8
Appendix

Here we will approach the problem from a Hamiltonian perspective and prove a Hamiltonian version of the theorem. The substantial difference is that the perturbation will be done by potentials, which simplifies the calculations. Along this chapter, equip M with a Riemannian metric and consider the Riemannian metric induced in T^*M.

Theorem 8.0.1. *Let $H : T^*M \to \mathbb{R}$ be a smooth 2-homogeneous reversible Hamiltonian and $\epsilon > 0$. Then there exist $\beta \in (0,1)$ and a potential $U : M \to \mathbb{R}$ such that*

i) U is $\epsilon-$close to zero in the $C^{1,\beta}$ topology;

ii) If $k > \sup_{x\in M} |U(x)|$ then

$$H^U = H + U$$

have no invariant continuous graphs in the level set associated with k.

We begin with the definition of Tonelli Hamiltonian [1].

Let $H : T^*M \to \mathbb{R}$ be a C^∞ *Tonelli Hamiltonian*, that is, a smooth function that satisfies:

i) *Convexity*: For all $x \in M$ and $p \in T^*_x M$ the Hessian $H_{p^i p^j}(x,p)$ is positive definite;

ii) *Superlinearity*:
$$\lim_{|p|_x \to \infty} \frac{H(x,p)}{|p|_x} = \infty.$$

In what follows, we will also require that H be 2-homogeneous, that is,

$$H(x, \lambda p) = \lambda^2 H(x,p) \quad \lambda \geq 0.$$

[1] Although the main theorem of this section is about 2-homogeneous, reversible Hamiltonians, the broader class of Tonelli Hamiltonians are very important because of their relation with Finsler metrics. See the discussion after lemma 8.0.2

It is easy to see that this property implies the superlinearity. These kind of Hamiltonians are obtained from the convex dual of Finsler metrics. Namely, given a Finsler metric F, $H(x,p) = \sup_{v\in T_xM}\{p(v) - F^2(x,v)\}$.

The natural symplectic structure induces a vector field X_H, called the *symplectic gradient* of H, by

$$dH = \iota_{X_H}\omega_0.$$

In local coordinates, the equation $\dot{\theta} = X_H(\theta)$, for $\theta \in T^*M$, becomes

$$\dot{x} = H_p \quad \dot{p} = -H_x \tag{8.0.1}$$

where H_x and H_p are the partial derivatives with respect to x and p. Observe that H is constant along the orbits of X_H then the *energy levels* $\Sigma_e = H^{-1}(e)$ contains the orbits of (8.0.1). Compactness of M and the superlinearity of H implies that Σ_e is compact. Since (8.0.1) is Lipschitz, its solutions are defined for all $t \in \mathbb{R}$. Denote the flow of X_H by ψ_t.

A *potential* is a smooth function $U : M \to \mathbb{R}$. The Hamiltonian $H^U = H + U$ still satisfies the properties of the Tonelli Hamiltonian, but it is no longer homogeneous.

Lemma 8.0.2. *Let $\alpha(t) = (x(t), p(t))$ be an orbit of X_H without self intersection and such that $H(\alpha) = 1$. If U is a potential such that $d_{x(t)}U = f(t)p(t)$, where f is a smooth positive real function, then there exists a reparametrization $t \to s(t)$ such that $\bar{\alpha}(s) = (\bar{x}(s), \bar{p}(s))$, where $\bar{x}(s) = x(t(s))$ and $\bar{p}(s) = \frac{dt}{ds}p(t(s))$, is an orbit of X_{H^U}.*

Proof. Suppose that α is defined on $[a,b]$. Let $E > \sup_{t\in[a,b]} U(x(t))$. Define the function s by

$$s(t) = c + \int_a^t (E - U(x(u)))^{-\frac{1}{2}}\, du. \tag{8.0.2}$$

We have that

$$\frac{d\bar{x}}{ds} = \frac{dt}{ds}\frac{dx}{dt} = \frac{dt}{ds}H_p(x(t(s)), p(t(s))).$$

