gamma(s), \bar\gamma'(s)) &= (\beta'(s))^2 e^{2\sigma(\bar\gamma(s))}(F^2)_{x^i}(\gamma(\beta(s)), \dot\gamma(\beta(s))) \\ &+ 2\sigma_{x^i}(\bar\gamma(s))(\beta'(s))^2 e^{2\sigma(\bar\gamma(s))}(F^2)(\gamma(\beta(s)), \dot\gamma(\beta(s)))\end{aligned} \tag{5.0.3}$$

where $' = \frac{d}{ds}$ and $\dot{} = \frac{d}{d\beta}$. The Euler-Lagrange equation for F_σ^2 is

$$\begin{aligned}\frac{d}{ds}\left((F_\sigma^2)_{y^i}\right) - (F_\sigma^2)_{x^i} =& e^{2\sigma}(\beta''(F^2)_{y^i} + 2\sigma'\beta'(F^2)_{y^i} + (\beta')^2\frac{d(F^2)_{y^i}}{d\beta} \\ &- (\beta')^2(F^2)_{x^i} - 2\sigma_{x^i}(\beta')^2F^2); \\ =& \left(\beta'' e^{2\sigma}(F^2)_{y^i} + 2\sigma'\beta' e^{2\sigma}(F^2)_{y^i}\right) - 2\sigma_{x^i}F^2; \\ =& \frac{d(e^\sigma)}{ds}(F^2)_{y^i} - 2\sigma_{x^i}; \\ =& \sigma' e^\sigma (F^2)_{y^i} - 2\sigma_{x^i}, \end{aligned} \tag{5.0.4}$$

where we used that γ is a geodesic of F. Since $F^2(\gamma, \dot\gamma) = g_{kj}\dot\gamma^k\dot\gamma^j$ we have that

$$\begin{aligned}(F^2)_{y^i} &= \frac{\partial g_{kj}}{\partial y^i}\dot\gamma^i\dot\gamma^j + 2g_{ik}\dot\gamma^k; \\ &= 2g_{ik}\dot\gamma^k,\end{aligned}$$

because $\frac{\partial g_{kj}}{\partial y^i}\dot\gamma^\iota = 0$ where $\iota = i, j$ or k. From

$$\sigma' = \sigma_{x^i}\dot\gamma^i\beta' = f(t)\beta' g_{ij}\dot\gamma^j\dot\gamma^i = f(t)\beta'$$

and the hypothesis of the lemma, we conclude that

$$\begin{aligned}\sigma' e^\sigma (F^2)_{y^i} - 2\sigma_{x^i} &= 2f(t)\beta' e^\sigma g_{ik}\dot\gamma^k - 2f(t)g_{ik}\dot\gamma^k \\ &= 0.\end{aligned} \tag{5.0.5}$$

Equations (5.0.4) and (5.0.5) proves the first part of the lemma. The fact that $\bar\gamma$ is unitary follows from $(g_\sigma)_{ij} = e^{2\sigma}g_{ij}$. □

Remark 5.0.11. *Choose a unitary geodesic $\gamma : [a, b] \to U$. We can regard every vector field $\xi : [a, b] \to \pi^*\widetilde{TM}$ along γ as a curve in $\mathbb{R}^n$ given by $t \mapsto (\xi^1(t), ..., \xi^n(t))$ where $\xi(t) = \xi^i(t)\, \frac{\partial}{\partial x^i}\big|_{(\gamma(t),\gamma'(t))}$. From now on, if it is not explicitly mentioned, the vector field ξ will be identified with its curve.*

Definition 5.0.12. *Let $\gamma : [a, b] \to U$ be a unitary geodesic. The Jacobi operator with respect to the Finsler metric F^2 is a differential operator $J : C^2([a, b], \mathbb{R}^n) \to C^0([a, b], \mathbb{R}^n)$ defined by*

$$J_{ij}\,\xi^i = \frac{d}{dt}\left((F^2)_{y^iy^j}\dot\xi^i + (F^2)_{x^iy^j}\xi^i\right) - \left((F^2)_{y^ix^j}\dot\xi^i + (F^2)_{x^ix^j}\xi^i\right), \tag{5.0.6}$$

where $\dot{} = \frac{d}{dt}$. We say that ξ is a Jacobi field if $J\xi = 0$.

