Pontifícia Universidade Católica
do Rio de Janeiro

José Victor Goulart Nascimento

Towards a combinatorial approach to the topology of spaces of nondegenerate spherical curves

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Matemática of the Departamento de Matemática of PUC-Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Nicolau Corção Saldanha
Co-advisor: Prof. Boris Aronovich Khesin

Rio de Janeiro
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To Samuel and Jacob, with all of my love.

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Abstract

Nascimento, José Victor Goulart; Saldanha, Nicolau Corção (advisor); Khesin, Boris Aronovich (co-advisor). **Towards a combinatorial approach to the topology of spaces of nondegenerate spherical curves**. Rio de Janeiro, 2016. 73p. PhD Thesis – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

The space of nondegenerate curves on the n-sphere subject to a fixed monodromy matrix (provided with a suitable Hilbert manifold structure) is decomposed into a countable collection of contractible cells parameterized by the SO_{n+1}-lifted curves' admissible itineraries through cells arriving from a stratification of SO_{n+1} closely related to the classical Bruhat decomposition of GL_{n+1}. The expression "admissible itinerary" herein stands for a finite sequence of cells subject to a few constraints that are otherwise naturally suggested by the geometry of the problem. The main interest of such a new approach is that this combinatorialization works homogeneously in any dimension n (with obvious computational difficulties), unlike the more geometry-flavoured *ad-hoc* methods for achieving topological information about these and related spaces of curves (which usually have had a good run only in low dimensions n). This approach can be regarded as a first attempt at a unified method for figuring out the homotopy-type of such spaces, and it helps to override some functional analysis arguments usually deployed in defining the "right" topology for these spaces of curves.

Keywords

topology of infinite-dimensional manifolds; nondegenerate curves; Bruhat decomposition; Coxeter-Weyl group;

Resumo

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Decompõe-se o espaço das curvas não-degeneradas sobre a n-esfera sujeitas a uma dada matriz de monodromia (munido de uma estrutura de variedade de Hilbert adequada) em uma coleção enumerável de células contráteis parametrizadas pelos itinerários admissíveis para os levantamentos a SO_{n+1} das referidas curvas através das células obtidas de uma estratificação de SO_{n+1} estreitamente relacionada com a clássica decomposição de Bruhat de GL_{n+1}. A expressão "itinerário admissível" significa aqui uma seqüência finita de células sujeitas a umas poucas restrições que, ademais, são naturalmente insinuadas pela geometria do problema. O principal interesse dessa nova abordagem é que essa combinatorialização funciona homogeneamente em todas as dimensões n (não obstante óbvias dificuldades computacionais), diferentemente dos métodos *ad-hoc*, de cunho mais geométrico, até aqui empregados para obter informações topológicas sobre esses e outros espaços de curvas relacionados (que têm sido bem sucedidos apenas em dimensões n baixas). Essa abordagem pode ser considerada como uma primeira tentativa de chegar a um método unificado para a determinação do tipo homotópico de tais espaços, e ajuda a dispensar certos argumentos de análise funcional usualmente empregados na definição da topologia "correta" para os referidos espaços de curvas.

Palavras-chave

topologia de variedades de dimensão infinita; curvas não-degeneradas; decomposição de Bruhat; grupo de Coxeter-Weyl;

Contents

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I have, alas, studied philosophy,
Jurisprudence and medicine, too,
And, worst of all, theology
With keen endeavor, through and through —
And here I am, for all my lore,
The wretched fool I was before.
Called Master of Arts, and Doctor to boot,
For ten years almost I confute
And up and down, wherever it goes,
I drag my students by the nose —
And see that for all our science and art
We can know nothing. It burns my heart.

Goethe's Faust, *in Walter Kaufmann's translation.*

1
Introduction

A curve on the two-dimensional sphere $\mathbb{S}^2$ is called nondegenerate (or locally convex) if its geodesic curvature is everywhere nonvanishing. Several aspects of the topology of the space of nondegenerate curves were studied in (1), (2), (3) and (4). The classification of closed nondegenerate curves up to homotopy was described in (1), where it was shown to exist exactly six connected components in the corresponding space. The slightly more general problem of the classification of quasiperiodic (but not necessarily closed) nondegenerate curves is closely related to certain problems of conformal field theory, namely, to the classification of symplectic leaves of the Gel'fand-Dikii Poisson structures. These structures are defined on the space of coefficients of n-th order linear differential operators on the circle. They are also called SL_n-Adler-Gel'fand-Dikii algebras or generalized n-KdV structures. In Physics these structures are also known as the classical W_n-algebras. (5) and (2) related the classification of the symplectic leaves of these Poisson brackets for operators of arbitrary order to the homotopy classification of nondegenerate curves on spheres or projective spaces. Roughly speaking, two differential operators belong to the same symplectic leaf if and only if the curves obtained by the projectivization of their fundamental solutions are homotopically equivalent within the class of nondegenerate quasiperiodic curves. After that, (3) classified up to homotopy the nondegenerated curves on the 2-sphere subject to a fixed monodromy operator $M \in \mathrm{GL}_3^+$. In (4), the homotopy type of the spaces $\mathcal{L}\mathbb{S}^2(Q)$ (cf. def. on page 17) of positive nondegenerate curves in $\mathbb{S}^2$ subject to a given monodromy $Q \in \mathrm{SO}_3$ was throughly studied. Each such space splits into two (not necessarily connected) components $\mathcal{L}\mathbb{S}^2(\pm z)$, where the unit quaternions $\pm z \in \mathbb{S}^3$ are the lifts of the matrix Q in the Lie group SO_3 to its universal (double) cover $\mathrm{SU}_2 \approx \mathbb{S}^3 \subset \mathbb{H}$. As z ranges over the unit quaternions, the homotopy types of the strata $\mathcal{L}\mathbb{S}^2(z)$ were shown to fall into the following three homotopy types:

$$\Omega\mathbb{S}^3, \Omega\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots, \Omega\mathbb{S}^3 \vee \mathbb{S}^0 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \cdots$$

The motivation for this later work grew up in the course of the study of the family of third order differential operators

$$\frac{d^3}{dt^3} + u\,(t)\,\frac{d}{dt} + v\,(t)$$

in the unit interval, since the set of pairs of coefficients (u, v) for which its kernel admits of a basis comprised by periodic solutions is homotopy equivalent to the space of closed nondegenerate curves with prescribed frame at $t = 0$. For the prehistory of this problem, see (6, 7, 8, 9).

Between the first results concerning the characterization of the connected components of these spaces of nondegenerate curves and the determination of their homotopy types, the interest of some authors shifted from nondegenerate curves on the two-dimensional sphere to their higher-dimensional analogues and to curves with constrained geodesic curvature on the 2-sphere. In the latter direction, (10) characterized the connected components of the space of curves on $\mathbb{S}^2$ with geodesic curvature in a prescribed interval in terms of geometrical properties of these curves.

In higher dimension, a curve γ on the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is said to be nondegenerate if its derivatives up to order n (including itself as a derivative of order zero) at each point span the whole surrounding space $\mathbb{R}^{n+1}$. In (11), it was shown that the space of closed nondegenerate curves in even-dimensional spheres has at least six connected components, generalizing thus the main result of (1). In (12), the question was finally settled down and the exact number was shown to be 6 for even-dimensional spheres and 4 for odd-dimensional spheres. Similar results were proved therein regarding nondegenerate curves in n-dimensional affine and projective spaces. In this spirit, (13) established the existence of a contractible connected component of convex curves for projective spaces of each dimension. In (14), the introduction of a well-tailored stratification of the orthogonal group SO_{n+1} as an aid to the study of the lifts of nondegenerate curves in $\mathbb{S}^n$ was explored, resulting in a classification of the spaces of positive nondegenerate curves subject to a monodromy $Q \in \mathrm{SO}_{n+1}$ up to homeomorphism in a finite number of classes for each n. This (somewhat improperly) so called Bruhat decomposition of SO_{n+1} has shown to be of great value in the determination of the homotopy type of the spaces of nondegenerate curves in the 2-sphere in (4) and the present work takes up the baton from where it was left, to wit: our initial project was to stratify the space $\mathcal{L}\mathbb{S}^n\,(Q)$ of nondegenerate curves on the n-sphere subject to a monodromy $Q \in \mathrm{SO}_{n+1}$ into a countable family of contractible strata of finite codimension – what was fully accomplished – and to derive from this decomposition a CW structure which would hopefully allow for topological information to be harvested in a combinatorial way. This last part is still on the way of being satisfactorily established as a theorem, and we hope to clarify matters in a future follow-up paper. As a matter of fact, we have good reasons to believe its feasibility, since the number of strata in the boundary of a given stratum is finite and the normal bundles of these finite-codimensional strata appear to be trivial (but for the time being we still do not have the argument ready to print). In any case, we found this introductory chapter to be a good place to hint at further directions.

This thesis is roughly divided as follows. In chapter 2 the reader is to find the basic definitions and facts regarding nondegenerate curves on the n-dimensional sphere for general n and the forementioned stratification of SO_{n+1}. We apologize for naming it "the Bruhat decomposition" of the orthogonal group since this term refers properly to a now classical result in the realm of algebraic groups, but in want for a better term we found this choice harmless in the strict limits of this work. Chapter 3 is devoted to a characterization of convex arcs in terms of Bruhat cells and therein we offer a new proof for the contractibility of the space of strictly convex curves. In chapter 4 the main theorem is stated and some of its consequences are drawn before a full proof is put forward, the most technical parts of it being moved to the last section in the form of two lemmas.

We close this introductory chapter with a brief overview of the main results proved in this thesis.

Any nondegenerate curve γ on $\mathbb{S}^n$ admits of a canonical lifting $\mathfrak{F}_\gamma$ to the orthogonal group O_{n+1} which we shall call its path of frames. This lifting is obtained by an orthonormalization procedure in the set of derivatives of γ. Let $\mathcal{L}\mathbb{S}^n$ be the space of nondegenerate curves on the n-sphere with fixed initial frame I. It admits of a Hilbert space structure. Given a matrix $Q \in \mathrm{SO}_{n+1}$, which we shall refer to as the monodromy matrix, let $\mathcal{L}\mathbb{S}^n(Q)$ be the Hilbert submanifold of $\mathcal{L}\mathbb{S}^n$ comprised by curves with final frame Q. As the curve γ travels along the sphere $\mathbb{S}^n$, its path of frames $\mathfrak{F}_\gamma$ wanders through the cells of the Bruhat decomposition of SO_{n+1} given by

$$\mathrm{SO}_{n+1} = \bigsqcup_{P \in \mathrm{B}_{n+1}^+} \mathrm{Bru}_P,$$

where B_{n+1}^+ is the group of orientation-preserving signed permutation matrices. The dimension of each cell Bru_P (cf. def. on page 24) is the number of inversions in the permutation obtained from P by dropping signs. An important feature of this cell decomposition is that a nondegenerate curve can only puncture a positive-codimensional cell in a well-defined transversal direction: leaving a uniquely determined open cell and entering a uniquely determined open cell. This condition places a geometric constraint in the nature of the acceptable itineraries the path of frames can follow. What we call an itinerary is a finite sequence $\mathcal{I}$ of positive-codimensional cells (the information about the open cells is redundant by the reasons we have just explained) which are successively traversed by $\mathfrak{F}_\gamma$ in its way from I to Q. The number of cells listed in an itinerary is what we call its length. As a matter of fact, the geometric constraint we have just described turns out being the only restriction to be observed by acceptable itineraries, so that it is very easy in practice to decide whether a particular itinerary is acceptable or not.

Now, the Hilbert manifold $\mathcal{L}\mathbb{S}^n(Q)$ admits of a decomposition that reads like

$$\mathcal{L}\mathbb{S}^n(Q) = \bigsqcup_{\substack{\mathcal{I}\text{ acceptable}\\ \text{itinerary}}} \mathcal{L}\mathbb{S}^n(Q|\mathcal{I}),$$

where $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ is the Hilbert submanifold of those curves that follow itinerary $\mathcal{I}$.

All in all, chapter 3 is devoted to establishing the major part of the following (slightly paraphrased) result (cf. theorems 3.2, 3.7 and corollary 4.2)

Theorem *There are convex curves in the space $\mathcal{L}\mathbb{S}^n(Q)$ if and only if the monodromy Q admits of an itinerary $\mathcal{I}_c$ of length zero or one. In this case, this shortest itinerary is unique and the submanifold $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ is a (non-empty) contractible connected component of $\mathcal{L}\mathbb{S}^n(Q)$ comprised by the corresponding convex curves.*

Our main theorem, stated and proved in chapter 4, reads roughly as follows (cf. theorem 4.1)

Main Theorem *Given a monodromy matrix $Q \in \mathrm{SO}_{n+1}$ and an acceptable itinerary $\mathcal{I}$, the set $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ is a contractible submanifold of $\mathcal{L}\mathbb{S}^n(Q)$ whose codimension can be easily computed from $\mathcal{I}$.*

As an application, it would be very nice if we could figure out the homotopy type of $\mathcal{L}\mathbb{S}^n(Q)$ by combinatorial reasoning regarding the way the cells $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ assemble together. As a first step, we could recover the result for $n = 2$ obtained in (4). We hope to be able to accomplish this task in a future paper.

2
Basic notions

In what follows, we shall always regard the n-sphere $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} | \ \langle \mathbf{x}|\mathbf{x} \rangle = 1\}$ as an inclusion-embedded submanifold of $\mathbb{R}^{n+1}$ endowed with its usual Euclidean structure given by the scalar product

$$\langle \mathbf{x}|\mathbf{y} \rangle = \sum_{j=0}^{n} x_j y_j \ ,$$

where $\mathbf{x} = (x_0, \cdots, x_n)\,, \mathbf{y} = (y_0, \cdots, y_n) \in \mathbb{R}^{n+1}$.

The general linear group of $\mathbb{R}^{n+1}$, denoted by GL_{n+1}, will almost always be regarded as the isomorphic group of invertible matrices of order $n+1$. Accordingly, any of its subgroups that are to make an appearance in this work shall be defined in terms of matrices alone. For instance, the group Up_{n+1}^{+} is defined as the group of upper-triangular matrices with positive diagonal entries and its subgroup Up_{n+1}^{1} as that of matrices with unit diagonal entries.

What explains (if hardly justifies) our rather cumbersome insistence in the index $n+1$ instead of the perhaps more natural choice n for the dimension of the surrounding Euclidean space is the fact that the natural *habitat* (so to speak) of the curves which are protagonists of this thesis is $\mathbb{S}^n$ and we find $n-1$ more awkward an index than $n+1$.

2.1
Nondegenerate spherical curves

A smooth curve $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{S}^n$ is called *nondegenerate* if its ordered frame of derivatives

$$\left(\gamma\,(t)\,, \gamma'\,(t)\,, \cdots, \gamma^{(n)}\,(t)\right)$$

is a basis for $\mathbb{R}^{n+1}$ for all $t \in [a, b]$. In particular it is an immersion. We shall call it *positive* (respectivelly, *negative*) *nondegenerate* if the ordered frame of derivatives above is a positive (resp., negative) basis for $\mathbb{R}^{n+1}$ for all $t \in [a, b]$.

The *Wronski matrix* of γ at time $t \in [a, b]$ is the square matrix of order

$n+1$ and real entries given by

$$W_\gamma(t) = \begin{pmatrix} | & | & & | \\ \gamma(t) & \gamma'(t) & \cdots & \gamma^{(n)}(t) \\ | & | & & | \end{pmatrix}$$

An alternate phrasing of the former definition is saying that γ is nondegenerate iff $W_\gamma(t)$ is non-singular for all $t \in [a,b]$, *i.e.*, for all t

$$\det W_\gamma(t) \neq 0,$$

being positive iff this Wroskian determinant is always positive and negative otherwise.

In dimension $n = 2$, a nondegenerate positive curve is one that is always turning left, while a nondegenerate negative curve turns right (see figure 2.1 below).