But H_p is 1-homogeneous and $\bar{p}(s) = \frac{dt}{ds}p(t(s))$, which implies that

$$\frac{d\bar{x}}{ds} = H_p^U(\bar{x}(s), \bar{p}(s)).$$

For the second equation,

$$\begin{aligned}\frac{d\bar{p}}{ds} &= \frac{d^2t}{ds^2}p(t(s)) + \left(\frac{dt}{ds}\right)^2\frac{dp}{dt};\\ &= \frac{d^2t}{ds^2}p(t(s)) - \left(\frac{dt}{ds}\right)^2 H_x(x(t(s)),p(t(s)));\\ &= \frac{1}{f(t(s))}\frac{d^2t}{ds^2}dU(x(t(s))) - H_x(\bar{x}(s),\bar{p}(s));\\ &= \frac{1}{f(t(s))}\frac{d^2t}{ds^2}dU(x(t(s))) + dU(x(t(s))) - H_x^U(\bar{x}(s),\bar{p}(s)),\end{aligned}$$

because H_x is 2-homogeneous. Remains to prove that the first two terms of the right-hand side of equation above cancel each other. Now, since

$$\frac{dt}{ds} = \sqrt{E - U(x(t(s)))},$$

we have that

$$\frac{d^2t}{ds^2} = -\frac{1}{2}(\frac{dt}{ds})^{-1}\frac{dt}{ds}\frac{d}{dt}(U(x(t))) = -\frac{1}{2}dU(x(t))\cdot\frac{dx}{dt}.$$

Because H is 2-homogeneous, we have $p(t)(\frac{dx}{dt}) = p \cdot H_p = 2$, therefore $\frac{d^2t}{ds^2} = -f(t)$. □

In the above lemma, with the reparametrization s, the new orbit $\bar{\alpha}$ will satisfy $H^U(\bar{\alpha}) = E$. Recall that our Hamiltonian H is the convex dual of a Finsler metric. So, this reparametrization suggests a Finsler like Maupertuis principle (cf [2]). In fact, this is part of a more general result: if L is the Lagrangian associated to a Tonelli Hamiltonian H then its flow restricted to some super critical level set of the energy is conjugated to the flow of a Finsler metric (cf [9]).

8.1 The Hamiltonian Jacobi equation

The Levi-Civita connection ∇ associated with the Riemannian metric on M can be extended to 1-forms in the following way

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

This is a torsion free linear connection on T^*M and can be used to construct a isomorphism

$$T_\theta T^*M \simeq T_{\bar{\pi}(\theta)}M \times T^*_{\bar{\pi}(\theta)}M.$$

Let's construct this isomorphism. For $\xi \in T_\theta T^*M$, let $t \mapsto \alpha(t) = (x(t), p(t)) \in T^*M$ be a curve satisfying $\alpha(0) = \theta$ and $\dot\alpha(0) = \xi$. The isomorphism is given by

$$\xi \mapsto \Phi(\xi) = (d_\theta \bar\pi(\xi), \nabla_{\dot x} P(0)) .$$

Here we can do another identification. Define the *vertical* subspace by $V(\theta) = \ker d_\theta \bar\pi$ and we have that

$$V(\theta) \simeq \{0\} \times T^*_{\bar\pi(\theta)} M \simeq T_{\bar\pi(\theta)} M.$$

The map $K_\theta(\xi) = \nabla_{\dot x} P(0)$ is called the *connection* map and we can define the *horizontal* subspace by $H(\theta) = \ker K_\theta$. From this, we have that

$$H(\theta) \simeq T_{\bar\pi(\theta)} M \times \{0\} \simeq T_{\bar\pi(\theta)} M.$$

Therefore $T_\theta T^*M \simeq H(\theta) \oplus V(\theta)$.

Now, let $\sigma : (-\epsilon, \epsilon) \to T^*M$ be a smooth curve. Define $(x_s(t), p_s(t)) = \psi_t \circ \sigma(s)$ be a variation of the orbit $(x(t), p(t)) = (x_0(t), p_0(t))$ by orbits of the of the vector field X_H. Using the decomposition $T_\theta T^*M \simeq T_{\bar\pi(\theta)} M \times T_{\bar\pi(\theta)} M$, let

$$\xi(t) = d_\theta \psi_t(\xi) = (h(t), v(t)),$$

where $\xi \in T_\theta T^*M$.