Definition 5.0.13. *Given two distinct points $p = \gamma(t_0)$ and $q = \gamma(t_1)$ along the geodeisc $\gamma : [a, b] \to U$ we say that p is* ***conjugated*** *to q if there exists* *a non-trivial Jacobi field ξ over γ such that $\xi(t_0) = \xi(t_1) = 0$ (cf. definition 4.0.7).*

A standard result from the calculus of variations (cf. [10] and [12]) asserts that if a geodesic has conjugate points then it can no longer be a minimizer between this points.

From now on, suppose that we are under the hypothesis of lemma 5.0.10 and we will follow the notation given in its proof.

We would like to know how the Jacobi operator J^σ, of the Finsler metric F_σ^2, along $\bar{\gamma}$ looks like when we take the derivative with respect to the parameter of γ.

Proposition 5.0.14. *Given a vector field η along $\bar{\gamma}$, the Jacobi operator J^σ when looked over γ is*

$$\begin{aligned} J^\sigma_{ij}\eta^i =& J_{ij}\eta^i + \left(2\sigma_{x^i}(F^2)_{y^j} + f\,(F^2)_{y^iy^j} - 2\sigma_{x^j}(F^2)_{y^i}\right)\dot{\eta}^i \\ &+ \left(2\frac{d}{d\beta}\left(\sigma_{x^i}(F^2)_{y^j}\right) + 2f\sigma_{x^i}(F^2)_{y^j} + f(F^2)_{x^iy^j}\right. \\ &\left. - 4\sigma_{x^i}\sigma_{x^j} - 2\sigma_{x^ix^j} - 2\sigma_{x^j}(F^2)_{x^i} - 2\sigma_{x^i}(F^2)_{x^j}\right)\eta^i. \end{aligned} \tag{5.0.7}$$

Proof. The Jacobi operator for the Finsler metric F_σ^2 is given by

$$J^\sigma_{ij}\eta^i = \frac{d}{ds}\left((F_\sigma^2)_{y^iy^j}\frac{d\eta^i}{ds} + (F_\sigma^2)_{x^iy^j}\eta^i\right) - \left((F_\sigma^2)_{y^ix^j}\frac{d\eta^i}{ds} + (F_\sigma^2)_{x^ix^j}\eta^i\right). \tag{5.0.8}$$

We will expand the terms above and then conclude the result.

Observe that the vector field η changes as $\frac{d\eta^i}{ds} = \beta'\dot{\eta}^i$. The first two terms on the left-hand side of equation (5.0.8) are given by

$$(F_\sigma^2)_{y^iy^j}\frac{d\eta^i}{ds} = e^\sigma(F^2)_{y^iy^j}\dot{\eta}^i$$

and

$$(F_\sigma^2)_{x^iy^j}\eta^i = e^\sigma\left(2\sigma_{x^i}(F^2)_{y^j}\eta^i + (F^2)_{x^iy^j}\eta^i\right).$$

The third and the fourth terms are

$$(F_\sigma^2)_{y^ix^j}\frac{d\eta^i}{ds} = 2\sigma_{x^j}(F^2)_{y^i}\dot{\eta}^i + (F^2)_{y^ix^j}\dot{\eta}^i$$

and

$$(F_\sigma^2)_{x^ix^j}\eta^i = 2\sigma_{x^ix^j}\eta^i + 4\sigma_{x^i}\sigma_{x^j}\eta^i + 2\sigma_{x^j}(F^2)_{x^i}\eta^i + 2\sigma_{x^i}(F^2)_{x^j}\eta^i + (F^2)_{x^ix^j}\eta^i.$$