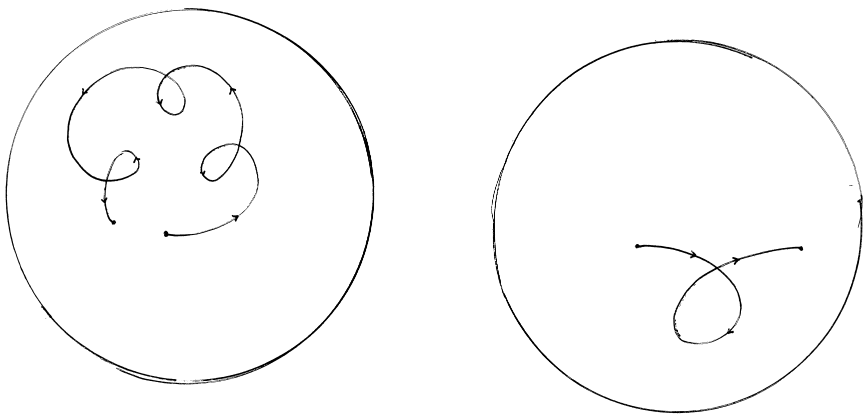

Figure 2.1: turning-left and turning-right curves on the 2-sphere

Any nondegerate curve $\gamma : [a,b] \to \mathbb{S}^n$ can be canonically lifted to a curve $\mathfrak{F}_\gamma : [a,b] \to \mathrm{O}_{n+1}$ by applying the Gram-Schmidt procedure to the column-vector space of its Wronski matrix W_γ. Explicitly, for each $t \in [a,b]$ there is a unique upper triangular matrix with positive diagonal entries $R_\gamma(t) \in \mathrm{Up}^+_{n+1}$ such that

$$W_\gamma(t) = \mathfrak{F}_\gamma(t)\, R_\gamma(t)$$

and $\mathfrak{F}_\gamma(t)$ is an orthogonal matrix. We'll often speak of the lifted curve $\mathfrak{F}_\gamma$ as the *path of frames* of γ. Clearly, the path of frames of a positive nondegenerate curve lives in the connected component SO_{n+1} of O_{n+1}. We shall be primarily concerned with positive nondegenerate curves $\gamma : [a,b] \to \mathbb{S}^n$ whose initial frames are fixed at the group identity, i.e., those curves satisfying

$$\mathfrak{F}_\gamma(a) = I$$

It is really not too serious a restriction for, given a nondegenerate curve γ on $\mathbb{S}^n$, its translated curve $\bar{\gamma} = \mathfrak{F}_\gamma(a)^{-1} \cdot \gamma$ has all the desired properties (the dot stands for the natural action of O_{n+1} on $\mathbb{S}^n$). In fact, its path of frames $\mathfrak{F}_{\bar{\gamma}}(t)$ is the curve $\mathfrak{F}_\gamma(a;t)$ defined by

$$\mathfrak{F}_\gamma(a;t) \equiv \mathfrak{F}_\gamma(a)^{-1} \mathfrak{F}_\gamma(t)$$

Notice that the first column of the matrix $\mathfrak{F}_\gamma(t)$ is the vector $\gamma(t) \in \mathbb{S}^n$ itself. More generally, every path $\Gamma : [a,b] \to \mathrm{SO}_{n+1}$ descends to a (in general not nondegenerate or even smooth) curve $\pi[\Gamma] : [a,b] \to \mathbb{S}^n$ given by the first column of Γ:

$$\pi[\Gamma](t) = \Gamma(t) \cdot e_1$$

Let $\mathcal{L}\mathbb{S}^n$ be the set of all (positive) nondegenerate curves $\gamma : [0,1] \to \mathbb{S}^n$ parameterized in the unit interval satisfying $\mathfrak{F}_\gamma(0) = I$. There is a way of providing it with the (in itself rather uninteresting) topology of a Hilbert space (14), which allows for a decomposition of $\mathcal{L}\mathbb{S}^n$ into infinite-dimensional Hilbert manifolds $\mathcal{L}\mathbb{S}^n(Q)$ obtained by fixing the final frame or *monodromy matrix* Q:

$$\mathcal{L}\mathbb{S}^n = \bigsqcup_{Q \in \mathrm{SO}_{n+1}} \mathcal{L}\mathbb{S}^n(Q)$$

$$\mathcal{L}\mathbb{S}^n(Q) = \{\gamma : [0,1] \to \mathbb{S}^n \text{ nondegenerate} \mid \mathfrak{F}_\gamma(0) = I \text{ and } \mathfrak{F}_\gamma(1) = Q\}.$$

The homotopy type of the spaces $\mathcal{L}\mathbb{S}^n(Q)$ for $n = 2$ was thoroughly described in (4). For $n > 2$ this is a wide open problem towards the solution of which we hopefully present the initial steps in this thesis, provided the path afterwards does not show prohibitively sinuous.

We've just seen above that there is a projection π from the space $\Omega\mathrm{SO}_{n+1}$ of continuous paths in the special orthogonal group onto the space $\Omega\mathbb{S}^n$ of continuous paths in the n-sphere along with a natural lifting $\mathfrak{F}$ from $\mathcal{L}\mathbb{S}^n \subset \Omega\mathbb{S}^n$ to $\Omega\mathrm{SO}_{n+1}$ taking $\mathcal{L}\mathbb{S}^n(Q)$ into $\Omega\mathrm{SO}_{n+1}(Q)$, the subspace of continuous paths ending up in $Q \in \mathrm{SO}_{n+1}$. What is the image of $\mathcal{L}\mathbb{S}^n$ under $\mathfrak{F}$? In other words: which smooth curves Γ in SO_{n+1} arise as paths of frames for nondegenerate curves on the n-sphere? Those Γ are called *holonomic paths*, and allow for a nice description in terms of their logarithmic derivatives, which we describe below.

$$\begin{tikzcd} & \Omega\mathrm{SO}_{n+1} \arrow[d, "\pi"] \\ \mathcal{L}\mathbb{S}^n \arrow[ur, "\mathfrak{F}"] \arrow[r, hook, "i"'] & \Omega\mathbb{S}^n \end{tikzcd}$$

Figure 2.2: commutative diagram

Given a smooth path $\Gamma : [a,b] \to G$ in a Lie group, its (left) *logarithmic derivative* is the path $\Lambda_\Gamma : [a,b] \to \mathfrak{g}$ in the corresponding Lie algebra defined by

$$\Lambda_\Gamma(t) = T_{\Gamma(t)} L_{\Gamma(t)^{-1}} \cdot \frac{d}{dt}\Gamma(t),$$

where $L_g : G \to G$ is the left-multiplication by group element g, $L_g(x) = gx$. Since we are dealing only with Lie groups of matrices, we are entitled to write more simply $\Lambda_\Gamma(t) = (\Gamma(t))^{-1}\Gamma'(t)$. In our particular case where $G = \mathrm{SO}_{n+1}$, given Γ, we have as logarithmic derivative a path $\Lambda_\Gamma : [a,b] \to \mathfrak{so}_{n+1}$ of skew-symmetric matrices. Let $\mathfrak{J} \subset \mathfrak{so}_{n+1}$ be the n-dimensional convex cone of the so-called *skew-Jacobian* matrices, *i.e.*, tridiagonal skew-symmetric matrices with positive subdiagonal entries:

$$\mathfrak{J} = \left\{ \begin{pmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & 0 & -c_n \\ & & & c_n & 0 \end{pmatrix} \in \mathfrak{so}_{n+1} \right\}_{c_1,\cdots,c_n>0}$$

Notice that $\mathfrak{J}$ generates the whole Lie algebra $\mathfrak{so}_{n+1}$ under the Lie-bracket. We have the following characterization for holonomic paths in SO_{n+1} (14).

Proposition 2.1 *Let* $\Gamma : [0,1] \to \mathrm{SO}_{n+1}$ *be a smooth path with* $\Gamma(0) = I$. *Then,* Γ *is holonomic if and only if* Λ_Γ *is skew-Jacobian,* i.e.*, if and only if* $\Lambda_\Gamma(t) \in \mathfrak{J}$ *for all* $t \in [0,1]$.

Proof. ($\to$) Given $\gamma \in \mathcal{L}\mathbb{S}^n$, differentiation of its Wronski matrix with respect to time yields

$$W_\gamma'(t) = W_\gamma(t)\,H(t)$$

with

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & * \\ 1 & 0 & 0 & \cdots & * \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & * \\ 0 & 0 & \cdots & 1 & * \end{pmatrix},$$

where the asterisks stand for some functions that shall be irrelevant for the argument.

There is a one parameter smooth family of upper matrices $R : [0,1] \to \mathrm{Up}_{n+1}^{+}$ such that

$$W_\gamma(t) = \mathfrak{F}_\gamma(t) R(t),$$

and therefore

$$\mathfrak{F}_\gamma'(t) R(t) + \mathfrak{F}_\gamma(t) R'(t) = \mathfrak{F}_\gamma(t) R(t) H(t).$$

Multiplying by $\mathfrak{F}_\gamma(t)^{-1}$ by the left and by $R(t)^{-1}$ by the right yields

$$\Lambda_{\mathfrak{F}_\gamma}(t) = R(t) H(t) R(t)^{-1} - R'(t) R(t)^{-1}.$$

Since $R'(t) R(t)^{-1} \in \mathfrak{up}_{n+1}^{+}$ is upper triangular and $R(t) H(t) R(t)^{-1}$ is of the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ + & * & * & \cdots & * \\ 0 & + & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & * & * \\ 0 & 0 & \cdots & + & * \end{pmatrix},$$

it follows from the fact that $\Lambda_{\mathfrak{F}_\gamma}(t) \in \mathfrak{so}_{n+1}$ is skew-symmetric that $\Lambda_{\mathfrak{F}_\gamma}(t) \in \mathfrak{J}$.

($\leftarrow$) Suppose that $\Gamma : [0,1] \to \mathrm{SO}_{n+1}$ is such that $\Gamma(0) = I$ and $\Lambda_\Gamma(t) \in \mathfrak{J}$ for all $t \in [0,1]$ and define $\gamma : [0,1] \to \mathbb{S}^n$ by $\gamma = \pi[\Gamma]$. Differentiation of γ with respect to time yields

$$\begin{aligned} \gamma'(t) &= \Gamma'(t) \cdot e_1 \\ &= \Gamma(t) \Lambda_\Gamma(t) \cdot e_1 \\ &= c_1(t) \Gamma(t) \cdot e_2 \end{aligned}$$

where $c_1 : [0,1] \to (0,+\infty)$ is a positive function. In general, for each $j \in [n] = \{1, 2, \cdots, n\} \subset \mathbb{Z}$, one has

$$\gamma^{(j)}(t) = c_j(t) \Gamma(t) \cdot e_{j+1} + v(t),$$

where $c_j : [0,1] \to (0,+\infty)$ is a positive function and $v(t) \in \operatorname{span}(e_1, \cdots, e_j)$. Applying the Gram-Schmidt procedure to the frame $\left(\gamma(t), \gamma'(t), \cdots, \gamma^{(n)}(t)\right)$ yields therefore

$$(\Gamma(t) \cdot e_1, \cdots, \Gamma(t) \cdot e_n)$$

and one has

$$\mathbf{Q}(W_\gamma(t)) = \mathfrak{F}_\gamma(t) = \Gamma(t),$$

where $\mathbf{Q}(X)$ stands for the orthogonal part in the QR decomposition of an invertible matrix X. ■

Therefore, given a smooth path $\Lambda : [0,1] \to \mathfrak{J} \subset \mathfrak{so}_{n+1}$ in the convex cone $\mathfrak{J}$, the initial value problem

$$\begin{cases} \Gamma'(t) = \Gamma(t)\Lambda(t) \\ \Gamma(0) = I \end{cases} \tag{2-1}$$

has a unique holonomic solution $\Gamma : [0,1] \to \mathrm{SO}_{n+1}$ which descends naturally to a nondegenerate curve $\gamma = \pi[\Gamma] \in \mathcal{L}\mathbb{S}^n$. On the other hand, given a smooth $\gamma \in \mathcal{L}\mathbb{S}^n$, its lifting $\Gamma = \mathfrak{F}_\gamma$ is the unique solution of the initial value problem above for $\Lambda = \Lambda_{\mathfrak{F}_\gamma}$. All in all, there is a homeomorphism $\gamma \mapsto \Lambda_{\mathfrak{F}_\gamma}$ between $\mathcal{L}\mathbb{S}^n$ and the convex set $C^\infty([0,1], \mathfrak{J})$. But we do not really need Λ to be smooth for equation (2-1) to have a unique solution Γ neither do we really want its projection γ to be of class C^∞: every now and then an argument will be put forward which involves building up a γ by juxtaposition of frame paths Γ, what could result in disagreeing high order derivatives of γ in the joint points, making thus the space of smooth curves $\mathcal{L}\mathbb{S}^n$ too narrow for our needs. If we allow Λ to dwell in the set $L^2([0,1], \mathfrak{J}) \subset L^2([0,1], \mathfrak{so}_{n+1})$, then equation (2-1) still has a unique solution Γ, but this time with projection $\pi[\Gamma]$ in the Sobolev space $H^1([0,1], \mathbb{R}^{n+1})$, where the very notion of nondegeneracy does not make sense at first sight but for a few elements (those of class C^n). It is not really too serious a hindrance, since we can relax the definition of nondegeneracy as in (14). Therein the map $\Lambda \in L^2([0,1], \mathfrak{J}) \mapsto \gamma \in H^1([0,1], \mathbb{R}^{n+1})$ is used to define a Hilbert-space topology on its image $\mathcal{H}\mathbb{S}^n$ via the idenfication $\gamma \sim \Lambda_{\mathfrak{F}_\gamma}$. The subspace topology of $\mathcal{L}\mathbb{S}^n$ viewed as sitting inside the larger space $\mathcal{H}\mathbb{S}^n$ disagrees with its original C^∞-structure, but those topologies turn out being homotopy equivalent. More details in (14), (10) (where the details are more throughly exposed) and (4), the later two for dimension $n = 2$ only. The corresponding discussions in each of the forementioned papers rely strongly on some general functional analysis results to be found in (6).

Inelegant as such kind of move is, we shall swiftly switch the definition of $\mathcal{L}\mathbb{S}^n$ and from now on refer to the larger space $\mathcal{H}\mathbb{S}^n$ endowed with its Hilbert structure as the actual $\mathcal{L}\mathbb{S}^n$. Accordingly, the fixed monodromy subspaces $\mathcal{L}\mathbb{S}^n(Q)$ acquire a natural Hilbert submanifold structure. As mentioned above, the main advantage of the new definition as compared to the former one is that it allows for discontinuities in the logarithmic derivatives and gets rid of tedious smoothening arguments.

One of the byproducts of the present thesis is to grant us the license to bypass unadvertently all of these topological preoccupations and related analytical arguments, since our main theorem will put forward a decomposition of $\mathcal{L}\mathbb{S}^n(Q)$ that combinatorializes its topology. We also expect that this cell decomposition will turn the wide open problem of finding the homotopy type of the spaces $\mathcal{L}\mathbb{S}^n$ for $n > 2$ into a combinatorial one.

2.2 A useful cell decomposition of the orthogonal group

In what follows we shall denote the symmetric $(n+1)-$group by S_{n+1}. It is the group of the permutations of $n+1$ objects, i.e., the group (under composition of functions) of the bijections of the subset $[n+1] = \{1, 2, \cdots, n+1\}$ of the integers, or, alternatively, of the subset $[n]_0 = \{0, 1, \cdots, n\}$. It admits of a faithful linear representation

$$\begin{array}{rccc} P: & S_{n+1} & \longrightarrow & \mathrm{GL}_{n+1} \\ & \pi & \mapsto & P_\pi \end{array}$$

as a group $\mathbf{S}_{n+1}$ of matrices P_π given by the rule

$$P_\pi \cdot e_i = e_{\pi(i)},$$

where $(e_1, \cdots, e_{n+1})$ is the canonical basis of $\mathbb{R}^{n+1}$, thought of as the space $\mathbb{R}^{(n+1)\times 1}$ of column matrices. Notice the $\mathbf{S}_{n+1}$ is a subgroup of the orthogonal group O_{n+1}. The group of invertible upper triangular matrices in GL_{n+1} shall be denoted by Up_{n+1}. Its subgroups of positive and unit diagonal matrices will be denoted by Up^+_{n+1} and Up^1_{n+1}, respectively. Similar notation will be used for subgroups of lower triangular matrices.

The following is one of the simplest instances of the now classical Bruhat decomposition theorem for algebraic groups. Of course a similar result is true for lower instead of upper matrices.

Theorem 2.2 *Given $X \in \mathrm{GL}_{n+1}$, there exist an unique permutation matrix $P \in \mathbf{S}_{n+1}$ and (non-unique) upper triangular matrices $U_1, U_2 \in \mathrm{Up}_{n+1}$ such that*

$$X = U_1 P U_2$$

In particular,

$$\mathrm{GL}_{n+1} = \bigsqcup_{\pi \in S_{n+1}} \mathcal{B}_\pi,$$

where

$$\mathcal{B}_\pi = \left\{ U_1 P_\pi U_2 \mid U_1, U_2 \in \mathrm{Up}_{n+1} \right\}$$

As the reader may know, this is nothing but Gauss-Jordan elimination in disguise. Indeed, we may consider $U_1 \in \mathrm{Up}^1_{n+1}$ in the equation $X = U_1 P U_2$ provided we absorbe its diagonal entries in U_2. Then, U_1 is a row echelon form of X that took a permutation P^{-1} of its columns to be performed, and U_2 is the coefficient matrix of the Gauss-Jordan algorithm. The non-uniqueness of the pair (U_1, U_2) for a given X is thus seen to be just a periphrasis of the familiar non-uniqueness of the row echelon form.

In an even more operational sense, what the theorem above is saying is that a given invertible matrix X may be transformed into a unique permutation matrix P by some row and column operations satisfying the following conditions:

(i) adding a multiple of row i_1 to row i_2 is allowed if and only if $i_1 > i_2$

(ii) adding a multiple of column j_1 to column j_2 is allowed if and only if $j_1 < j_2$

Condition *(i)* and *(ii)* correspond to left and right multiplication by upper matrices, respectively. This operational characterization provides us with an algorithm for deciding what $\mathcal{B}_\pi$ a given invertible matrix X belongs to.

