7
Proof of the main theorem

In order to use the methods developed in the first section we have to reduce the formula of the Jacobi equation in corollary 5.1.4 to a first order equation. So, consider the Fermi coordinates along a radial geodesic α_v passing through p.

Given a vector field along α_v of the type $\eta(t) = x(t)V(t)$ where $x : (-\rho - \delta, \rho + \delta) \to \mathbb{R}$ and V is the parallel vector field along α_v introduced in the definition of the Fermi coordinates. This vector field was chosen to be perpendicular along α_v.

Lemma 7.0.3. *If $x(-\rho) = 0$ and x is not trivial then, in the Fermi coordinates of corollary 5.1.4, the Jacobi operator applied on the perpendicular vector field $\eta(t) = x(t)V(t)$ is given by*

$$J^\sigma \eta = \Bigg(x''(t) - \frac{f'(t)}{\Lambda_0} x'(t) \tag{7.0.1}$$

$$+ \left(K_v(t) + \frac{\lambda^2(t)}{\Lambda_0^2} \frac{f'(t)}{|t|} - \frac{f'(t)}{2\Lambda_0}(C_{112})_{x^2}\right) x(t) \Bigg) V(t),$$

where λ is a smooth positive function defined on $[-\rho, \rho]$ with $\lambda(0) = 1$.

Proof. First of all, a Jacobi field η along a geodesic α_v satisfies the relation

$$g_{\alpha_v'}(\alpha_v'(t), \eta(t)) = g_{\alpha_v'}(\alpha_v'(-\rho), \eta(-\rho)) + (t + \rho) g_{\alpha_v'}(\alpha_v'(-\rho), \eta'(-\rho)). \tag{7.0.2}$$

If we multiply both sides of the equation above by $e^{2\sigma}$ we obtain the same relation for the Finsler metric F_σ. Then, the vector field η is perpendicular to α_v for both metrics, g and g^σ. Therefore, given a parallel vector field W along α_v associated with the metric g^σ such that $g^\sigma_{\dot\alpha_v}(\dot\alpha_v'(s), W(s)) = 0$, there exists a function $y = y(s)$ such that $\eta = yW$, where the dot corresponds to the derivative in the new time parameter. Now, using the Chern-Rund connection formalism

$$J^\sigma \eta = D_{\dot\alpha_v} D_{\dot\alpha_v} \eta + R^\sigma(\eta, \dot\alpha_v)\dot\alpha_v$$

but, from the parallelism of W, $D_{\dot\alpha_v} W = 0$ and

$$R^\sigma(\eta, \dot\alpha_v)\dot\alpha_v = K_v^\sigma(s) y(s) W(s).$$

So, the vector $J^\sigma \eta$ is parallel to W which is parallel to V because of equation (7.0.2).

Since $\eta = x(t)V(t)$ we have that $J^\sigma_{1i}\eta^1 = 0$. From what was done above we have that $J^\sigma_{21}\eta^2 = 0$ because the vector $J^\sigma\eta$ is parallel to V. After all this reductions we conclude that

$$J^\sigma \eta = J^\sigma_{22}\eta^2\, V(t).$$

To finish the proof we have to calculate the derivative $\sigma_{x^2x^2}$. According to the definition of σ, using the Fermi coordinate system defined on equation (5.1.2) on a square contained inside the ball $\mathcal{B}_p(\rho)$,

$$\sigma(x^1, x^2) = \sigma_0\left((\exp_p)^{-1}(x^1, x^2)\right),$$

where $(x^1(s,t), x^2(s,t)) = \pi \circ \phi_s\left(\alpha_v(t), V(t)\right)$. Consider the vector field along tv defined by $\bar{V}(t) = d(\exp_p^{-1})_{\alpha_v(t)}V(t)$. Gauss lemma implies that $\bar{V}(t) = \lambda(t)\bar{V}_0$, where $\bar{V}_0$ is a unitary vector perpendicular to v and $\lambda > 0$ such that $\lambda(0) = 1$.

Therefore, we have that

$$\begin{aligned}\sigma_{x^2x^2}(x^1(t,0), x^2(t,0)) &= \left.\frac{d^2}{ds^2}\right|_{s=0} \left(\sigma_0(tv + s\bar{V}(t))\right) \\ &= -\frac{\lambda^2(t)}{2\Lambda_0^2}\frac{f'(t)}{|t|}.\end{aligned}$$

□

Lemma 7.0.4. *If there is a non trivial solution y of equation*

$$y''(t) + \left(K_v(t) + \frac{f'(t)}{|t|}\right)y(t) = 0 \qquad (7.0.3)$$

such that $y(-\rho) = y(t_0) = 0$, $t_0 \in (-\rho, \rho]$, then the geodesic α_v has conjugate points.