Substituting this results in (5.0.8) we have that

$$\frac{d}{ds}\left((F_\sigma^2)_{y^iy^j}\frac{d\eta^i}{ds}+(F_\sigma^2)_{x^iy^j}\eta^i\right)=\frac{d\sigma}{d\beta}M_{ij}+\frac{d}{d\beta}\left((F^2)_{y^iy^j}\dot\eta^i+(F^2)_{x^iy^j}\eta^i\right)$$
$$+2\sigma_{x^i}(F^2)_{y^j}\dot\eta^i+2\frac{d}{d\beta}\left(\sigma_{x^i}(F^2)_{y^j}\right)\eta^i,$$

where $M_{ij}=(F^2)_{y^iy^j}\dot\eta^i+2\sigma_{x^i}(F^2)_{y^j}\eta^i+(F^2)_{x^iy^j}\eta^i$, and

$$(F_\sigma^2)_{y^ix^j}\frac{d\eta^i}{ds}+(F_\sigma^2)_{x^ix^j}\eta^i=\left((F^2)_{y^ix^j}\dot\eta^i+(F^2)_{x^ix^j}\eta^i\right)+2\sigma_{x^j}(F^2)_{y^i}\dot\eta^i$$
$$+2\sigma_{x^ix^j}\eta^i+4\sigma_{x^i}\sigma_{x^j}\eta^i$$
$$+\left(2\sigma_{x^j}(F^2)_{x^i}+2\sigma_{x^i}(F^2)_{x^j}\right)\eta^i.$$

The result follows from the fact that $\frac{d\sigma}{d\beta}=f$. □

5.1 Fermi coordinates

From now on we will restrict ourselves to the case of surfaces. Recall that the covariant derivative along the geodesic c is locally given by

$$D_T\xi=\left(\frac{d\xi^i}{dt}+T^j\xi^k\Gamma^i_{jk}(c,T)\right)\frac{\partial}{\partial x^i}\bigg|_{\bar\gamma}, \quad (5.1.1)$$

where $T=c'$ and Γ^i_{jk} are the Chern-Rund connection coefficients [4].

Proposition 5.1.1 (Finslerian Fermi coordinates)**.** *Let $c:[a,b]\to(M,F)$ be a unitary Finslerian geodesic. There is a $\epsilon-$tubular neighborhood $\mathcal{N}_\epsilon$ and a coordinate chart $\psi:\mathcal{N}_\epsilon\to(a-\delta,b+\delta)\times(-\epsilon,\epsilon)$, with $0<\delta<\epsilon$, such that in these new coordinates we have*

i) $\psi(c(t))=(t,0)$ *and* $\frac{\partial}{\partial t}\big|_{c(t)}=c'(t)$;

ii) $g_{(c,c')}(\frac{\partial}{\partial t},\frac{\partial}{\partial s})=0$;

iii) $\Gamma^i_{jk}(c(t),c'(t))=0$ *for all* i,j,k;

iv) $\forall\, t\in(a-\delta,b+\delta)$ *the curve* $s\mapsto\psi^{-1}(t,s)$ *is a unitary geodesic.*

Proof. To avoid clutter, $(c(t),c'(t))$ will be written $c'(t)$. Let $v\in T_{c(0)}M$ such that $g_{c'(0)}(v,v)=1$ and $g_{c'(0)}(c'(0),v)=0$. If $V(t)$ is the parallel transport of v along c, then $t\mapsto(c(t),V(t))$ is a curve in the unitary tangent bundle T^1M. Define φ by

$$\varphi(t,s)=\pi\circ\phi_s(c(t),V(t)), \quad (5.1.2)$$

where $\phi_s : T^1M \to T^1M$ is the geodesic flow of F. Immediately we have that $\frac{\partial\varphi}{\partial t}\Big|_{(t,0)} = c'(t)$ and that