Given $X \in \mathrm{GL}_{n+1}$, in order to determine the unique $\pi \in S_{n+1}$ such that $X \in \mathcal{B}_\pi$, one takes the lowest non-zero entry in the first column of X and divides the whole column by this number in order to produce an 1 there. Then one uses that 1 to kill off the remaining entries of both the first column and the row it belongs to. Then one proceeds likewise with the second column and so on, till the very last one, eventually arriving at the desired permutation matrix P_π. Since row and column operations satisfying conditions *(i)* and *(ii)* above preserve the ranks of southwest blocks, which integers uniquely identify P_π among all other permutation matrices, one sees that the algorithm just described defines an onto map $X \in \mathrm{GL}_{n+1} \mapsto \pi_X \in S_{n+1}$ such that $X \in \mathcal{B}_{\pi_X}$.

Let $\rho = (1, n+1)\,(2, n) \cdots \in S_{n+1}$, that is

$$R = P_\rho = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \in \mathbf{S}_{n+1}.$$

It is easy to convince oneself that the set $\mathcal{B}_\rho$ is open and dense, since the matrices that do admit of LU decomposition constitute a dense subset of GL_{n+1}, as is readily shown. Explicitly, given GL_{n+1}, that's how one decides if $X \in \mathcal{B}_\rho$:

$$\begin{aligned} X \ \in\ \mathcal{B}_\rho &\leftrightarrow \exists U_1 \in \mathrm{Up}^1_{n+1}\ \exists U_2 \in \mathrm{Up}_{n+1}\ \left(X = U_1 R U_2\right) \\ &\leftrightarrow \exists U_1 \in \mathrm{Up}^1_{n+1}\ \exists U_2 \in \mathrm{Up}_{n+1}\ RX = (RU_1R)U_2 \\ &\leftrightarrow \exists L_1 \in \mathrm{Lo}^1_{n+1}\ \exists U_2 \in \mathrm{Up}_{n+1}\ \left(RX = L_1 U_2\right), \end{aligned}$$

for the conjugated matrix RU_1R is invertible lower triangular with unit diagonal. That last condition is met by every matrix RX whose northwest minor determinants are all non-vanishing. Therefore, $X \in \mathcal{B}_\rho$ if and only if its southwest minor determinants are all non-vanishing, and that is an open dense condition over GL_{n+1}. It is therefore evident that $\mathcal{B}_\rho$ have 2^{n+1} connected components, given by the distinct sequences of signs of southwest minor determinants, each one of those is diffeomorphic to $\mathbb{R}^{(n+1)^2}$. This example suggests a finer decomposition of GL_{n+1} into the connected components of the Bruhat cells $\mathcal{B}_\pi$ defined as follows.

Consider the Coxeter-Weyl group of signed permutation matrices B_{n+1} and the "forgetful map"

$$\begin{array}{rccc} \Pi: & \mathrm{B}_{n+1} & \twoheadrightarrow & \mathbf{S}_{n+1} \\ & P & \mapsto & \Pi\left[P\right] \end{array},$$

which is a homomorphism of B_{n+1} onto its subgroup $\mathbf{S}_{n+1}$ defined by dropping signs, having kernel $\mathrm{Diag}_{n+1} = \Pi^{-1}\left[I\right]$ isomorphic to the additive group $\mathbb{Z}_2^{n+1}$. Each fibre $\Pi^{-1}\left[P_\pi\right]$ is then given by the left coset $P_\pi \mathrm{Diag}_{n+1}$ and has 2^{n+1} elements and therefore $|\mathrm{B}_{n+1}| = (n+1)!2^{n+1}$.

Absorbing the diagonal entries of U_1 into U_2 and the signs of the resulting U_2 into P_π in the Bruhat decomposition $X = U_1 P_\pi U_2$ yields a slightly different decomposition

$$\begin{aligned} X &= U_1 P_\pi U_2 \\ &= U_1 P_\pi (D\tilde{U}_2) \\ &= U_1 \left(P_\pi D\right) \tilde{U}_2 \\ &= U_1 P \tilde{U}_2 \quad , \end{aligned}$$

where $U_1 \in \mathrm{Up}^1_{n+1}$, $D \in \mathrm{Diag}_{n+1}$, $\widetilde{U}_2 = DU_2 \in \mathrm{Up}^+_{n+1}$ has now positive diagonal, and $P = P_\pi D \in \mathrm{B}_{n+1}$ is now in the Coxeter-Weyl group. It is clear that P is uniquely determined by X (although U_1 and $\widetilde{U}_2$ are non-unique just as before). Thus, we can state the following

Corollary 2.3 *Given $X \in \mathrm{GL}_{n+1}$, there exist an unique signed permutation matrix $P \in \mathrm{B}_{n+1}$ and (non-unique) upper triangular matrices $U_1 \in \mathrm{Up}^1_{n+1}$ and $U_2 \in \mathrm{Up}^+_{n+1}$ such that*

$$X = U_1 P U_2$$

In particular,

$$\mathrm{GL}_{n+1} = \bigsqcup_{P \in \mathrm{B}_{n+1}} \mathrm{Up}^1_{n+1} P \ \mathrm{Up}^+_{n+1}$$

It can be shown that the cells $\mathcal{C}_P = \mathrm{Up}^1_{n+1} P \ \mathrm{Up}^+_{n+1}$ are now contractible and that the decomposition given by the corollary above yields a CW complex structure for GL_{n+1}. This fact is not to be used in what follows (for the contractibility of the analogous cells $\mathrm{Bru}_P \subset \mathrm{O}_{n+1}$ of the orthogonal group, cf. the upper-tringular system of coordinates defined on page 26). It is worth noticing that

$$\mathcal{B}_\pi = \bigsqcup_{\Pi[P] = P_\pi} \mathcal{C}_P$$

Since signed permutation matrices are orthogonal, we see that each cell $\mathcal{C}_P$ has non-trivial intersection with O_{n+1}. Abusing the classical terminology, we shall call the intersection $\mathcal{C}_P \cap \mathrm{O}_{n+1}$ the *Bruhat cell* of $P \in \mathrm{B}_{n+1}$ and denote it by Bru_P. At first sight, one could suspect connectedness was lost, let alone contractibility, when passing from $\mathcal{C}_P$ to Bru_P, but fortunately this is not the case. In fact, given an invertible matrix X, one can perform the Gram-Schmidt algorithm on its column-vectors producing its unique QR decomposition that reads as follows:

$$\forall X \in \mathrm{GL}_{n+1} \exists! Q \in \mathrm{O}_{n+1} \exists! R \in \mathrm{Up}^+_{n+1} \ (X = QR)$$

We shall denote by $\mathbf{Q}(X) = Q$ the unique orthogonal part of X and by $\mathbf{R}(X) = R$ its unique positive upper triangular part. Now, one sees that in the

Bruhat decomposition

$$Q = U_1 P U_2$$

of $Q \in \mathrm{O}_{n+1}$ the upper $U_2 \in \mathrm{Up}_{n+1}^+$ is uniquely determined by $U_1 \in \mathrm{Up}_{n+1}^1$ and Q itself for QU_2^{-1} is the unique QR decomposition of $U_1 P \in \mathrm{GL}_{n+1}$. It suggests the definition of an Up_{n+1}^1-action on O_{n+1} given by

$$\begin{array}{rccc} \mathcal{B}: & \mathrm{Up}_{n+1}^1 \times \mathrm{O}_{n+1} & \to & \mathrm{O}_{n+1} \\ & (U, Q) & \mapsto & \mathbf{Q}\,(UQ) \end{array},$$

the orbits of which are exactly the Bruhat cells Bru_P. It will follow immediately from the system of coordinates we are about to introduce in Bru_P that these Up_{n+1}^1-orbits are contractible submanifolds of O_{n+1}. Henceforth the *Bruhat decomposition* (classical terminology is being abused again)

$$\mathrm{O}_{n+1} = \bigsqcup_{P \in \mathrm{B}_{n+1}} \mathrm{Bru}_P$$

endows the orthogonal group with a nice cell structure. It turns out that this is indeed a CW complex structure, but since this fact is not to play any role in this work, we shall omit the details, which are to be found in a subsequent paper (joint work with Nicolau C. Saldanha).

The Up_{n+1}^1-action on O_{n+1} defined above, to which we shall refer from now on as the *Bruhat action*, induces a very handy system of coordinates in each Bruhat cell Bru_P. To wit, fixed a signed permutation matrix $P \in \mathrm{B}_{n+1}$, its right multiplication by an upper matrix $U \in \mathrm{Up}_{n+1}^1$ performs a $\Pi\,[P]$-permutation of U's columns along with appropriate changes of sign. The result is a matrix $UP \in \mathrm{GL}_{n+1}$ that has ± 1 entries in the exact same positions as P but possibly some non-zero entries in each slot above each of the forementioned ± 1. Henceforth, if we allow U to range over $\mathrm{Up}_{n+1}^1 \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$ we get $\frac{n(n+1)}{2}$ free variables to define $UP \in \mathrm{GL}_{n+1}$. Now, taking the Q-part of UP is tantamount to left multiplying it by the well-defined $R = \mathbf{R}(UP) \in \mathrm{Up}_{n+1}^+$. Thus,

$$\mathcal{B}\,(U, P) = UPR$$

is defined by the forementioned $\frac{n(n+1)}{2}$ variables only up to some ambiguity, *i.e.*, these variables, if taken as coordinates of $\mathcal{U}\,(U, P)$ in Bru_P are not independent. To select a subset of independent coordinates that is a chart for Bru_P, it is worth resuming to the cell $\mathcal{C}_P$ in GL_{n+1}, where one can allow V to range freely over Up_{n+1}^+ so as to get $UPV \in \mathcal{C}_P$. Since the right multiplication by V allows for column operations to the right, one can use the ± 1 entries in UP to kill off all the entries to the right of them, preserving only a minimal subset of independent U-coordinates in the final expression of UPV for a suitable chosen V. Now, it is easy to see that

$$\mathbf{Q}\,(UPV) = \mathbf{Q}\,(UP)$$

Henceforth, the surviving coordinates can be taken as independent coordinates generating the whole Bru_P, and since they come from amongst the canonical coordinates of the contractible Lie group $\mathrm{Up}_{n+1}^{+} \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$, each of them ranging over the whole real line, it is clear that the cells Bru_P are contractible submanifolds of O_{n+1}.

The simplest system of coordinates is that of sigleton cells $\mathrm{Bru}_D = \{D\}$ with diagonal $D \in \mathrm{Diag}_{n+1}$: the killing off procedure is frantic, so as to "make no prisoners". This is a 0-dimensional submanifold.

At the other hand of the spectrum one has the systems of coordinates of open cells Bru_{DR}, where $D \in \mathrm{Diag}_{n+1}$ and $R = P_\rho$ is defined on page 23 (not to be confused with the upper triangular part of a QR decomposition), to wit:

$$\mathrm{Bru}_{DR} = \left\{ \mathbf{Q} \begin{pmatrix} \pm u_{1(n+1)} & \pm u_{1n} & \cdots & \pm u_{12} & \pm 1 \\ \vdots & \vdots & & \pm 1 & \\ \vdots & \pm u_{(n-2)n} & \iddots & & \\ \pm u_{n(n+1)} & \pm 1 & & & \\ \pm 1 & & & & \end{pmatrix} \right\}_{u_{ij} \in \mathbb{R}} \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$$

Notice that Bru_{DR} has top dimension $\frac{n(n+1)}{2}$ in the $\frac{n(n+1)}{2}$-dimensional Lie group O_{n+1}. This happens because the number of inversions in the permutation $\rho = (0, n)\,(1, n-1) \cdots$ is the maximum $\frac{n(n+1)}{2}$ and there is no killing off of redundant coordinates. If we undid the inversion in the last two columns of R then the coordinate above the last column's 1 would get killed off by the nearby 1 in the n-th column, resulting in only $\left(\frac{n(n+1)}{2} - 1\right)$ free coordinates:

$$\begin{pmatrix} u_{1(n+1)} & u_{1n} & \cdots & u_{13} & 1 & 0 \\ \vdots & \vdots & & \vdots & 0 & 1 \\ \vdots & \vdots & & 1 & & \\ \vdots & u_{(n-2)n} & \iddots & & & \\ u_{n(n+1)} & 1 & & & & \\ 1 & & & & & \end{pmatrix}$$

It is no coincidence that $\dim \mathrm{Bru}_P = \mathrm{inv}(P)$, where $\mathrm{inv}(P)$ is defined as the number of inversions in the permutation π such that $\Pi\,[P] = P_\pi$.

Indeed, one can produce any $P \in \mathrm{B}_{n+1}$ from an appropriate *full inverted* signed permutation matrix DR by means of a finite sequence of transpositions of contiguous columns, each of which drops the number of free variables in Bru_P by one unit. Notice that one goes all the way down from a full inverted to a diagonal matrix in exactly $\frac{n(n+1)}{2}$ steps, although there are plenty of paths of contiguous transpositions one can follow. This suggests the following poset structure on B_{n+1} (closely related to the now classical Bruhat order in S_n) that turns out to have a nice topological counterpart. In a nutshell, the partial order we are about to define yields the attaching maps of the Bruhat CW structure of O_{n+1}. But, as we pointed out earlier in this section, this fact shall be of no consequence to the remaining of the text.

Let $P, Q \in \mathrm{B}_{n+1}$ be such that $\Pi[P] = P_\pi$ and $\Pi[Q] = P_\sigma$ for $\pi, \sigma \in S_{n+1}$. We write

$$P \lessdot Q$$

if there are $i < j$ in $[n+1]$ such that

(i) $\forall k \in [n+1] \;\; \forall l \in [n+1] \setminus \{i, j\} \;\; (P_{kl} = Q_{kl})$
(ii) $\pi(i) = \sigma(j) < \pi(j) = \sigma(i)$
(iii) $\nexists k \in (i, j) \;\; (\pi(k) \in (\pi(i), \pi(j)))$
(iv) $\det P = \det Q$

(The symbol (i, j) above stands for the open interval of integers with endpoints $i < j$: $(i, j) = \{k \in \mathbb{N} \mid i < k < j\}$)

The conditions *(i)* to *(iv)* mean that P and Q differ only by one inversion and one sign at the opposite corners of a block full of zeroes, as in the following ilustration.

$$\begin{pmatrix} & & & -1 & & \\ & -1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & 1 & \\ & & 1 & & & \end{pmatrix} \lessdot \begin{pmatrix} & & & -1 & & \\ & 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & -1 \\ & 1 & 0 & 0 & 0 & \\ & & 1 & & & \end{pmatrix}$$

It is straightfoward to prove that

$$P \lessdot Q \longrightarrow \mathrm{inv}(Q) = \mathrm{inv}(P) + 1$$

We shall omit the argument.

Now we define the partial order $<$ in B_{n+1} as the transitive closure of the relation $\lessdot$. The following is a very interesting result that is akin to a classical Bruhat decomposition theorem.

Proposition 2.4 $\forall\ P, Q \in\ \mathrm{B}_{n+1}\ (P \leq Q\ \longrightarrow\ \overline{\mathrm{Bru}_P} \subseteq\ \overline{\mathrm{Bru}_Q})$

The reader acquainted with the classical Bruhat decomposition of GL_{n+1} and the related Bruhat order in S_n may wonder about the validity of the converse implication. Indeed the converse is true, but we shall omit its proof since it is much more involved and is not to be needed in anything that follows. The thorough statement of the theorem, including its full proof, is to be found in a subsequent paper (joint work with Nicolau C. Saldanha).