Lemma 8.1.1. *The functions $h(t), v(t)$ satisfies system of the equations*

$$\begin{aligned} \dot h &= H_{px} h(t) + H_{pp} v(t); \\ \dot v &= -H_{xx} h(t) - H_{xp} v(t), \end{aligned} \tag{8.1.1}$$

where H_{xx}, H_{xp}, H_{px} and H_{pp} are linear operators on $T_{\bar\pi(\theta)} M$ which coincides in a local coordinate system with the matrices of partial derivatives $\left(\frac{\partial^2 H}{\partial x^i \partial x^j}\right)$, $\left(\frac{\partial^2 H}{\partial x^i \partial p_j}\right)$, $\left(\frac{\partial^2 H}{\partial p_i \partial x^j}\right)$ and $\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)$.

8.2 The perturbed Jacobi equation

Consider $U : M \to \mathbb{R}$ a potential such that, if $(x(t), p(t))$ is an orbit of X_H, then $d_{x(t)} U = f(t) p(t)$ for some smooth positive real function f. We would like to see how the Hamiltonian Jacobi equation behave for the Hamiltonian H^U when considered over the orbit $(\bar x(s), \bar p(s))$ obtained as in lemma 8.0.2.

Proposition 8.2.1. *Let $\bar\xi(s) = (\bar h(s), \bar v(s))$ be a vector field along $(\bar x(s), \bar p(s))$ which satisfies the equations* (8.1.1) *for the Hamiltonian H^U. Using the change* *of coordinates $t \mapsto s(t)$ from lemma 8.0.2 these equations looked over the orbit $(x(t), p(t))$ of X_{H^U} become*

$$\begin{aligned} \dot{h} &= H_{px}h(t) + \frac{ds}{dt}H_{pp}v(t) \\ \dot{v} &= -\left(\frac{ds}{dt}\right)^{-1} H_{xx}h(t) - H_{xp}v(t) - \frac{ds}{dt}U_{xx}h(t). \end{aligned} \tag{8.2.1}$$

Proof. From lemma 8.0.2, we know that $\bar{x}(s) = x(t(s))$ and $\bar{p}(s) = \frac{dt}{ds}p(t(s))$. If Y is a vector field along $\bar{x}$ then

$$\frac{d}{ds}Y = \frac{dt}{ds}\frac{d}{dt}Y.$$

If H is 2-homogeneous then H_{xx} is 2-homogeneous, H_{xp} and H_{px} are 1-homogeneous and H_{pp} is 0-homogeneous.

The first equation became

$$\begin{aligned} \frac{d}{ds}\bar{h} &= H^U_{px}\bar{h}(s) + H^U_{pp}\bar{v}(s) \\ &= \frac{dt}{ds}H_{xp}h(t(s)) + H_{pp}v(t(s)). \end{aligned}$$

Then, multiplying by $\frac{ds}{dt}$, we have that the first equation over $(x(t), p(t))$ is

$$\dot{h} = H_{px}h(t) + \frac{ds}{dt}H_{pp}v(t). \tag{8.2.2}$$

The second equation became

$$\begin{aligned} \frac{d}{ds}\bar{v} &= -H^U_{xx}\bar{h}(s) - H^U_{xp}\bar{v}(s) \\ &= -\left(\frac{dt}{ds}\right)^2 H_{xx}h(t) - U_{xx}h(t) - \frac{dt}{ds}H_{xp}v(t). \end{aligned}$$

Again, multiplying by $\frac{ds}{dt}$, we obtain

$$\dot{v} = -\left(\frac{ds}{dt}\right)^{-1} H_{xx}h(t) - H_{xp}v(t) - \frac{ds}{dt}U_{xx}h(t). \tag{8.2.3}$$

□

The following result is found in [11]. It can be regarded as an Hamiltonian version of the well known Fermi coordinates[2] and we will call them *Hamiltonian Fermi coordinates*.