$$\frac{\partial\varphi}{\partial s}\Big|_{(t,0)} = d\pi\left(\frac{\partial}{\partial s}\Big|_{(t,0)} \phi_s(c(t), V(t))\right) = V(t)$$

because $s \mapsto \phi_s(c(t), V(t))$ is the unitary geodesic that when $s = 0$ is at $c(t)$ with velocity $V(t)$. From the Inverse Function Theorem and from the compacity of $[a, b]$ that $\exists\delta, \epsilon > 0$ with $\epsilon > \delta$ such that φ is a diffeomorphism from $(a-\delta, b+\delta)\times(-\epsilon,\epsilon)$ to $\mathcal{N}_\epsilon := \varphi((a-\delta, b+\delta)\times(-\epsilon,\epsilon))$. Let $\psi = \varphi^{-1}$.

From the definition of ψ we immediately conclude a and d. Letter b follows from $g_{c'(t)}(c'(t), V(t))$ and $\frac{\partial}{\partial s}\Big|_{c(t)} = V(t)$.

The Chern-Rund connection coefficients along the geodesic c are given by

$$\Gamma^i_{jk}(c'(t)) = g_{c'(t)}(D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}),$$

where $\frac{\partial}{\partial x^1} = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^2} = \frac{\partial}{\partial s}$. But $D_{c'}c' = 0$ and $D_{c'}V = D_V c' = 0$, so the remaining cases are those involving $D_V V$:

$$g_{c'(t)}(D_V V, c') = \frac{\partial}{\partial t} g_{c'(t)}(V, c') - g_{c'(t)}(V, D_V c') = 0$$

and

$$g_{c'(t)}(D_V V, V) = \frac{1}{2}\frac{\partial}{\partial s} g_{c'(t)}(V, V) = 0.$$

This proves letter c. □

Lift the coordinates ψ to the coordinates $\bar{\psi}$ on the tangent space in the same way we have done on remark 3.0.1.

Corollary 5.1.2. *In the coordinates $\bar{\psi}$ we have that*

$$(F^2)_{x^1}(c(t), c'(t)) = 0; \qquad (F^2)_{x^2}(c(t), c'(t)) = 0; \tag{5.1.3}$$

and

$$(F^2)_{y^1}(c(t), c'(t)) = 2; \qquad (F^2)_{y^2}(c(t), c'(t)) = C_{112}(c(t), c'(t)). \tag{5.1.4}$$

The second order derivatives are given by

$$(F^2)_{x^1y^1}(c(t), c'(t)) = 0; \qquad (F^2)_{x^2y^1}(c(t), c'(t)) = 0; \tag{5.1.5}$$

and

$$(F^2)_{y^2x^1}(c(t), c'(t)) = (C_{112})_{x^1}(c(t), c'(t));$$
$$(F^2)_{y^2x^2}(c(t), c'(t)) = (C_{112})_{x^2}(c(t), c'(t)).$$

Proof. Since there's no loss, we will carry out the calculations considering the pulled-back metric $\psi_* F^2$ and call it F^2 along this proof.

The equalities on 5.1.3 follow from the fact that $t \mapsto (t + t_0)e_1$ and $s \mapsto (t_0 e_1 + s e_2)$ are unitary geodesics with velocity e_1. The first equality of 5.1.4 follows from the homogeneity of F^2. The second equality is given by

$$\begin{aligned}(F^2)_{y^2}(te_1, e_1) &= \frac{d}{ds}\bigg|_{s=0} \big(F^2(te_1, e_1 + se_2)\big) \\ &= \frac{d}{ds}\bigg|_{s=0} (g_{11}(te_1, e_1 + se_2) + 2sg_{12}(te_1, e_1 + se_2) \\ &\quad + s^2 g_{22}(te_1, e_1 + se_2)\big) \\ &= C_{112}(te_1, e_1).\end{aligned}$$

Let's proceed with the calculations of the second order derivatives.