Proof. Since $P < Q$ means that we have a path of coverings $P = P_0 \lessdot P_1 \lessdot \cdots \lessdot P_k = Q$, it clearly suffices to prove that $P \lessdot Q\ \longrightarrow\ \overline{\mathrm{Bru}_P} \subset\ \overline{\mathrm{Bru}_Q}$. Referring to conditions (i) to (iv) above, we see that there is a free coordinate u in the $\pi\,(j)$-th row and the i-th column of $\mathbf{Q}(UQ)$. Then, one of the two limits

$$L_{\pm} = \lim_{u \to \pm\infty} \mathbf{Q}\,(UQ)$$

is an element of Bru_P and it is easy to see that every element of Bru_P is obtainable as such a limit. ■

At last we turn our attention to the orientation preserving orthogonal matrices, which comprises the connected component $\mathrm{SO}_{n+1} \subset \mathrm{O}_{n+1}$. Consider the group of orientation preserving signed permutation matrices

$$\mathrm{B}_{n+1}^{+} =\ \mathrm{B}_{n+1} \cap\ \mathrm{SO}_{n+1}$$

It is a subgroup of B_{n+1} of order $\left|\mathrm{B}_{n+1}^{+}\right| = (n+1)!2^n$, since it is the kernel of the epimorphism $\det :\ \mathrm{B}_{n+1} \to \{\pm 1\}$. Since for each $P \in\ \mathrm{B}_{n+1}$ we have

$$\begin{aligned} \mathrm{Bru}_P &= \mathcal{B}\left(\mathrm{Up}_{n+1}^{1}, P\right) \\ &= \mathbf{Q}\left(\mathrm{Up}_{n+1}^{1} P\right), \end{aligned}$$

it is clear that

$$\mathrm{Bru}_P \cap \mathrm{SO}_{n+1} = \begin{cases} \mathrm{Bru}_P, \text{ if } P \in \mathrm{B}_{n+1}^{+} \\ \emptyset, \text{ if } P \notin \mathrm{B}_{n+1}^{+} \end{cases} .$$

Therefore we have a stratification of SO_{n+1} given by the following *Bruhat cell decomposition* (herein it is intended to be a definition, not just an abuse of terminology)

$$\mathrm{SO}_{n+1} = \bigsqcup_{P \in\ \mathrm{B}_{n+1}^{+}} \mathrm{Bru}_P$$

into contractible cells of dimension

$$\dim \ \mathrm{Bru}_P = \mathrm{inv}\,(P)\,.$$

Again, this is indeed a CW complex structure, a fact that will not be used in anything that follows and whose details shall accordingly be omitted. Let us just point out that using the equivalence between the combinatorial order $P < Q$ in B_{n+1}^{+} and the topological order $\overline{\mathrm{Bru}_P} \subset \overline{\mathrm{Bru}_Q}$, we can chase boundary cells through the Hasse diagram of the poset B_{n+1}^{+} and use this combinatorial information to assemble them together into the closure of each (top-dimensional) open cell. For instance, following the Hasse diagram of figure 2.3 all the way down from a top-dimensional cell to the bottom singletons, through every available path of coverings, one can figure out the singular structure of the closures of open cells in SO_3 (cf. figure 2.4).

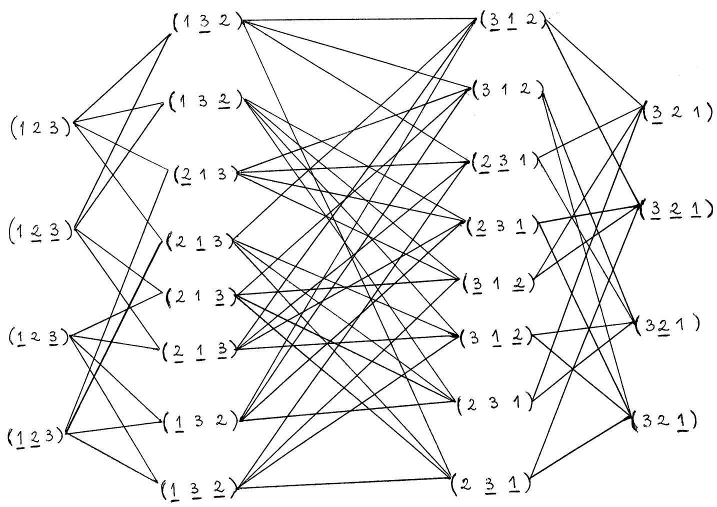

Figure 2.3: Hasse diagram for the Bruhat partial ordering of B_3^{+}

Sometimes we shall denote by $Bru\,(Q)$ the only signed permutation matrix Bruhat equivalent to a given orthogonal matrix Q.

2.3 The "chopping" and "advancing" operations

In (14) an operation on the group B_{n+1}^{+} was introduced that allows for a deep geometrical insight about the behavior of holonomic paths in SO_{n+1}. It corresponds naïvely to chopping a small tip at the end of a nondegenerate curve in order to see which open Bruhat cell its path of frames is coming

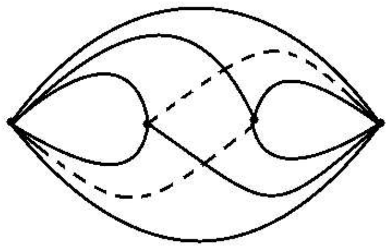

Figure 2.4: boundary of a 3-D cell in SO_3: four 0-D, eight 1-D and four 2-D cells

from. Fairly enough, this operation was baptized the "chopping" operation. In this section, we are going to recall the main definitions and a very useful practical way of computing the chop of orthogonal matrices which relies on the forementioned naïve approach. Besides, we will define the opposite operation, which we call "advancing" and corresponds roughly to trimming a small piece at the beginning of the curve, in order to tell where its path of frames is going to.

Given, $P \in \mathrm{B}_{n+1}^{+}$ and $i, j \in [n+1]$ such that $i = \pi(j)$, where $\Pi[P] = P_\pi \in \mathbf{S}_{n+1}$, we define the integer $\mathbf{NE}(P, i, j)$ to be the number of nonzero entries lying to the northeast of P_{ij}:

$$\mathbf{NE}(P, i, j) = \operatorname{card}\left\{(\mu, \nu) \in [n+1]^2 \mid \nu > j,\, \mu = \pi(v) < i\right\}$$

Now we define a map $\Delta : \mathrm{B}_{n+1}^{+} \to \mathrm{Diag}_{n+1}^{+}$ by

$$\Delta(P) = \begin{pmatrix} \delta_1(P) & & & \\ & \delta_2(P) & & \\ & & \ddots & \\ & & & \delta_{n+1}(P) \end{pmatrix},$$

where

$$\delta_i(P) = (-1)^{\mathbf{NE}\left(P, i, \pi^{-1}(i)\right)} P_{i\pi^{-1}(i)}$$

for each $i \in [n+1]$ and extend it to a map $\Delta : \mathrm{SO}_{n+1} \to \mathrm{Diag}_{n+1}^{+}$ by

$$\Delta(Q) = \Delta(Bru(Q)).$$

A matrix that will play a major role in the remaining of this work is the

Arnol'd matrix:

$$A = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \iddots & & & \\ (-1)^n & & & & \end{pmatrix}$$

It owes its name to an operation on nondegenerate spherical curves – the Arnol'd duality – that is related to the projective duality of curves in the projective plane. We use this matrix to define the chopping operation

$$\begin{array}{rccl} \mathrm{chop}: & \mathrm{SO}_{n+1} & \to & \mathrm{B}^+_{n+1} \\ & Q & \mapsto & \mathrm{chop}\,(Q) = \Delta\,(Q)\,A \end{array}$$

Notice that, since $\mathrm{chop}(Q)$ is always a full inverted matrix, $\mathrm{Bru}_{\mathrm{chop}(Q)}$ is always an open cell, moreover dense in the fibre $\mathrm{chop}^{-1}\,(\mathrm{chop}\,(Q))$.

Now for the geometric content of the chopping operation. In (14), the reader is to find the following important result.

Proposition 2.5 *For any $Q \in SO_{n+1}$ and for any $\gamma \in \mathcal{L}\mathbb{S}^n\,(Q)$ there exists $\epsilon > 0$ such that for all $t \in (1-\epsilon, 1)$ we have that $\mathfrak{F}_\gamma\,(t) \in \mathrm{Bru}_{\mathrm{chop}(Q)}$.*

For instance, suppose $\gamma : [0,1] \to \mathbb{S}^3$ is nondegenerate such that $\mathfrak{F}_\gamma\,(t_0) = Q$, where

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Expanding γ in Taylor series around t_0, we have

$$\gamma\,(t_0 + \Delta t) \approx \left(\Delta t, \frac{\Delta t^3}{6}, 1, -\frac{\Delta t^2}{2}\right),$$

hence

$$W_\gamma\,(t_0 + \Delta t) \approx \begin{pmatrix} \Delta t & 1 & 0 & 0 \\ \frac{\Delta t^3}{6} & \frac{\Delta t^2}{2} & \Delta t & 1 \\ 1 & 0 & 0 & 0 \\ -\frac{\Delta t^2}{2} & -\Delta t & -1 & 0 \end{pmatrix}.$$

For a small $\Delta t < 0$, we have the following sequence of signs for the southwest minor determinants of $W_\gamma\,(t_0 + \Delta t)$, which are the same for $\mathfrak{F}_\gamma\,(t_0 + \Delta t)$:

$$-\frac{\Delta t^2}{2} < 0, \quad \begin{Vmatrix} 1 & 0 \\ -\frac{\Delta t^2}{2} & -\Delta t \end{Vmatrix} > 0, \quad \begin{Vmatrix} \frac{\Delta t^3}{6} & \frac{\Delta t^2}{2} & \Delta t \\ 1 & 0 & 0 \\ -\frac{\Delta t^2}{2} & -\Delta t & -1 \end{Vmatrix} < 0,$$

$$\text{and } \det W_\gamma(t_0+\Delta t) = \det Q > 0$$

Since these are the exact sequence of southwest minor determinants of the Arnol'd matrix, we have $\operatorname{chop}(Q) = A$ and there exists $\epsilon > 0$ such that $\mathfrak{F}_\gamma(t) \in \mathrm{Bru}_A$ for all $t \in (t_0-\epsilon, t_0)$.

Analogously, for a small $\Delta t > 0$, the sequence of signs above switches to

$$-\frac{\Delta t^2}{2} < 0, \quad \begin{Vmatrix} 1 & 0 \\ -\frac{\Delta t^2}{2} & -\Delta t \end{Vmatrix} < 0, \quad \begin{Vmatrix} \frac{\Delta t^3}{6} & \frac{\Delta t^2}{2} & \Delta t \\ 1 & 0 & 0 \\ -\frac{\Delta t^2}{2} & -\Delta t & -1 \end{Vmatrix} < 0,$$
$$\text{and } \det W_\gamma(t_0+\Delta t) = \det Q > 0,$$

which is the same as the corresponding sequence for the full inverted matrix

$$DA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

where $D = \operatorname{diag}(1,-1,-1,1) \in \mathrm{Diag}^+_{n+1}$ Accordingly, we would like to define the "advancing" operation

$$\begin{array}{rccc} \operatorname{adv}: & \mathrm{SO}_{n+1} & \to & \mathrm{B}^+_{n+1} \\ & Q & \mapsto & \operatorname{adv}(Q) \end{array}$$

in such a way that

$$\operatorname{adv}(Q) = DA.$$

Even if we will not have occasion in this work to use combinatorial formulae for chop and adv, preferring always the geometric method for figuring out $\operatorname{chop}(Q)$ and $\operatorname{adv}(Q)$, let us point out that the "advancing" operation can be defined by the extension to SO_{n+1} of the rule

$$\operatorname{adv}(P) = \bar{\Delta}(P)\, A^T,$$

where $\bar{\Delta}(P) = \operatorname{diag}\Big(\bar{\delta}_1(P), \cdots, \bar{\delta}_n(P)\Big)$ is given by

$$\bar{\delta}_i(P) = (-1)^{\mathbf{SW}\left(P,i,\pi^{-1}(i)\right)} P_{i\pi^{-1}(i)},$$

and $\mathbf{SW}(P,i,j)$ is the number of nonzero entries of P to the southwest of P_{ij}.

The discussion above suggests a result for the advancing operation analogous to proposition 2.5 above, to wit:

Proposition 2.6 *If* $\gamma : [0,1] \to \mathbb{S}^n$ *is nondegenerate and* $t_0 \in [0,1)$ *is such that* $\operatorname{adv}(\mathfrak{F}_\gamma(t_0)) = DA$*, then there exists* $\epsilon > 0$ *such that* $\mathfrak{F}_\gamma(t) \in \mathrm{Bru}_{DA}$ *for all* $t \in (t_0, t_0 + \epsilon)$*.*

It is clear that if $\operatorname{chop}(P) = DA$ or $\operatorname{adv}(P) = DA$ then, $\mathrm{Bru}_P \subset \partial \mathrm{Bru}_{DA}$.

Given a full inverted matrix $DA \in \mathrm{B}^+_{n+1}$ and its open Bruhat cell Bru_{DA}, the cells Bru_P such that $\operatorname{adv}(P) = DA$ are called *entering cells* of Bru_{DA}. Analogously, the cells Bru_P such that $\operatorname{chop}(P) = DA$ are called *exit cells* of Bru_{DA}. The path of frames of a nondegenerate curve can get into an open cell only through one of its entering cells, and can get out of it only through one of its exit cells. The remaining cells in the boundary of Bru_{DA} are called its *tangency cells*, but we won't have occasion to deal with them in this work.

We close this section gathering some chops and advances of notorious signed permutation matrices.

It is worth noticing that our definitions are such that the advance and chop of any full inverted matrix DA is DA itself.

Besides, $\operatorname{chop}(I) = A$ and $\operatorname{adv}(I) = A^T$. Therefore, each newborn $\gamma \in \mathcal{L}\mathbb{S}^n$ immediately enters the open cell Bru_{A^T} and can be viewed as coming from Bru_A. Also, let $\Omega = (-1)^n I$. Then, $\operatorname{chop}(\Omega) = A^T$ and $\operatorname{adv}(\Omega) = A$. When n is even, one has $A = A^T$ and $\Omega = I$.

3
Convex curves

An important and natural notion is that of a convex curve on $\mathbb{S}^n$. In somewhat loose terms, we say that a smooth immersion $\gamma : [a, b] \to \mathbb{S}^n$ is *convex* if, for any n-dimensional vector subspace $H \subset \mathbb{R}^{n+1}$, the intersection numbers between H and the restricted immersions $\gamma_0 = \gamma|[a, b)$ and $\gamma_1 = \gamma|(a, b]$ (multiplicities taken into account: a tangency counts as two intersections; an osculation counts as three, etc.) do not exceed n:

$$\forall H \subset \mathbb{R}^{n+1}, \#(H \cap \gamma_0) \leq n \text{ and } \#(H \cap \gamma_1) \leq n.$$

We shall clarify the meaning of this provisional but otherwise very intuitive definition in a while. In dimension $n = 2$, its meaning is clear: that the curve intersects each great circle at most twice even if one takes into account tangencies as double intersections, with the possible exception of a great circle that contains both its initial and final points and is tangent to the curve in one of them, if there is such a great circle. If there is no great circle as that, then we would like to say that the convex curve is *strictly convex* (see figure 3.1). Another more precise and general definition of strict convexity will be given right after the statement of theorem 3.2 below.

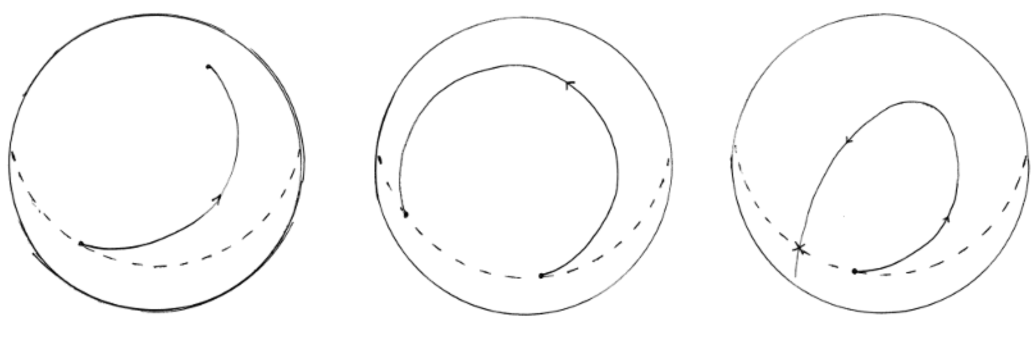

Figure 3.1: a strictly convex curve, a non-strictly convex curve and a nonconvex curves on the 2-sphere

A smooth curve $\gamma : [a, b] \to \mathbb{S}^n$ is *convex* if given any $\tau_1 < \cdots < \tau_k$ all in $[a, b)$ or all in $(a, b]$ and positive integers $\mu_1, \cdots, \mu_k$ whose sum is $n + 1$, then the ordered frame of derivatives

$$\left(\gamma\left(\tau_1\right),\cdots,\gamma^{(\mu_1-1)}\left(\tau_1\right),\cdots,\gamma\left(\tau_k\right),\cdots,\gamma^{(\mu_k-1)}\left(\tau_k\right)\right)$$

is a basis for $\mathbb{R}^{n+1}$. A convex curve on $\mathbb{S}^n$ is called *positive* (respectivelly, *negative*) if the ordered frame above is a positive (resp., negative) basis for $\mathbb{R}^{n+1}$ no matter what increasing sequence $\tau_1<\cdots<\tau_k$ of instants in $[a,b)$ or in $(a,b]$ we choose and regardless of the *multiplicities* $\mu_1,\cdots,\mu_k$ adding up to $n+1$.

The geometric content of the definition above is that no n-dimensional subspace $H\subset\mathbb{R}^{n+1}$ can intersect $\gamma|[a,b)$ or $\gamma|(a,b]$ in points $\gamma\left(\tau_1\right),\cdots,\gamma\left(\tau_k\right)$ with respective multiplicities $\mu_1,\cdots,\mu_k$ adding up to $n+1$, for one would then have

$$\operatorname{span}\left(\gamma\left(\tau_1\right),\cdots,\gamma^{(\mu_1-1)}\left(\tau_1\right),\cdots,\gamma\left(\tau_k\right),\cdots,\gamma^{(\mu_k-1)}\left(\tau_k\right)\right)\subseteq H$$

and hence this set of $n+1$ vectors would be linearly dependent.