Lemma 8.2.2. *Let $(x(t), p(t))$, $t \in [a, b]$, be a non-singular orbit of the flow of H without self intersections. Suppose that $(x(0), p(0)) = (x, p)$. Up to*

[2]See proposition 5.1.1.

translation in the x variable and to take a smaller interval, there exist local coordinates $\Phi : \mathcal{O} \subset M \to \mathbb{R}^n$, $\Phi = (x_1, x_2, ..., x_n)$, where $\mathcal{O}$ is an open neighbourhood of x and $\Phi(x) = 0$, such that the Hamiltonian $\bar{H} = \Phi_* H$ defined in an open set $\mathcal{V} \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ satisfies the following properties:

i) The orbit of $\bar{H}$ through $(0, e_1)$ is $\bar{\psi}_t(0, e_1) = (te_1, e_1)$ for every $t \in [a, b]$.

ii) In the coordinates (x, p) the Hamiltonian satisfies

(a) $\frac{\partial^2 \bar{H}}{\partial x^i \partial p_j}(\bar{\psi}_t(0, e_1)) = 0$ for any $i, j = 1, ..., n$.

(b) $\frac{\partial^2 \bar{H}}{\partial p_1 \partial p_j}(\bar{\psi}_t(0, e_1))$ for any $j = 2, ..., n$.

(c) The $(n-1) \times (n-1)$ matrix whose entries are $\frac{\partial^2 \bar{H}}{\partial p_i \partial p_j}(\bar{\psi}_t(0, e_1))$, $2 \le i, j \le n$, is the identity matrix I_{n-1}.

They allow us to rewrite the Hamiltonian Jacobi equation in a much simpler way. In these coordinates we have that $H_{px} = H_{xp} = 0$. Since

$$\frac{\partial^2 H}{\partial x^1 \partial x^i} = \frac{d}{dt}\frac{\partial H}{\partial x^i} = \ddot{p}^i = 0,$$

because $\Phi_*(x(t), p(t)) = (te_1, e_1)$, the operator H_{xx} becomes

$$H_{xx} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 H}{\partial x^i \partial x^j} \end{pmatrix},$$

where $i, j = 2, ..., n$.

Suppose that $\xi(t) = (h(t), v(t))$ is a solution of (8.1.1) with initial conditions in Hamiltonian Fermi coordinates $h(0) = (0, h_0(0))$ and $v(0) = (0, v_0(0))$, where $h_0(0)$ and $v_0(0)$ are generated by $\frac{\partial}{\partial x^i}$, $i = 2, ..., n$. From equation (8.2.1), lemma 8.2.2 and the matrix of the operator H_{xx} we have that $h(t) = (0, h_0(t))$ and $v(t) = (0, v_0(t))$, that is, in the new coordinates, the space

$$\Pi = \{x^i \frac{\partial}{\partial x^i} | i = 2, ..., n\} \times \{x^i \frac{\partial}{\partial x^i} | i = 2, ..., n\} \subset T_{x(t)}M \times T_{x(t)}M$$

is invariant by $d\psi_t$. We will work only on the space Π.

If we use the Hamiltonian Fermi coordinates over $(x(t), p(t))$ with Hamiltonian H, the (projected) Hamiltonian Jacobi equation for H^U obtained on (8.2.1) become

$$\dot{h} = \frac{ds}{dt} v(t),$$
$$\dot{v} = -\left(\frac{ds}{dt}\right)^{-1} H_{xx} h(t) - \frac{ds}{dt} U_{xx} h(t),$$

where all the operators are obtained from the derivatives on the projected space Π.