Proof. The equation

$$x''(t) - \frac{f'(t)}{\Lambda_0}x'(t) + (K_v(t) + \frac{\lambda^2(t)}{\Lambda_0^2}\frac{f'(t)}{|t|} - \frac{f'(t)}{2\Lambda_0}(C_{112})_{x^2})x(t) = 0 \qquad (7.0.4)$$

can be seen as a first order linear system of the type

$$X'(t) = F(t, X, \epsilon), \qquad (7.0.5)$$

where $X = (x, p)$, $p = x'$, and F is linear in the X variable. Observe that, although the dependence on the parameter ϵ is no explicit in (7.0.4), it appears if we look to the definition of f. Moreover, $f, f' \to 0$ uniformly when $\epsilon \to 0$.

Change the ϵ dependence associated with the terms $\frac{f'(t)}{\Lambda_0}$ and $\frac{f'(t)}{2\Lambda_0}(C_{112})_{x^2}$ by $\bar{\epsilon}$ such that $\bar{\epsilon} \in [0, \epsilon)$. Call these new terms $g_i(t, \bar{\epsilon})$, $i = 1, 2$, respectively. It is clear that this term also goes uniformly to zero when $\bar{\epsilon} \to 0$. We will obtain another second order equation (now with parameters $(\epsilon, \bar{\epsilon})$)

$$x''(t) - g_1(t, \bar{\epsilon})x'(t) + \left(K_v(t) + \frac{\lambda^2(t)}{\Lambda_0^2}\frac{f'(t)}{|t|} - g_2(t, \bar{\epsilon})\right)x(t) = 0 \tag{7.0.6}$$

which will linearize to

$$X'(t) = \bar{F}(t, X, \epsilon, \bar{\epsilon}). \tag{7.0.7}$$

Recall that we have to choose the constant ρ with respect to ϵ and this will only depend on the infimum of the curvatures inside the coordinate ball chosen.

Consider the solution $\bar{X}(t) = \bar{X}(t; \epsilon, \bar{\epsilon})$ of (7.0.7) with initial conditions $\bar{X}(-\rho) = (0, 1)$. The continuous dependence on initial conditions and parameters (cf. [8], p. 58) implies that, if, for some $t_0 \in (-\rho, \rho)$ and $\bar{\epsilon} = 0$, $\bar{X}(t_0) = (0, -\tau)$ then there exists $\delta > 0$ such that for $\bar{\epsilon} \in [0, \delta)$ there is $\bar{t}_0 = \bar{t}_0(\bar{\epsilon})$ with $\bar{X}(\bar{t}_0; \epsilon, \bar{\epsilon}) = (0, -\bar{\tau})$. Actually, by continuity, this will work also for $\bar{\epsilon} = \delta$. Then, applying several times the continuous dependence theorem we can cover the interval $[0, \epsilon)$ and finally conclude that the solution $X(t) = X(t; \epsilon)$ of 7.0.5 with the same initial conditions of $\bar{X}$ also has a zero at some $t_1 > -\rho$.

To finish the proof, just observe that $\frac{\lambda^2}{\Lambda_0^2} \geq 1$ and apply Sturm comparison theorem. □

7.1 Proof of theorem 1.0.1

Let $\epsilon > 0$. If $\epsilon > \bar{\epsilon} > 0$, let $\beta \in (0, 1)$, $\rho(\bar{\epsilon}, \beta)$ and $\alpha(\bar{\epsilon}, \beta)$ be the ones obtained in proposition 4.0.11. These constants together with the Finsler metric F determine the functions σ_0 and σ. Rename σ as σ_β. Because of 6.1.2 we can suppose that

$$||1 - e^{\sigma_\beta}||_1 < \frac{\epsilon}{||F||_1}.$$

Define $\bar{F} = \frac{\rho}{a}F$. Since $K_v(t) \geq K_0^\rho$, we have that $K_v(t) + \frac{f'(t)}{|t|} \geq K^\rho_{\bar{\epsilon},\alpha,\beta}$ and, therefore, by proposition 4.0.11, equation (7.0.3) has conjugate points. So, lemma 7.0.4, implies that the geodesic has conjugate points for every $v \in T^1_pM$.

Given a geodesic γ of $\bar{F}_\beta = e^{\sigma_\beta}\bar{F}$ such that $\gamma(0) = p$, there exist $0 < t_0 \leq \rho$ such that the length of gamma restricted to $[-t_0, t_0]$ is strictly bigger that $d(\gamma(-t_0), \gamma(t_0))$. This fact is invariant by rescaling, so, all the geodesics of $e^{\sigma_\beta}F = F_\beta$ containing p have conjugate points.

Finally, observe that

$$||F - e^{\sigma}F||_1 \leq ||1 - e^{\sigma}||_1 \, ||F||_1 < \epsilon.$$

To finish the proof, Lemma 6.1.2 implies that $||\sigma_\beta|| < \epsilon$.