Given an increasing sequence of instants $\tau_1<\cdots<\tau_k$ sampled in $[a,b)$ or in $(a,b]$ and multiplicities $\mu_1,\cdots,\mu_k$ adding up to $n+1$, we might consider the *sampled Wronski matrix* of γ with *time-sample* $\tau=(\tau_1,\cdots,\tau_k)$ and *multiplicity vector* $\mu=(\mu_1,\cdots,\mu_k)$ defined by

$$W_\gamma\left(\tau,\mu\right)=\begin{pmatrix} | & & | & & | & & | \\ \gamma\left(\tau_1\right) & \cdots & \gamma^{(\mu_1-1)}\left(\tau_1\right) & \cdots & \gamma\left(\tau_k\right) & \cdots & \gamma^{(\mu_k-1)}\left(\tau_k\right) \\ | & & | & & | & & | \end{pmatrix}$$

Of course the convexity condition can be recast in matricial terms as

$$\det W_\gamma\left(\tau,\mu\right)\neq 0$$

for all μ,τ meeting the appropriate conditions stated above. In the same spirit, γ is positive convex iff

$$\det W_\gamma\left(\tau,\mu\right)>0$$

and negative convex iff

$$\det W_\gamma\left(\tau,\mu\right)<0$$

for all admissible choices of τ and μ. There is no intermediate situation with positive and negative determinants occuring for different choices of τ and μ and the same convex γ since every convex curve is nondegenerate by definition and nondegeneracy is tantamount to local convexity in the sense of the following definition.

A smooth curve $\gamma : [a,b] \to \mathbb{S}^n$ is said to be *locally convex* if for every $t_0 \in [a,b]$ there is an $\varepsilon > 0$ such that the arc $\gamma|\left([t_0-\varepsilon, t_0+\varepsilon] \cap [a,b]\right)$ is convex. We state the equivalence between nondegeneracy and local convexity as a proposition for the sake of reader's easy reference and postpone its full proof to the end of section 3.1. We shall from now on use the terms "nondegenerate" and "locally convex" interchangeably when referring to spherical curves.

Proposition 3.1 *A smooth curve $\gamma : [a,b] \to \mathbb{S}^n$ is nondegenerate if and only if it is locally convex.*

Proof. Local convexity implies nondegeneracy by definition. For the reciprocal, see remark 3.6 at the end of section 3.1. ■

3.1 Characterization of convex arcs

In this section we shall provide an equivalent condition on the lifted SO_{n+1}-curve $\mathfrak{F}_\gamma$ to the convexity of an arc $\gamma : [t_0, t_1] \to \mathbb{S}^n$, to wit: that the quotient frame

$$\mathfrak{F}_\gamma\left(t_0; t\right) = \mathfrak{F}_\gamma\left(t_0\right)^{-1} \mathfrak{F}_\gamma\left(t\right)$$

belong, for all $t \in (t_0, t_1)$, in the Bruhat cell of A^T, the transpose of *Arnol'd matrix*

$$A = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \iddots & & & \\ (-1)^n & & & & \end{pmatrix}.$$

It is a necessary and sufficient condition for $X \in \mathrm{Bru}_{A^T}$ that all of the southwest minors of X be positive. We are going to use this criterion every now and then in what follows.

Given $X \in \mathrm{SO}_{n+1}$, we shall denote by $Bru\,(X)$ the unique signed permutation matrix Bruhat-equivalent to X. That is,

$$X \in \mathrm{Bru}_P \to Bru\,(X) = P$$

Since $\mathrm{adv}(I) = A^T$, we know that after leaving I the quotient frame $\mathfrak{F}_\gamma\left(t_0; \cdot\right)$ associated to any nondegenerate curve γ imediatelly enters the cell Bru_{A^T}. What we are going to show is that γ describes a convex arc as long as its quotient remains in the open cell Bru_{A^T}. More precisely:

Theorem 3.2 *Consider a nondegenerate curve $\gamma : [t_0, t_1] \to \mathbb{S}^n$. The following conditions are equivalent:*

(a) γ *is a convex arc;*
(b) $\forall t \in (t_0, t_1)$, $\mathfrak{F}_\gamma\,(t_0; t) \in \ \mathrm{Bru}_{A^T}$.
(c) $\forall t_a < t_b$ *in* $[t_0, t_1)$, $\mathfrak{F}_\gamma\,(t_a; t_b) \in \ \mathrm{Bru}_{A^T}$
(d) $\forall t_a < t_b$ *in* $(t_0, t_1]$, $\mathfrak{F}_\gamma\,(t_b; t_a) \in \ \mathrm{Bru}_{A}$

Let us postpone the proof of theorem 3.2 just for a while and comment a little upon it.

The characterization of convexity in terms of condition (b) suggests that we define a convex curve to be *strictly convex* iff it satisfies the more restrictive condition

$$\forall t \in (t_0, t_1], \mathfrak{F}_\gamma\,(t_0; t) \in \ Bru_{A^T},$$

i.e., iff γ abides by rule (b) till the very end of its life (though not from crib to grave, since its crib is always $I \notin \mathrm{Bru}_{A^T}$). In other words, γ is non-strictly convex iff it is convex and has final quotient frame $\mathfrak{F}_\gamma\,(t_0; t_1)$ in some exit cell Bru_Q of Bru_{A^T}. Figure 3.2 exhibits the five ways a nondegenerate curve $\gamma \in \mathcal{L}\mathbb{S}^2$ can lose its innate convexity.

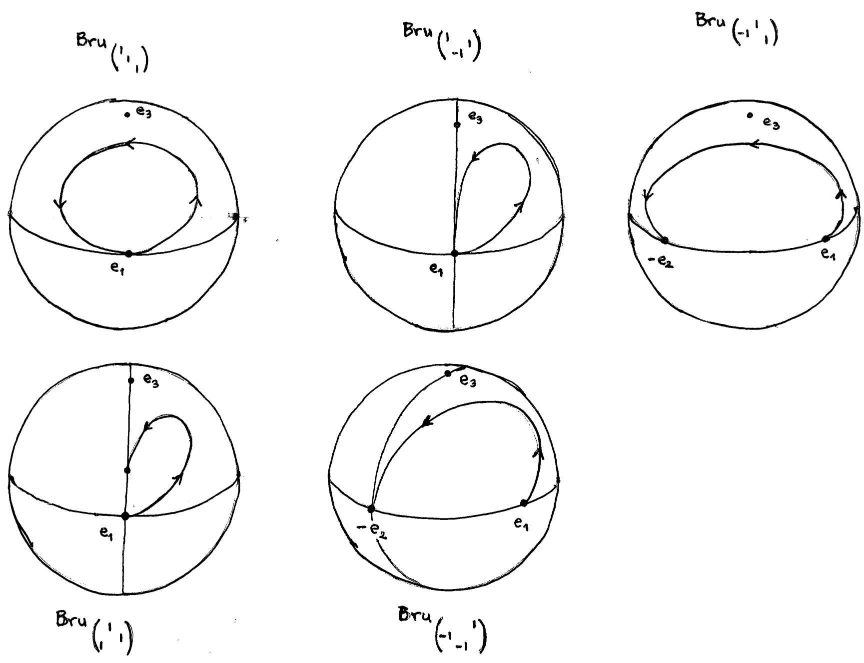

Figure 3.2: the five exit cells of $\mathrm{Bru}_{A^T} \subset \mathrm{SO}_3$

It was about time that we had some examples of convex curves on spheres of dimension greater than 2. Well, theorem 3.2 above give us them all at once. As for the first example, recalling that $\mathrm{adv}(I) = A^T$ and taking $\epsilon > 0$

sufficiently small so that the holonomic path

$$\Gamma(t) = \mathbf{Q}\left(\begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ & & \ddots & & \\ \frac{t^{n-1}}{(n-1)!} & \cdots & t & 1 & \\ \frac{t^n}{n!} & \cdots & \frac{t^2}{2} & t & 1 \end{pmatrix}\right), \; 0 \leq t \leq \epsilon$$

issuing from I stay within Bru_{A^T}, we have got a strictly convex curve $\gamma_Y = \pi[\Gamma] : [0, \epsilon] \to \mathbb{S}^n$ ending up in some $Y = \Gamma(\epsilon) \in \mathrm{Bru}_{A^T}$. This example is much more general than it seems: using the Bruhat action $\mathcal{B}$ of a suitably chosen $U \in \mathrm{Up}^1_{n+1}$ we can take the final frame Y into whatever $X \in \mathrm{Bru}_{A^T}$ we want in a way that respects strict convexity. In fact, there is a unique $U_{X,Y} \in \mathrm{Up}^1_{n+1}$ such that

$$\mathcal{B}(U_{X,Y}, Y) = X$$

and we get a strictly convex curve γ_X with frame path issuing from I and ending up in X defining it by

$$\gamma_X(t) = \pi[\mathcal{B}(U_{X,Y}, \Gamma(t))]$$

Next section's main theorem will show that this is the unique convex curve on $\mathbb{S}^n$ with initial frame I and final frame X up to a homotopy preserving local convexity. So this construction is very general indeed.

Moreover, we can get a (non-strictly) convex curve γ with initial frame I and whatever final frame we choose in any exit cell of Bru_{A^T}. In fact, suppose Bru_P is an exit cell of Bru_{A^T} and that we want a fixed $Q \in \mathrm{Bru}_P$ to be the final frame of our convex curve γ with initial frame I. Since $\mathrm{chop}(P) = A^T$, we can take $\epsilon > 0$ sufficiently small so that the holonomic path

$$\Gamma_Q(t) = Q \cdot \mathbf{Q}\left(\begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ & & \ddots & & \\ \frac{t^{n-1}}{(n-1)!} & \cdots & t & 1 & \\ \frac{t^n}{n!} & \cdots & \frac{t^2}{2} & t & 1 \end{pmatrix}\right), \; -\epsilon \leq t \leq 0$$

ending up in $Q \in \mathrm{Bru}_P$ begin in some $X \in \mathrm{Bru}_{A^T}$. Now, we just have to weld together (after a reparameterization) γ_X and $\pi[\Gamma_Q]$ to obtain a convex curve $\gamma = \gamma_X * \pi[\Gamma_Q]$ with intial frame is I and final frame Q.

At this point the reader may have felt uncomfortable with this "welding together" procedure if he has read the statement of theorem 3.2 as assuming smoothness from γ, what he could have rigthfully done, since its proof relies on a lemma where smoothness is explicitly stated as a hypothesis. Fortunately, this is not the case: the theorem is much more general than that and can be read as regarding curves that are nondegenerate in the sense of the more comprehensive definition given at the end of section 2.1, when we were discussing which topology to endow $\mathcal{L}\mathbb{S}^n$ with. In that discussion, we have pointed out that the original "smoothly" defined space $\mathcal{L}\mathbb{S}^n$ has the same homotopy type as viewed sitting inside its "enlarged" Hilbert space version $\mathcal{H}\mathbb{S}^n$, which now encompasses not only smooth curves, but also curves γ with H^1 path of frames $\mathfrak{F}_\gamma$ – in particular, curves γ with continuous and piecewise smooth paths of frames $\mathfrak{F}_\gamma$, exactly as in our examples above. In particular, the connected components of the strata $\mathcal{L}\mathbb{S}^n(Q)$ can't get disconnected when passing to the corresponding stratum $\mathcal{H}\mathbb{S}^n(Q)$. Now, corollary 4.2 in chapter 4 will tell us that the space of convex curves in $\mathcal{H}\mathbb{S}^n(Q)$ is either empty or consists of a connected component of curves satisfying that very condition (b) in theorem 3.2 above. The reader can therefore be reassured – provided he indulges in some suspension of judgement regarding the not yet proven corollary 4.2 – that the examples we have just exhibited are indeed convex curves, and could even be slightly homotoped into a smooth convex curve with the same initial and final frames, for $\mathcal{L}\mathbb{S}^n$ is dense in $\mathcal{H}\mathbb{S}^n$. [We have herein rehabilitated the long gone symbol $\mathcal{H}\mathbb{S}^n$ only to make a clear distinction between the space it once represented – our updated $\mathcal{L}\mathbb{S}^n$ – and the provisionaly defined $\mathcal{L}\mathbb{S}^n$ as the space of smooth nondegenerate curves. From now on we pledge ourselves to abide again by our relaxed definition of $\mathcal{L}\mathbb{S}^n$.]

It is worth pointing out in any case that there is a *prêt-à-porter* example of convex smooth curve $\xi : [0, \pi] \to \mathbb{S}^n$ with initial frame I and final frame $\Omega = (-1)^n I$ in the only zero-dimensional exit cell $\mathrm{Bru}_\Omega = \{\Omega\}$ of Bru_{A^T}, to wit:

$$\xi(t) = \exp\left(t\begin{pmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & 0 & -c_n \\ & & & c_n & 0 \end{pmatrix}\right) \cdot e_1,$$

for $0 \le t \le \pi$, with $c_j = \sqrt{j(n-j+1)}$ for all $j \in [n]$. Moreover, it has the very pleasing property that

$$\mathfrak{F}_\xi\left(\frac{\pi}{2}\right) = A^T$$

These curves made their first appearance in lemma 2.2 of (14) as examples of locally convex curves. Their convexity follows now as a consequence of theorem 3.2: it takes only a straightforward computation on southeast minor determinants to show that

$$\mathfrak{F}_{\xi}\left(t\right) \in \text{ Bru}_{A^T}$$

for all $t \in (0, \pi)$.

If we allow ξ to follow its course beyond $t = \pi$, we get a π-periodic locally convex curve if n is even and a 2π-periodic locally convex curve if n is odd, satisfying

$$\mathfrak{F}_{\xi}\left(0\right) = I, \;\; \mathfrak{F}_{\xi}\left(\frac{\pi}{2}\right) = A^T, \;\; \mathfrak{F}_{\xi}\left(\pi\right) = \Omega, \;\; \mathfrak{F}_{\xi}\left(\frac{3\pi}{2}\right) = A, \;\; \mathfrak{F}_{\xi}\left(2\pi\right) = I, \;\; \cdots$$

For $n = 2$, we have

$$\xi\left(t\right) = \frac{1}{2}\Big(1 + \cos(2t), \sqrt{2}\sin\left(2t\right), 1 - \cos\left(2t\right)\Big)$$

which is nothing but the circle of diameter $e_1 e_3$ in $\mathbb{S}^2$.

Proof of theorem 3.2. $(\neg c \to \neg a)$ Suppose that γ violates condition (c). If we take $t_a < t_b$ minimal such that $\mathfrak{F}_{\gamma}\left(t_a; t_b\right) \notin \text{ Bru}_{A^T}$, then $\mathfrak{F}_{\gamma}\left(t_a; t_b\right)$ will exhibit a first non-invertible southwest block of order $k+1$ for some $k \in \{0, 1, \cdots, n-1\}$, and the signed permutation matrix $Bru\left(\mathfrak{F}_{\gamma}\left(t_a; t_b\right)\right)$ will have a $(k+1) \times (k+1)$ southwest block of the form

$$\begin{bmatrix} & & & 0 \\ & & (-1)^{k+1} & \\ & \iddots & & \\ -1 & & & \\ 1 & & & \end{bmatrix}$$

This means that the $(k+1)$-th column of $Bru\left(\mathfrak{F}_{\gamma}\left(t_a; t_b\right)\right)$ is one of the previous vectors of the standard basis of $\mathbb{R}^{n+1}$, some e_j with $j \in \{1, 2, \cdots, n-k\}$. Explicitly, one has

$$Bru\left(\mathfrak{F}_{\gamma}\left(t_a; t_b\right)\right) \cdot e_{k+1} = e_j,$$

which boils down to the following relation between the Wronski matrices of γ in t_a and t_b:

$$W_{\gamma}\left(t_b\right) U_b \cdot e_{k+1} = W_{\gamma}\left(t_a\right) U_a \cdot e_j$$

for some $U_a, U_b \in \text{ Up}^{+}_{n+1}$ and $j \in \{1, 2, \cdots, n-k\}$. Now, that means a linear dependence between the first $k+1$ columns of $W_{\gamma}\left(t_b\right)$ and the first $n-k$

columns of $W_\gamma(t_a)$:

$$\sum_{\alpha=0}^{n-k-1} A_\alpha \gamma^{(\alpha)}(t_a) + \sum_{\beta=0}^{k} B_\beta \gamma^{(\beta)}(t_b) = 0, \text{ with } B_k \neq 0,$$

hence

$$\det \begin{pmatrix} | & | & & | & | & | & & | \\ \varphi(t_a) & \varphi'(t_a) & \cdots & \varphi^{(n-k-1)}(t_a) & \varphi(t_b) & \varphi'(t_b) & \cdots & \varphi^{(k)}(t_b) \\ | & | & & | & | & | & & | \end{pmatrix} = 0$$

ruling out convexity for the arc $\gamma|[t_a, t_b]$.