$$\begin{aligned}(F^2)_{x^1 y^1}(te_1, e_2) &= \frac{d^2}{ds\,dw}\bigg|_{s=w=0} \big(F^2((t+s)e_1, (w+1)e_1)\big) \\ &= \frac{d^2}{ds\,dw}\bigg|_{s=w=0} \big((w+1)^2 F^2((t+s)e_1, e_1)\big) \\ &= 2\,\frac{d}{ds}\bigg|_{s=0} \big(F^2((t+s)e_1, e_1)\big) \\ &= 0.\end{aligned}$$

$$\begin{aligned}(F^2)_{x^2 y^1}(te_1, e_2) &= \frac{d^2}{ds\,dw}\bigg|_{s=w=0} \big(F^2(te_1 + se_2, (1+w)e_1)\big)\,; \\ &= 2\,\frac{d}{ds}\bigg|_{s=0} \big(F^2(te_1 + se_2, e_1)\big) \\ &= 4(g_{11})_{x^2}(te_1, e_1),\end{aligned}$$

but this coefficient is zero because V is parallel along the geodesic c.

$$
\begin{aligned}
(F^2)_{x^1y^2}(te_1, e_2) = & \left.\frac{d^2}{ds\,dw}\right|_{s=w=0} \left(F^2((t+s)e_1, e_1 + we_2)\right); \\
= & \left.\frac{d^2}{ds\,dw}\right|_{s=w=0} (g_{11}((t+s)e_1, e_1 + we_2) \\
& + 2wg_{12}((t+s)e_1, e_1 + we_2) \\
& = w^2 g_{22}((t+s)e_1, e_1 + we_2)\big); \\
= & (C_{112})_{x^1}(te_1, e_1) + 2(g_{12})_{x^2}(te_1, e_1); \\
= & (C_{112})_{x^1}(te_1, e_1).
\end{aligned}
$$

The calculations of $(F^2)_{x^2y^2}$ proceed in the same way as the above.

□

Remark 5.1.3. *The use of Finslerian Fermi coordinates gives a more geometric frame, quite close to the Riemannian geometry setting, to study conformal perturbations of the Jacobi equation that is our main goal. Once we have established the Jacobi equation in terms of the unperturbed fundamental tensor and the unperturbed flag curvature, Fermi coordinates helps us to simplify the resulting expression as well as the expression of the covariant derivative, which will be important ahead (see lemma 7.0.3). These simplifications makes clear the relation between the unperturbed geometric form of the Jacobi equation with its conformal perturbed one. A Hamiltonian version of the Fermi coordinates for Finsler metrics can be found in [16] and a general version is in the Appendix, see lemma 8.2.2.*

The next corollary is just the Jacobi equation for the conformal metric in terms of the old metric. It's proof is a straight forward application of propositions 5.0.14 and 5.1.1 and of the corollary 5.1.2.

Corollary 5.1.4. *The Jacobi equations obtained on proposition 5.0.14 in Fermi coordinates are given by*

$$
\begin{aligned}
J^{\sigma}_{11}\eta^1 = & J_{11}\eta^1 + 2f\dot{\eta}^1 + \left(4\dot{f} - 2\sigma_{x^1x^1}\right)\eta^1; \\
J^{\sigma}_{12}\eta^1 = & J_{12}\eta^1 + 2fC_{112}\dot{\eta}^1 + \Big(2\dot{f}C_{112} + 2f\frac{d}{d\beta}(C_{112}) \\
& + 2f^2C_{112} + f(C_{112})_{x^1} - 2\sigma_{x^1x^2}\Big)\eta^1; \\
J^{\sigma}_{21}\eta^2 = & J_{21}\eta^2 - 2fC_{112}\dot{\eta}^2 - 2\sigma_{x^1x^2}\eta^2; \\
J^{\sigma}_{22}\eta^2 = & J_{22}\eta^2 + 2f\dot{\eta}^2 + \left(f(C_{112})_{x^2} - 2\sigma_{x^2x^2}\right)\eta^2.
\end{aligned}
$$