Since $\dot{h}(t) = \frac{ds}{dt}v(t)$, we have that h satisfies the second order equation

$$\ddot{h}(t) - \frac{d^2s}{dt^2}\left(\frac{ds}{dt}\right)^{-1}\dot{h}(t) + H_{xx}h(t) + \left(\frac{ds}{dt}\right)^2 U_{xx}h(t) = 0. \tag{8.2.4}$$

When the $\dim M = 2$, the projected horizontal vector field is a scalar function. Call this scalar function h_x. In this case, equation (8.2.4), is the one dimensional second order linear equation

$$\ddot{h_x}(t) - \frac{d^2s}{dt^2}\left(\frac{ds}{dt}\right)^{-1}\dot{h_x}(t) + H_{x^2x^2}h_x(t) + \left(\frac{ds}{dt}\right)^2 U_{x^2x^2}h_x(t) = 0. \tag{8.2.5}$$

The function $H_{x^2x^2}$ will play the role of the flag curvature in this case.

8.3 Proof of the Hamiltonian version of the main theorem

Suppose that $M = T^2$, the Hamiltonian H is reversible and 2-homogeneous. Let $\epsilon > 0$ and $k > \epsilon^{-1}$.

Define the exponential map of the energy level $H_k = H^{-1}\{k\}$ as $\exp_x : T^*M \to M$, $\exp_x(t\theta) = \bar{\pi} \circ \psi_t(\theta)$. The reversibility together with the homogeneity of H, implies that $\psi_{-t}(x,p) = \psi_t(x,-p)$.

Now we will give another definition of conjugate point which is more suited to the Hamiltonian case. Of course it coincides with the other given.

Definition 8.3.1. *Given $\theta \in T^*M$. The point $\eta \in T^*M$ is conjugated to θ if there is $t_0 \in \mathbb{R}$ such that $\psi_{t_0}(\theta) = \eta$ and*

$$d_\theta\psi_{t_0}(V(\theta)) \cap V(\eta) \neq 0.$$

Let $H_k(\bar{\pi}(\theta)) = H_k \cap T^*_{\bar{\pi}(\theta)}M$. let $x_0 \in M$. For every $\theta \in H_k(x_0)$ there is $a_\theta > 0$ such that the orbit $\psi_t(\theta)$ does not have conjugate points on $[0, a_\theta]$. Moreover,

$$a = \inf_{\theta \in H_k(x)} a_\theta > 0.$$

Let $K(\theta) = \sup_{u \in H_k(x_0)} H_{xx}(\theta)uu$, where $\theta = (x_0, \theta_0)$. Define

$$K_0 = \inf_{t \in [-a,a]} \{K(\psi_t(x,u)) | \, (x,u) \in H_k(x_0)\}.$$

Let $\beta(\epsilon) \in (0,1)$. Define the constant $\rho = \rho(\epsilon, \beta, a, K_0)$ exactly as in (4.1.2). We will perturb the Hamiltonian

$$H^\rho = \frac{\rho^2}{a^2}H.$$

Use the constants β, ρ and the exponential map to build the potentials U_0 and U round the point x_0 in the same way that was done in the subsection 6.1. Define the perturbed Hamiltonian by $H^\rho_{U_0} = H^\rho + U_0$ and $H^\rho_U = H^\rho + U$.

Since $\sup_{x\in M} |U(x)| = \sup_{x\in M\setminus\{x_0\}} |U_0(x)| < \epsilon$, there is no inconsistency in choose the constant E to be k at lemma 8.0.2.

Because of lemma 8.0.2 and equation (8.0.2) we have that $\frac{ds}{dt} < \epsilon$ and $(\frac{d^2s}{dt^2})(\frac{ds}{dt})^{-1} < \epsilon^2$. So, use the same proof that was done in lemma 7.0.4 on equation (8.2.5) to use proposition 4.0.11 and show that every orbit containing x_0 is not a minimizer of H^ρ_U. Note that, in the Finsler setting, the perturbed Jacobi equation is simpler and this passage becomes easier.

Since the orbits of $H_{\frac{a^2}{\rho^2}U} = H + \frac{a^2}{\rho^2}U$ are just a reparametrization of a orbit of H^ρ_U, we conclude that, for the potential $\bar{U} = \frac{a^2}{\rho^2}U$, H_U has no invariant continuous graphs. For the remaining, just observe that by a reparametrization, the C^1 norm increases, but not the C^0 norm. This is sufficient in this case because ρ is of order $\epsilon^{\frac{1}{2+\beta}}$.