The implication $(c \to b)$ is trivial.

$(c \leftrightarrow d)$ Let $\mathfrak{F}_\gamma(t_a; t_b) \in Bru_{A^T}$ hold for $t_a < t_b$ in $[t_0, t_1)$. Then, there exist $U_0 \in \mathrm{Up}_{n+1}^+$ and $U_1 \in \mathrm{Up}_{n+1}^+$ such that

$$\mathfrak{F}_\gamma(t_a; t_b) = \mathfrak{F}_\gamma(t_a)^{-1} \mathfrak{F}_\gamma(t_b) = U_0 A^T U_1,$$

hence, taking inverses,

$$\mathfrak{F}_\gamma(t_b)^{-1} \mathfrak{F}_\gamma(t_a) = U_1^{-1} A U_0^{-1}$$

and $\mathfrak{F}_\gamma(t_b; t_a) \in \mathrm{Bru}_A$ for $t_a < t_b$ in $[t_0, t_1)$, and in particular for $t_a < t_b$ in (t_0, t_1). The case $t_0 < t_a < t_b = t_1$, can be treated by an argument analogous to the one used in the proof of $(\neg c \to \neg a)$ above.

In order to prove the remaining direction $(b \to a)$, we are going to rely on the following technical result, proven rightaway after this theorem.

Lemma 3.3 *Let a smooth path $\varphi : \mathbb{R} \to \mathbb{R}^{n+1}$ be such that*

$$\Phi(s) = \begin{pmatrix} | & | & & | \\ \varphi(s) & \varphi'(s) & \cdots & \varphi^{(n)}(s) \\ | & | & & | \end{pmatrix}$$

admits of an LU-decomposition for all $s \in \mathbb{R}$. Then, given an increasing sequence of real numbers $s_0 < s_1 < \cdots < s_k$ of length $k < n$ and positive integers $\mu_0, \mu_1, \cdots, \mu_k$ whose sum is $n+1$, the matrix $\Phi(s_0, \mu_0; \cdots; s_k, \mu_k)$ given by

$$\begin{pmatrix} | & | & & | & & | & | & & | \\ \varphi(s_0) & \varphi'(s_0) & \cdots & \varphi^{(\mu_0-1)}(s_0) & \cdots & \varphi(s_k) & \varphi'(s_k) & \cdots & \varphi^{(\mu_k-1)}(s_k) \\ | & | & & | & & | & | & & | \end{pmatrix}$$

admits of an LU-decomposition as well. In particular,

$$\det \Phi\left(s_0, \mu_0; \cdots ; s_k, \mu_k\right) \neq 0.$$

For the time being, let us go back to the proof of theorem 3.2.

Start choosing an increasing smooth parameterization h of the open interval (t_0, t_1) by the real line. For instance, one can choose explicitly

$$\begin{array}{rccl} h: & \mathbb{R} & \to & (t_0, t_1) \\ & s & \mapsto & h(s) = t_0 + \left(\frac{1}{2} + \frac{\arctan s}{\pi}\right)(t_1 - t_0) \end{array}.$$

Then, one takes $\varphi : \mathbb{R} \to \mathbb{R}^{n+1}$ to be

$$\varphi(s) = A\mathfrak{F}_\gamma(t_0)^{-1} \cdot \gamma(h(s))$$

(where A is the Arnol'd matrix mentioned in the first paragraph of this section) and

$$\Phi(s) = \begin{pmatrix} | & | & & | \\ \varphi(s) & \varphi'(s) & \cdots & \varphi^{(n)}(s) \\ | & | & & | \end{pmatrix}$$

as in lemma 3.3 above. A straightforward computation shows that for all $s \in \mathbb{R}$ there exists $U(s) \in \mathrm{Up}_{n+1}^{+}$ such that

$$\Phi(s) = A\mathfrak{F}_\gamma(t_0; h(s))\, U(s)$$

(the entries of $U(s)$ are functions of the derivatives of the parameterization $h = h(s)$).

Now, suppose that $\mathfrak{F}_\gamma(t_0; t) \in \mathrm{Bru}_{A^T}$ for all $t \in (t_0, t_1)$. Then , $\forall s \in \mathbb{R}$, there are $U_0(s) \in \mathrm{Up}_{n+1}^{1}$ and $U_1(s) \in \mathrm{Up}_{n+1}^{+}$ such that

$$\Phi(s) = AU_0(s)\, A^T U_1(s)$$

Since $AU_0(s)\, A^T \in \mathrm{Lo}_{n+1}^{1}$ $\forall s \in \mathbb{R}$, lemma 3.3 guarantees that γ is convex. ■

Before proving lemma 3.3, we pause to observe that the same argument used to prove the implication $(b \to a)$ can be slightly adapted to show the following general result, that states that a nondegenerate curve born into a (possibly translated) open Bruhat cell does not lose its convexity before leaving it. In particular, an arc whose path of frames is contained in a single (possibly translated) open Bruhat cell – except maybe for its endpoints – is convex.

Proposition 3.4 *Let $\gamma : [t_0, t_1] \to \mathbb{S}^n$ be a nondegenerate curve. If there is an orthogonal matrix $Q \in \mathrm{O}_{n+1}$ such that*

$$\mathfrak{F}_\gamma (t) \in \ Q \cdot \mathrm{Bru}_{A^T}$$

for all $t \in (t_0, t_1)$, then γ is convex.

Proof. We begin by considering the special case $Q = I$. The reasoning is exatcly the same used to prove implication $(b \to a)$ above.

Suppose $\mathfrak{F}_\gamma (t) \in \mathrm{Bru}_{A^T}$ for all $t \in (t_0, t_1)$. Use an increasing smooth parameterization $h = h(s)$ of the open interval (t_0, t_1) by $s \in \mathbb{R}$ as above and define $\varphi : \mathbb{R} \to \mathbb{R}^{n+1}$ this time to be

$$\varphi (s) = A \cdot \gamma (h (s))$$

so that

$$\Phi (s) = \begin{pmatrix} | & | & & | \\ \varphi (s) & \varphi' (s) & \cdots & \varphi^{(n)} (s) \\ | & | & & | \end{pmatrix}$$

be given by

$$\Phi (s) = A \mathfrak{F}_\gamma (h (s)) U (s)$$

for some $U : \mathbb{R} \to \mathrm{Up}_{n+1}^+$. The conclusion follows as before from lemma 3.3 by noticing that $AU_0A^T \in \mathrm{Lo}_{n+1}^1$ if $U_0 \in \mathrm{Up}_{n+1}^1$.

For the general case, suppose γ is such that $\mathfrak{F}_\gamma (t) \in Q{\cdot}\mathrm{Bru}_{A^T}$ for all $t \in (t_0, t_1)$ and consider the curve $Q^T \cdot \gamma$. Since orthogonal transformations preserve convexity, γ is convex if and only if $Q^T \cdot \gamma$ is convex. Now,

$$\begin{aligned} \mathfrak{F}_{Q^T \cdot \gamma} (t) &= W_{Q^T \cdot \gamma} (t) R (t) \\ &= Q^T W_\gamma (t) R (t), \end{aligned}$$

for some $R (t) \in \mathrm{Up}_{n+1}^+$ and from the unicity of QR decomposition it follows that

$$\mathfrak{F}_{Q^T \cdot \gamma} (t) = Q^T \mathfrak{F}_\gamma (t),$$

hence

$$\mathfrak{F}_{Q^T \cdot \gamma} (t) \in \mathrm{Bru}_{A^T}$$

and $Q^T \cdot \gamma$ is convex by the special case above. ■

Proof of lemma 3.3. We write $\varphi = (\varphi_0, \varphi_1, \cdots, \varphi_n)$ for the coordinate functions of the path φ satisfying

$$\Phi(s) = \begin{pmatrix} | & | & & | \\ \varphi(s) & \varphi'(s) & \cdots & \varphi^{(n)}(s) \\ | & | & & | \end{pmatrix} = L(s)\,U(s)$$

with $L(s) \in \mathrm{Lo}_{n+1}$, $U(s) \in \mathrm{Up}_{n+1}$ for all $s \in \mathbb{R}$, and we want to show that $\Phi(s_0, \mu_0; \cdots; s_k, \mu_k)$ given by

$$\begin{pmatrix} | & | & & | & & | & | & & | \\ \varphi(s_0) & \varphi'(s_0) & \cdots & \varphi^{(\mu_0-1)}(s_0) & \cdots & \varphi(s_k) & \varphi'(s_k) & \cdots & \varphi^{(\mu_k-1)}(s_k) \\ | & | & & | & & | & | & & | \end{pmatrix}$$

admits of an LU-decomposition as well, regardless of the real numbers $s_0 < s_1 < \cdots < s_k$ sampled and the positive integers $\mu_0, \mu_1, \cdots, \mu_k$ whose sum is $n+1$.

We proceed by induction in n. The cases $n = 0$ and $n = 1$ are trivial. There is no loss of generality in considering $\varphi_0 \equiv 1$, for, since $\Phi(s)$ admits of an LU-decomposition for all s, $\varphi_0(s) \neq 0$ for all s and we can always multiply $\Phi(s)$ by an upper triangular matrix with non-vanishing diagonal entries

$$\begin{pmatrix} \frac{1}{\varphi_0} & -\frac{\varphi_0'}{\varphi_0^2} & \frac{2\varphi_0'}{\varphi_0^3} - \frac{\varphi_0''}{\varphi_0^2} & \cdots & * \\ 0 & \frac{1}{\varphi_0} & -\frac{2}{\varphi_0^2} & \cdots & * \\ 0 & 0 & \frac{1}{\varphi_0} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\varphi_0} \end{pmatrix}(s) \in \mathrm{Up}_{n+1},$$

where the omitted terms above the diagonal are certain rational functions whose denominators are powers of $\varphi_0(s)$ and the numerators are polynomials in $\varphi_0(s)$, $\varphi_0'(s)$, $\cdots$, $\varphi_0^{(n)}(s)$. Besides, we only need to prove that

$$\det \Phi(s_0, \mu_0; \cdots; s_k, \mu_k) \neq 0,$$

since the minor determinants are automatically non-vanishing by the induction hypothesis.

Suppose, to the contrary, that there is a non-zero linear functional $\omega \in (\mathbb{R}^{n+1})^* \setminus \{0\}$ such that, for all $j \in \{0, 1, \cdots, k\}$, one has

$$\omega(\varphi(s_j)) = \omega(\varphi'(s_j)) = \cdots = \omega\left(\varphi^{(\mu_j-1)}(s_j)\right) = 0,$$

and define

$$\widehat{\varphi} : \mathbb{R} \to \mathbb{R}^n$$

by

$$\widehat{\varphi}(s) = \left(\varphi_1'(s), \cdots, \varphi_n'(s)\right).$$

Note that

$$\widehat{\Phi}(s) = \begin{pmatrix} | & | & & | \\ \widehat{\varphi}(s) & \widehat{\varphi}'(s) & \cdots & \widehat{\varphi}^{(n-1)}(s) \\ | & | & & | \end{pmatrix}$$

is the $n \times n$ southeast block of $\Phi(s)$ and therefore admits of an LU decomposition, just like $\Phi(s)$ itself (since we agreed to consider the first line of $\Phi(s)$ to be constant $(1, 0, \cdots, 0)$, the $\widehat{L}$ and $\widehat{U}$ factors of $\widehat{\Phi}(s)$ are exactly the $n \times n$ southeast blocks of the L and U factors, respectively, of $\Phi(s)$). Consider now the restricted linear functional

$$\widehat{\omega} = \omega|\mathbb{R}^n$$

where we identify $\mathbb{R}^n \simeq \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Since

$$\omega\left(\varphi\left(s_j\right)\right) = \omega\left(\varphi\left(s_{j+1}\right)\right) = 0,$$

by Rolle's theorem there is a $s_{j+\frac{1}{2}} \in (s_j, s_{j+1})$ such that

$$\omega\left(\varphi'\left(s_{j+\frac{1}{2}}\right)\right) = 0$$

and therefore

$$\widehat{\omega}\left(\widehat{\varphi}\left(s_{j+\frac{1}{2}}\right)\right) = 0.$$

Besides, $\widehat{\omega}$ inherits from ω the zeros

$$\widehat{\omega}\left(\widehat{\varphi}\left(s_j\right)\right) = \widehat{\omega}\left(\widehat{\varphi}'\left(s_j\right)\right) = \cdots = \widehat{\omega}\left(\widehat{\varphi}^{(\mu_j-2)}\left(s_j\right)\right) = 0,$$

for all $j \in \{0, 1, \cdots, k\}$. Now, take $\widehat{k} = 2k$ and define the refined increasing sequence

$$\widehat{s}_0 < \widehat{s}_1 < \cdots < \widehat{s}_{\widehat{k}}$$

by

$$\widehat{s}_j = s_{\frac{j}{2}}$$

with associated multiplicities

$$\widehat{\mu}_j = \begin{cases} \mu_{\frac{j}{2}} - 1, & \text{if } j \text{ is even} \\ 1, & \text{if } j \text{ is odd} \end{cases}.$$

Then, $\det \widehat{\Phi}\left(\widehat{s}_0, \widehat{\mu}_0; \cdots ; \widehat{s}_{\widehat{k}}, \widehat{\mu}_{\widehat{k}}\right) = 0$, which contradicts our induction hypothesis, since this is an $n \times n$ determinant. ■

At first sight, it seems to be no point in stating the conclusion of lemma 3.3 in terms of LU decompositions, since what was required for the proof of theorem 3.2 was only the non-vanishing of $\det \Phi\left(s_0, \mu_0; \cdots ; s_k, \mu_k\right)$. The proof by induction we just gave makes clear the point of our rather cumbersome statement: the LU decomposition turned out being handier an induction hypothesis, though a little too strong a conclusion.

Before we close this section with a remark that will complete the proof of proposition 3.1 left behind, let us state a lemma that will be used in chapter 4 to prove the existence of a connected component of $\mathcal{L}\mathbb{S}^n(Q)$ comprised solely by its convex curves, provided the monodromy Q allows for some convex curve.

Lemma 3.5 *Let $Q \in \mathrm{O}_{n+1}$ be an orthogonal matrix and Q_0, $Q_1 \in Q{\cdot}\mathrm{Bru}_{A^T}$. One of the following three mutually exclusive alternatives is to hold:*

(i) *Q_0 and Q_1 are not the endpoints of the path of frames of any convex arc,* i.e.,

$$\nexists\ \gamma : [t_0, t_1] \to \mathbb{S}^n \textit{ such that } \gamma \textit{ is convex and } \mathfrak{F}_\gamma(t_0) = Q_0 \textit{ and } \mathfrak{F}_\gamma(t_1) = Q_1$$

(ii) *Q_0 and Q_1 are the endpoints of the path of frames of a convex arc and every such arc has path of frames fully contained in the translated Bruhat cell $Q{\cdot}\mathrm{Bru}_{A^T}$,* i.e.,

$$\exists\ \gamma : [t_0, t_1] \to \mathbb{S}^n \textit{ such that } \gamma \textit{ is convex and } \mathfrak{F}_\gamma(t_0) = Q_0 \textit{ and } \mathfrak{F}_\gamma(t_1) = Q_1$$

and

$$\gamma \textit{ convex with } \mathfrak{F}_\gamma(t_0) = Q_0 \textit{ and } \mathfrak{F}_\gamma(t_1) = Q_1 \to \forall t \in [t_0, t_1]\left(\mathfrak{F}_\gamma(t) \in Q{\cdot}\mathrm{Bru}_{A^T}\right)$$

(iii) *Q_0 and Q_1 are the endpoints of the path of frames of a convex arc and every such arc has path of frames escaping the translated Bruhat cell $Q{\cdot}\mathrm{Bru}_{A^T}$,* i.e.,

$$\exists\ \gamma : [t_0, t_1] \to \mathbb{S}^n \textit{ such that } \gamma \textit{ is convex and } \mathfrak{F}_\gamma(t_0) = Q_0 \textit{ and } \mathfrak{F}_\gamma(t_1) = Q_1$$

and

$$\gamma \text{ convex with } \mathfrak{F}_\gamma(t_0) = Q_0 \text{ and } \mathfrak{F}_\gamma(t_1) = Q_1 \to \exists t \in [t_0, t_1](\mathfrak{F}_\gamma(t) \notin Q \cdot \mathrm{Bru}_{A^T})$$

Proof. It clearly suffices to prove the case $Q = I$.

