



Rodrigo Pereira Pacheco

**Destruction of Invariant Graphs by
 C^1, β Perturbations**

Tese de Doutorado

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC–Rio, as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Rafael Oswaldo Ruggiero Rodriguez

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Abstract

Pacheco, Rodrigo P.; Ruggiero, Rafael O.. **Destruction of Invariant Graphs by $C^{1,\beta}$ Perturbations**. Rio de Janeiro, 2014. 66p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

According to the theory developed by Kolmogorov, Arnold and Moser in the sixties, the majority of invariant tori persists under a C^3 perturbation of an integrable Hamiltonian. A natural question is if a perturbation in the C^k topology, $k < 3$, still preserves such tori. Bangert showed that, in the C^1 topology, what happens is the opposite: there are metrics with no invariant torus arbitrarily close to any given Riemannian metric. Ruggiero extended these results to mechanical Lagrangians in the torus and showed that for Riemannian metrics this phenomenon is C^1 generic. In this work, we show that, given $\epsilon > 0$, $E \in \mathbb{R}$ and a reversible Tonelli Hamiltonian $H : T\mathbb{T}^2 \rightarrow \mathbb{R}$, there exists $\beta \in (0, 1)$ and an ϵ perturbation H_0 of H in the $C^{1,\beta}$ topology such that H_0 has no continuous invariant graphs. The result is achieved by explicitly exhibiting a Finsler metric, without any continuous field of minimizers, constructed after an analytic study of the Jacobi operator.

Keywords

Conformal Finsler Geometry; Jacobi equation; Lagrangian graphs; Euler-Lagrange equation; Conjugate points.

Resumo

Pacheco, Rodrigo P.; Ruggiero, Rafael O.. **Destruição de gráficos invariantes por perturbações C^1** . Rio de Janeiro, 2014. 66p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Segundo a teoria desenvolvida por Kolmogorov, Arnold e Moser na década de sessenta, a grande maioria dos toros invariantes persistem após uma perturbação C^3 de um Hamiltoniano integrável. Uma pergunta natural é se perturbações em topologias C^k , para $k < 3$, ainda preservam tais toros. Bangert mostrou que a situação é a oposta na topologia C^1 : arbitrariamente próximo de uma métrica Riemanniana plana no toro existem métricas sem nenhum toro invariante. Ruggiero estendeu esses resultados para Lagrangeanos mecânicos no toro e mostrou que, no caso de métricas Riemannianas, esse fenômeno é C^1 genérico. Neste trabalho, mostramos que, dado $\epsilon > 0$, $E \in \mathbb{R}$ e um Hamiltoniano de Tonelli reversível $H : T\mathbb{T}^2 \rightarrow \mathbb{R}$, existe $\beta \in (0, 1)$ e uma ϵ perturbação H_0 de H tal que H_0 não possui gráficos contínuos invariantes. Para tal, construímos explicitamente uma métrica Finsler, sem nenhum campo contínuo de minimizantes, através de um estudo analítico do operador de Jacobi.

Palavras-chave

Geometria Finsler conforme; Equação de Jacobi; Gráficos Lagrangeanos; Equação de Euler-Lagrange; Pontos conjugados.

Contents

1	Introduction	10
2	A geometric approach	14
2.1	Perturbations by cones	16
2.2	Proof of theorem 2.0.2	19
3	Preliminaries	21
3.1	Lagrangian viewpoint	22
3.2	Geometrical viewpoint	25
3.3	On Lagrangian graphs	29
4	Some ODE results	33
4.1	Proof of proposition 4.0.11	36
5	Conformal Jacobi equation	43
5.1	Fermi coordinates	46
6	Construction of conformal radial perturbations	50
6.1	Building the σ function	50
7	Proof of the main theorem	53
7.1	Proof of theorem 1.0.1	55
8	Appendix	57
8.1	The Hamiltonian Jacobi equation	59
8.2	The perturbed Jacobi equation	60
8.3	Proof of the Hamiltonian version of the main theorem	63
	Bibliography	65

List of Figures

- 2.1 In the right, the curve connecting x_1 and x_2 is a minimizing geodesic of length 2ρ that not intersect V_p . 15
- 2.2 On the interval $[-r_a, r_a]$ we used a function of order $x^{\frac{1}{3}}$ to smooth f'_a . 18
- 4.1 $u_1(t) = \frac{\bar{y}(0)}{\rho}(t + \rho)$ and $u_2(t) = t + \rho$. 39
- 4.2 \hat{x} cannot intersect \hat{y} on $(0, \rho)$, so there is $t_0 \in (0, \rho)$ where $\hat{x}(t_0) = 0$. 41