Suppose by contradiction that $Q_0, Q_1 \in \mathrm{Bru}_{A^T}$ are the endpoints of the paths of frames of two convex curves α and β one of which is fully contained in Bru_{A^T} while the other escapes this open cell:

$$\alpha, \beta : [t_0, t_1] \to \mathbb{S}^n \text{ convex}$$

$$\begin{aligned} \mathfrak{F}_\alpha(t_0) &= \mathfrak{F}_\beta(t_0) = Q_0 \\ \mathfrak{F}_\alpha(t_1) &= \mathfrak{F}_\beta(t_1) = Q_1 \end{aligned}$$

$$\begin{aligned} \forall t &\in [t_0, t_1] (\mathfrak{F}_\alpha(t) \in \mathrm{Bru}_{A^T}) \\ \exists t_* &\in [t_0, t_1] (\mathfrak{F}_\beta(t_*) \notin \mathrm{Bru}_{A^T}) \end{aligned}$$

Let $\Omega = (-1)^n I$ be the only element of the zero-dimensional exit cell $\mathrm{Bru}_\Omega = \{\Omega\}$ of Bru_{A^T}. There is a convex curve γ with initial frame Q_1 and final frame Ω: for instance, take $\epsilon > 0$ sufficiently small so that the holonomic path

$$\Gamma(t) = (-1)^n \mathbf{Q} \left(\begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ & & \ddots & & \\ \frac{t^{n-1}}{(n-1)!} & \cdots & t & 1 & \\ \frac{t^n}{n!} & \cdots & \frac{t^2}{2} & t & 1 \end{pmatrix} \right), \; -\epsilon \le t \le 0$$

ending up in Ω begin in some $X \in \mathrm{Bru}_{A^T}$. Now use the Bruhat action $\mathcal{B}$ of a suitable $U \in \mathrm{Up}_{n+1}^1$ to take the initial frame X into Q_1:

$$\mathcal{B}(U, X) = Q_1$$

The desired curve can now be taken as

$$\gamma(t) = \pi\left[\mathcal{B}(U, \Gamma(t))\right]$$

By construction, γ is convex and, under a reparameterization $[-\epsilon, 0] \to [t_1, t_2]$,

$$\forall t \in [t_1, t_2) (\mathfrak{F}_\gamma(t) \in \mathrm{Bru}_{A^T})$$

Let us consider the nondegenerate curves

$$\begin{aligned} \alpha * \gamma : \; [t_0, t_2] &\to \mathbb{S}^n \\ t &\mapsto \begin{cases} \alpha(t), \text{ if } t_0 \le t \le t_1 \\ \gamma(t), \text{ if } t_1 \le t \le t_2 \end{cases} \end{aligned}$$

$$\begin{aligned} \beta * \gamma : \; [t_0, t_2] &\to \mathbb{S}^n \\ t &\mapsto \begin{cases} \beta(t), \text{ if } t_0 \le t \le t_1 \\ \gamma(t), \text{ if } t_1 \le t \le t_2 \end{cases} \end{aligned}$$

obtained by welding toghether at the joint Q_1 the arcs α and γ, and β and γ, respectively. From Ω's point of view, $\alpha*\gamma$ is convex and $\beta*\gamma$ is not convex. The claim follows from the fact that the arc described by $\mathfrak{F}_{\alpha*\gamma}$ is wholly contained in Bru_{A^T}, while the arc of $\mathfrak{F}_{\beta*\gamma}$ is not. In fact, the equivalence $(a \leftrightarrow d)$ in theorem 3.2 tells us that a nondegenerate arc $\omega : [t_0, t_2] \to \mathbb{S}^n$ with final frame Ω is convex if and only if

$$\mathfrak{F}_{\omega}(t) \in \Omega \cdot Bru_A = Bru_{A^T}$$

for all $t \in (t_0, t_2)$. But – from Q_0's perspective – $\mathfrak{F}_{\beta*\gamma}$ has to be convex, since, by theorem 3.2, we have $\mathfrak{F}_{\beta*\gamma}(t) = \mathfrak{F}_{\beta}(t) \in Q_0 \cdot \text{Bru}_{A^T}$ for all $t \in (t_0, t_1)$, since it is convex issuing from Q_0 and also $\mathfrak{F}_{\beta*\gamma}(t) = \mathfrak{F}_{\gamma}(t) \in Q_0 \cdot \text{Bru}_{A^T}$ for all $t \in (t_1, t_2)$, since $\mathfrak{F}_{\gamma}$ is the final piece of the convex curve $\mathfrak{F}_{\alpha*\gamma}$ issuing from Q_0. Contradiction. ■

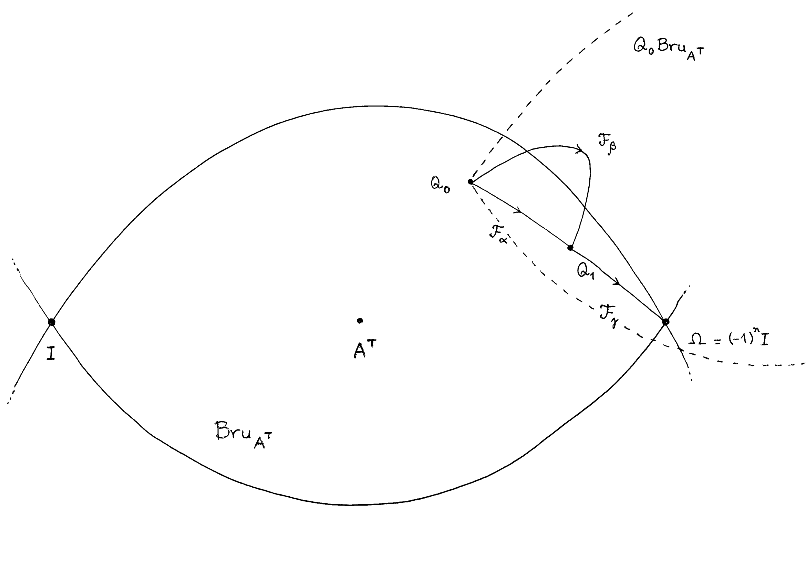

Figure 3.3: proof of lemma 3.5

Remark 3.6 *It can be seen as a consequence of theorem 3.2 that nondegeneracy implies local convexity. In fact, since* $\mathrm{adv}(I) = A^T$*, we have that for each time* t_* *there is* $\epsilon > 0$ *such that*

$$\mathfrak{F}_\gamma\left(t_*; t\right) \in \mathrm{Bru}_{A^T}$$

for all $t \in [t_*, t_* + \epsilon]$*, provided* γ *is nondegenerate. By theorem 3.2,* $\gamma| \left[t_*, t_* + \epsilon\right]$ *is convex.*

3.2 Contractibility of the set of strictly convex curves

In this section we are going to prove that the set

$$\mathcal{SCS}^n\left(Q\right) = \left\{\gamma \in \mathcal{LS}^n\left(Q\right) \ \mid \forall t \in (0,1] \left(\mathfrak{F}_\gamma\left(t\right) \in \ \mathrm{Bru}_{A^T}\right)\right\}$$

is non-empty and contractible if $Q \in \mathrm{Bru}_{A^T}$. By theorem 3.2 in the former section, this is the set of all strictly convex curves with initial frame I and final frame Q. Of course it is empty if $Q \notin \mathrm{Bru}_{A^T}$.

We begin by introducing a handy system of coordinates in the Bruhat cell of the transpose Arnol'd matrix. Since

$$Q \in \mathrm{Bru}_{A^T} \leftrightarrow \exists! U_0 \in \mathrm{Up}^1_{n+1}, \exists! U_1 \in \mathrm{Up}^+_{n+1} \left(Q = U_0 A^T U_1\right),$$

we have

$$Q \in \mathrm{Bru}_{A^T} \leftrightarrow \exists! L_0 \in \mathrm{Up}^1_{n+1}, \exists! U_1 \in \mathrm{Up}^+_{n+1} \left(AQ = L_0 U_1\right),$$

with $L_0 = AU_0A^T$. Therefore, we can define the diffeomorphism

$$\begin{array}{cccc} \lambda: & \mathrm{Bru}_{A^T} & \to & \mathrm{Lo}^1_{n+1} \\ & Q & \mapsto & \mathbf{L}\left(AQ\right) \\ & \mathbf{Q}\left(A^T L\right) & \leftarrow & L \end{array},$$

where $\mathbf{L}\left(X\right)$ stands for the L-part of the LU decomposition of X.

Now, let $\Gamma : [0,1] \to \mathrm{Bru}_{A^T}$ be a smooth path wholly contained in the open cell Bru_{A^T} and denote by $L : [0,1] \to \mathrm{Lo}^1_{n+1}$ this same path as seen through λ:

$$L\left(t\right) = \lambda\left(\Gamma\left(t\right)\right) = \mathbf{L}\left(A\Gamma\left(t\right)\right)$$

There exists a smooth one-parameter family of upper matrices $R : [0,1] \to \mathrm{Up}^+_{n+1}$ such that

$$L\left(t\right) = A\Gamma\left(t\right) R\left(t\right)$$

Differentiating with respect to time yields

$$\begin{aligned} L'(t) &= A\Gamma'(t)\,R(t) + A\Gamma(t)\,R'(t) \\ &= A\Gamma(t)\left(\Gamma(t)^{-1}\Gamma'(t)\,R(t) + R'(t)\right) \\ &= L(t)\left(R(t)^{-1}\Lambda_\Gamma(t)\,R(t) + R(t)^{-1}R'(t)\right) \end{aligned}$$

Therefore, we have the following relation between logarithmic derivatives:

$$L(t)^{-1}L'(t) = R(t)^{-1}\Lambda_\Gamma(t)\,R(t) + R(t)^{-1}R'(t)$$

The left hand side $L(t)^{-1}L'(t)$ belongs to the Lie algebra $\mathfrak{lo}^1_{n+1}$ and is therefore lower-triangular with zero diagonal; the term $R(t)^{-1}R'(t)$ belongs to $\mathfrak{up}^+_{n+1}$ and is upper-triangular. Therefore, by proposition 2.1, Γ is holonomic if and only if the logarithmic derivative of $L(t)$ is of the special form

$$L(t)^{-1}L'(t) = \begin{pmatrix} 0 & & & & \\ \beta_1(t) & 0 & & & \\ & \beta_2(t) & \ddots & & \\ & & \ddots & 0 & \\ & & & \beta_n(t) & 0 \end{pmatrix},$$

for some positive smooth functions $\beta_1, \cdots, \beta_n : [0,1] \to (0, +\infty)$. The omitted entries are all identically zero. This is so because if $\Lambda_\Gamma(t) \in \mathfrak{J} < \mathfrak{so}_{n+1}$ is skew-Jacobi with positive subdiagonals, then the term $R(t)^{-1}\Lambda_\Gamma(t)\,R(t)$ is of the form

$$\begin{pmatrix} * & & & & \\ + & * & & * & \\ 0 & + & \ddots & & \\ \vdots & & \ddots & * & \\ 0 & \cdots & 0 & + & * \end{pmatrix},$$

meaning that the entries in the diagonal and above are arbitrary functions, the entries in the subdiagonal are positive functions and the remaining ones are identically zero. We shall from now on say that a smooth path $L : [0,1] \to \mathrm{Lo}^1_{n+1}$ is *holonomic* if it is the λ-transform $\lambda(\Gamma)$ for a holonomic path $\Gamma : [0,1] \to \mathrm{Bru}_{A^T}$, *i.e.*, iff its logarithmic derivative $L(t)^{-1}L'(t)$ is of the form above.

If $L : [0,1] \to \mathrm{Lo}^1_{n+1}$ is holonomic satisfying $L(0) = I$, one can integrate its logarithmic derivative obtaining

$$L_{ij}(t) = \begin{cases} 0, & \text{if } i < j \\ 1, & \text{if } i = j \\ \underset{0 \le t_{i-1} \le \cdots \le t_j \le t}{\int \cdots \int} \beta_j(t_j) \cdots \beta_{i-1}(t_{i-1})\, dt_{i-1} \cdots dt_j, & \text{if } i > j \end{cases}$$

For instance, integrating a constant logarithmic derivative Λ_1 with unit subdiagonal one has

$$\exp\left(t \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ \frac{t^2}{2} & t & \ddots & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{t^n}{n!} & \frac{t^{n-1}}{(n-1)!} & \cdots & t & 1 \end{pmatrix}$$

We are goint to use extensively this representation of strictly convex curves to proof the following result.

Theorem 3.7 *If* $Q \in \mathrm{Bru}_{A^T}$*, then the space of strictly convex curves* $\mathcal{SCS}^n(Q) \subset \mathcal{LS}^n(Q)$ *is non-empty and contractible.*

Proof. We have already seen in the first example right after the statement of theorem 3.2 that if $Q \in \mathrm{Bru}_{A^T}$, then there is a strictly convex curve γ_Q with initial frame I and final frame Q, hence the non-emptiness of $\mathcal{SCS}^n(Q)$. To prove its contractibility, we shall digress a little and prove the following equivalent statement.

Claim 3.8 *Let* $L_1 = \exp(\Lambda_1)$*. The space*

$$\mathcal{L} = \left\{ L : [0,1] \to \mathrm{Lo}^1_{n+1} \mid L \text{ is holonomic, } L(0) = I \text{ and } L(1) = L_1 \right\}$$

is contractible.

This space can be regarded as a subset of $L^1([0,1], \mathbb{R})^n$ comprised by n-tuples of absolutely continuous measures $\mu_j = \beta_j(t)\, dt$ with respect to the Lebesgue measure in the real line satisfying the normalization conditions:

$$1 \le j < i \le n+1 \longrightarrow \underset{0 \le t_{i-1} \le \cdots \le t_j \le t}{\int \cdots \int} \beta_j(t_j) \cdots \beta_{i-1}(t_{i-1})\, dt_{i-1} \cdots dt_j = \frac{1}{(i-j)!}$$

This space is star-shaped with base point in the diagonal element in the Lebesgue measure $(dt, \cdots, dt)$, or $\exp(t\Lambda_1)$, in its original embodiment as a set of paths of matrices: given $(\mu_1, \cdots, \mu_n) \in \mathcal{L}$ we can exhibit a continuous path

$s \in [0,1] \mapsto (\mu_1^s, \cdots, \mu_n^s) \in \mathcal{L}$ with end points $(\mu_1^0, \cdots, \mu_n^0) = (\mu_1, \cdots, \mu_n)$ and $(\mu_1^1, \cdots, \mu_n^1) = (dt, \cdots, dt)$. Explicitly, for $\mu_j = \beta_j(t)\, dt$ define $\mu_j^s = \beta_j^s(t)\, dt$ by

$$\beta_j^s(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq s \\ \beta_j\left(\frac{t-s}{1-s}\right), & \text{if } s \leq t \leq 1 \end{cases}$$

It is a matter of straightforward computations to show that the measures $\beta_j^s(t)\, dt$ satisfy the normalization conditions. We shall omit the details.

Turning back to the group SO_{n+1}, we have just shown that the space of holonomic paths $\Gamma : [0,1] \to \mathrm{Bru}_{A^T}$ with $\Gamma(0) = A^T$ and $\Gamma(1) = \lambda^{-1}(L_1)$ is contractible. These are, of course, paths of frames of strictly convex curves, and, by implication $(a \to b)$ of theorem 3.2, $A\Gamma(t) \in \mathrm{Bru}_{A^T}$ for all $t \in (0,1]$, with $A\Gamma(0) = I$. Now, given $U \in \mathrm{Up}_{n+1}^1$ the unique upper matrix such that $\mathcal{B}(U, A\lambda^{-1}(L_1)) = Q \in \mathrm{Bru}_{A^T}$, we get the homeomorphism

$$\begin{array}{ccc} \mathcal{L} & \rightleftarrows & \mathcal{SCS}^n(Q) \\ L & \mapsto & \mathcal{B}(U, A\lambda^{-1}(L)) \end{array},$$

which shows that $\mathcal{SCS}^n(Q)$ is contractible. ■

4
The itinerary-cell stratification

Let $Q \in \mathrm{SO}_{n+1}$ be a fixed monodromy matrix. It is our aim in this chapter to decompose $\mathcal{L}\mathbb{S}^n(Q)$ into countably many contractible cells $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ parameterized by certain finite sequences $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ of elements in the group B^+_{n+1}. Given a finite sequence $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ of signed permutation matrices of order $n+1$ preserving orientation, one can define the (possibly empty) subset $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ of $\mathcal{L}\mathbb{S}^n(Q)$ of curves *along the itinerary* $\mathcal{I}$ as follows: given $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$, take the increasing sequence of times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k \leq 1$ for which $\mathfrak{F}_\gamma(\tau_j)$ belongs to a Bruhat cell of positive codimension in SO_{n+1} for all $j \in [k]_0$. We say that γ *has itinerary* $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ if the following conditions are met:

(a) $\forall j \in [k]_0 \ (\mathfrak{F}_\gamma(\tau_j) \in \mathrm{Bru}_{Q_j})$
(b) $\forall j \in [k] \ \forall t \in (\tau_{j-1}, \tau_j) \ (\mathfrak{F}_\gamma(t) \in \mathrm{Bru}_{\mathrm{chop}(Q_j)} \setminus \mathrm{Bru}_{Q_j})$
(c) $\forall t \in (\tau_k, 1] \ (\mathfrak{F}_\gamma(t) \in \mathrm{Bru}_{\mathrm{adv}(Q_k)})$

From *(a)*,*(b)* and *(c)* above, it is readily seen that the following are necessary conditions on the itinerary $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ for $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ to be non-empty:

(i) $\forall j \in [k]_0 \ (\mathrm{inv}(Q_j) < \frac{n(n+1)}{2})$
(ii) $Q_0 = I$
(iii) $\forall j \in [k] \ (\mathrm{adv}(Q_{j-1}) = \mathrm{chop}(Q_j))$
(iv) $\begin{cases} \mathrm{inv}(Q) < \frac{n(n+1)}{2} \longrightarrow Q \in \mathrm{Bru}_{Q_k} \\ \mathrm{inv}(Q) = \frac{n(n+1)}{2} \longrightarrow Q \in \mathrm{Bru}_{\mathrm{adv}(Q_k)} \end{cases}$

Itineraries $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ satisfying conditions *(i)* to *(iv)* above shall be called *acceptable itineraries*, the (countable) collection of which we shall denote by $\mathcal{A}(Q)$. The integer k is to be called the *length* of the itinerary $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$ and denoted by $|\mathcal{I}|$.

We are going to show shortly that, if these conditions are met by some itinerary $\mathcal{I}$, then the subset $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ is nonempty and contractible. The decomposition we are after will then read as

$$\mathcal{L}\mathbb{S}^n(Q) = \bigsqcup_{\mathcal{I} \in \mathcal{A}(Q)} \mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$$

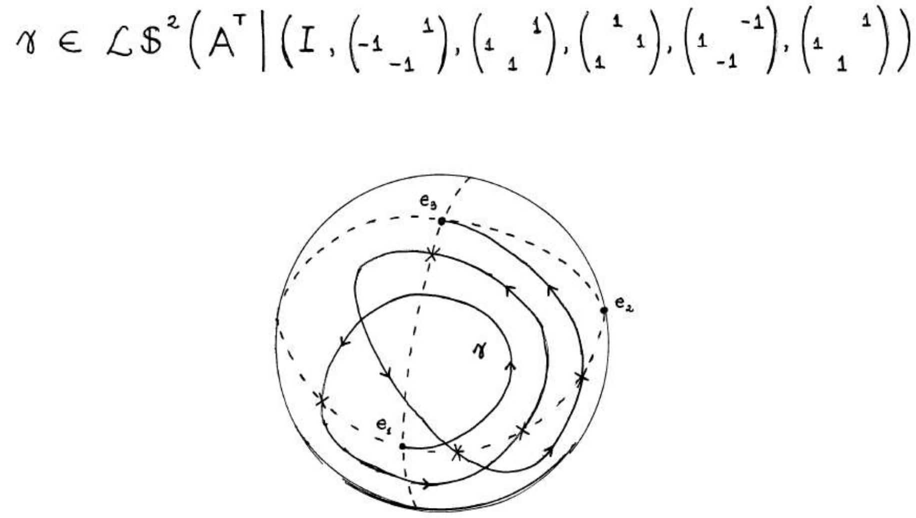

Figure 4.1: an acceptable itinerary of length 5. The crosses mark the moments of transition between open cells

These itinerary-cells $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ turn out to be finite-codimensional contractible Hilbert submanifolds of $\mathcal{L}\mathbb{S}^n(Q)$ whose closures are obtained by attaching finitely many itinerary-cells of greater codimension. Maybe an example in dimension $n = 2$ is in order so that the reader could have a glimpse of what comes next. Cf. figure 4.2 below

It is about time to state the main result of this thesis, the proof of which we postpone to the next section.

Theorem 4.1 *Given a monodromy matrix* $Q \in \mathrm{SO}_{n+1}$ *and an acceptable itinerary*

$$\mathcal{I} = (Q_0, Q_1, \cdots, Q_k) \in \mathcal{A}(Q),$$

the set $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ *is a contractible submanifold of* $\mathcal{L}\mathbb{S}^n(Q)$ *of codimension*

$$\operatorname{codim} \mathcal{L}\mathbb{S}^n(Q|\mathcal{I}) = \sum_{j=1}^{k-1}\left(\frac{n(n+1)}{2} - \operatorname{inv}(Q_j) - 1\right)$$

A consequence of theorem 4.1 is the existence of a contractible connected component comprised by the convex curves of $\mathcal{L}\mathbb{S}^n(Q)$, provided the monodromy Q allows for such curves to exist. This interesting result was first obtained in (13). In particular, the convex closed curves, which our work in

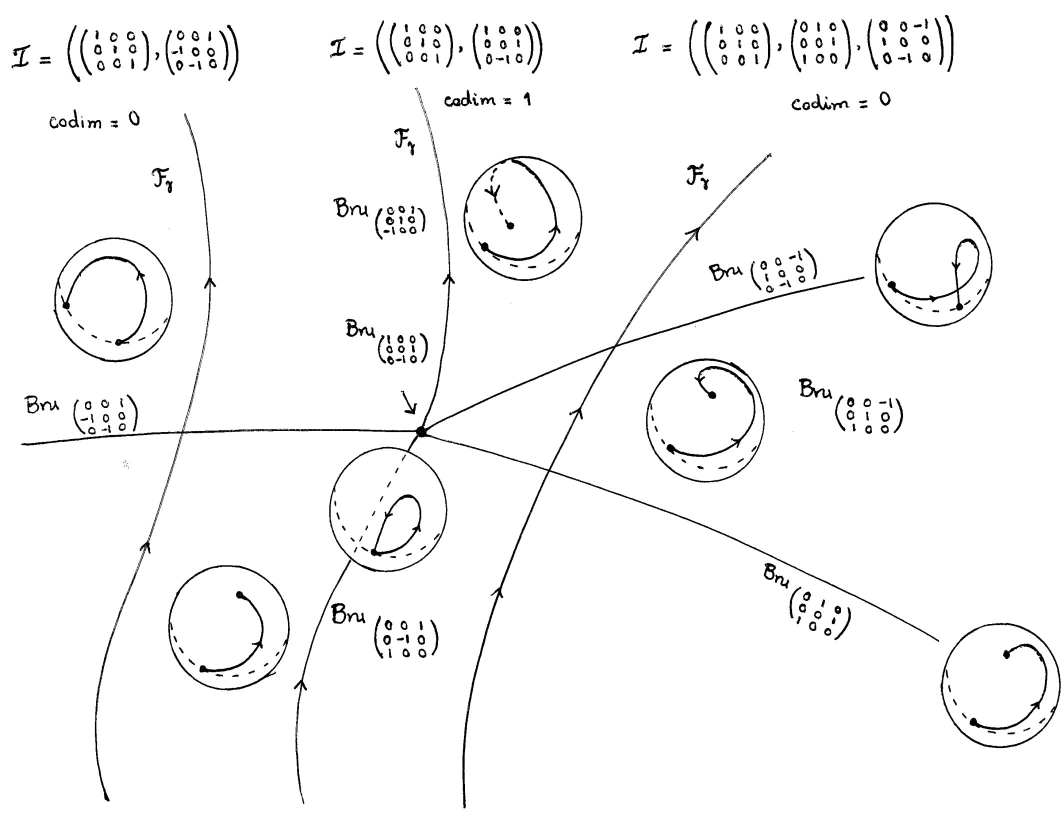

Figure 4.2: example in dimension $n = 2$: an itinerary-cell in the closure of two lower codimensional cells

chapter 3 showed to exist if and only if n is even, comprise a non-empty connected component of $\mathcal{L}\mathbb{S}^n(I)$, provided n is even.

Theorem 3.2 of chapter 3 yields at once that if the monodromy Q is in the Bruhat cell Bru_{A^T} of the transpose of the Arnol'd matrix then the shortest itinerary $\mathcal{I}_c = (I)$ yields a (non-empty) contractible space $\mathcal{SC}\mathbb{S}^n(Q) = \mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ comprised by the strictly convex curves in $\mathcal{L}\mathbb{S}^n(Q)$. Assuming theorem 4.1 for the time being, we can show that $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ is indeed a connected component. Moreover, even in the case we have only $\mathrm{chop}(Q) = A^T$ but $Q \notin \mathrm{Bru}_{A^T}$ (*i.e.*, Q is in an exit cell of Bru_{A^T}), the shortest itinerary $\mathcal{I}_c = (I, Bru(Q))$ still correspond to a (non-empty) contractible space of convex curves in $\mathcal{L}\mathbb{S}^n(Q)$ given by $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$. In any case, $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ is indeed a contractible connected component of $\mathcal{L}\mathbb{S}^n(Q)$.

Corollary 4.2 *If the monodromy matrix $Q \in \mathrm{SO}_{n+1}$ satisfies $\mathrm{chop}(Q) = A^T$, then there is a (non-empty) contractible connected component of convex curves in $\mathcal{L}\mathbb{S}^n(Q)$*

Proof. It suffices to show that there is no cell $\mathcal{L}\mathbb{S}^n(Q|\mathcal{J})$ in the closure of the shortest itinerary cell $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ other than $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$ itself. If there was such a cell $\mathcal{L}\mathbb{S}^n(Q|\mathcal{J})$, one could pick a curve γ in it and perturb its path of frames $\mathfrak{F}_\gamma$ holonomically in order to obtain Γ such that $\widetilde{\gamma} = \pi[\Gamma] \in \mathcal{L}\mathbb{S}^n(Q|\mathcal{I}_c)$. The main point of the argument is that in passing from γ to $\widetilde{\gamma}$ the length of the itinerary cannot generically decrease, that is, for a generic perturbation, one has

$$\mathcal{I}(\gamma) \leq \mathcal{I}(\widetilde{\gamma})$$

This follows from lemma 3.5, near the end of section 3.1. Indeed, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k \leq 1$ be the times at which $\mathfrak{F}_\gamma$ traverses positive codimension cells of capitals $Q_0, \cdots, Q_k \in \mathrm{B}^+_{n+1}$. The only way the itinerary could be shortened by a generic perturbation would be in the event that $\mathfrak{F}_\gamma$ get deflected from a positive codimension cell into an open cell, in the following exact terms: $\exists j \in [k]$ such that $\mathrm{chop}(Q_j) = \mathrm{adv}(Q_j)$, $\mathfrak{F}_\gamma(\tau_j) \in \mathrm{Bru}_{Q_j}$ and $\exists \epsilon > 0$ such that $\forall t \in [\tau_j - \epsilon, \tau_j + \epsilon]$ one has $\Gamma(t) \in \mathrm{Bru}_{\mathrm{chop}(Q_j)}$. If such a perturbation would exist then it could be made to affect only a small vincinity of $\mathfrak{F}_\gamma(\tau_j)$ so as to leave the endpoints $\mathfrak{F}_\gamma(\tau_j - \epsilon), \mathfrak{F}_\gamma(\tau_j + \epsilon) \in \mathrm{Bru}_{\mathrm{chop}(Q_j)}$ fixed, *i.e.*, so that

$$\mathfrak{F}_\gamma(\tau_j - \epsilon) = \Gamma(\tau_j - \epsilon) \text{ and } \mathfrak{F}_\gamma(\tau_j + \epsilon) = \Gamma(\tau_j + \epsilon)$$

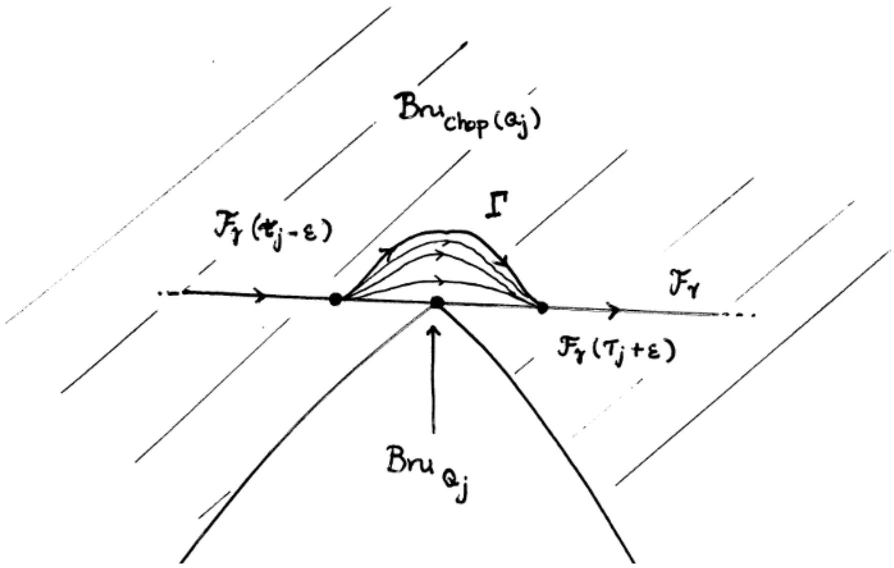

Figure 4.3: illustration of the forbidden perturbation

Since nondegeneracy is tantamount to local convexity (proposition 3.1), and convexity is ensured as long as a curve does not leave a open cell (proposition 3.4) , one would have a convex arc $\gamma : [\tau_j - \epsilon, \tau_j + \epsilon] \to \overline{\mathrm{Bru}_{\mathrm{chop}(Q_j)}}$ leaving the open cell $\mathrm{Bru}_{\mathrm{chop}(Q_j)}$ in τ_j and a convex arc $\widetilde{\gamma} : [\tau_j - \epsilon, \tau_j + \epsilon] \to \mathrm{Bru}_{\mathrm{chop}(Q_j)}$

with same endpoints in the open cell $\mathrm{Bru}_{\mathrm{chop}(Q_j)}$ that never gets to leave this cell. This contradicts lemma 3.5.

Since $\mathcal{I}_c$ is the shortest itinerary ever, we can never get to $\mathcal{L}\mathbb{S}^n\,(Q|\mathcal{I}_c)$ by a perturbation from a hypothetical boundary cell $\mathcal{L}\mathbb{S}^n\,(Q|\mathcal{J})$, and this concludes the proof. ■

Another result we obtain almost for free from theorem 4.1 is the emptiness or contratibility of each space of multiconvex curves in $\mathbb{S}^2$. In (4) the space of k-multiconvex curves in $\mathcal{L}\mathbb{S}^2\,(Q)$ is denoted by $\mathcal{M}_k$ and shown to be a contractible submanifold of codimension $2k-2$ (if non-empty). These multiconvex components $\mathcal{M}_k$, with k ranging over the positive integers, play a major role in determining the homotopy type of $\mathcal{L}\mathbb{S}^2\,(Q)$ in that paper. Therein, we find a definition of $\mathcal{M}_k$ that can be directly translated in terms of itineraries as

$$\mathcal{M}_k = \mathcal{L}\mathbb{S}^2\,(Q|\,(I, I, I, \cdots))$$

where there are k identity matrices and possibly a last entry $Bru\,(Q)$ (if the number of inversions of Q does not exceed 5) in the itinerary above. We are assuming that $\mathrm{chop}(Q) = A^T$, so that $(I, I, \cdots)$ is an admissible itinerary and $\mathcal{M}_k$ is not empty.

Inspired in the dual triangulation of a triangulated surface we conjecture the existence of abstract finite-dimensional manifolds $\mathcal{C}_Q\,(\mathcal{I})$ which assemble to yield a CW complex

$$\mathcal{C}_Q = \bigsqcup_{\mathcal{I}\in\mathcal{A}(Q)} \mathcal{C}_Q\,(\mathcal{I})$$

of the same homotopy type as $\mathcal{L}\mathbb{S}^n\,(Q)$. The cells $\mathcal{C}_Q\,(\mathcal{I})$ – which we imagine can be obtained exploring the triviality of the normal bundle of each itinerary-cell $\mathcal{L}\mathbb{S}^n\,(Q|\mathcal{I})$ (a feature yet to be proved) – are meant to exhibit the following properties:

1. for each $\mathcal{I} \in \mathcal{A}\,(Q)$, the subset $\mathcal{C}_Q\,(\mathcal{I})$ is a contractible manifold of finite dimension

$$\dim \mathcal{C}_Q\,(\mathcal{I}) = \mathrm{codim}\ \mathcal{L}\mathbb{S}^n\,(Q|\mathcal{I})$$

2. given $\mathcal{I} \in \mathcal{A}\,(Q)$, the closure of $\mathcal{C}_Q\,(\mathcal{I})$ is obtained by attaching finitely many cells $\mathcal{C}_Q\,(\mathcal{J})$ of strictly lower dimension than $\mathcal{C}_Q\,(\mathcal{I})$.

We hope to clarify in a subsequent work if such a homotopy equivalent CW model $\mathcal{C}_Q$ of $\mathcal{L}\mathbb{S}^n\,(Q)$ does exist. If it does, one might expect to be able to harvest topological information regarding $\mathcal{L}\mathbb{S}^n\,(Q)$ from it in a combinatorial fashion.

4.1 Proof of the main theorem

We shall prove theorem 4.1 by induction in the length k of the itinerary $\mathcal{I} = (Q_0, Q_1, \cdots, Q_k)$. In the course of this endeavour we are going to rely repeatedly on the fact that if a Hilbert manifold M fibres over a contractible basis B with contractible fibres modeled after a (contractible) space F, then it is diffeomorphic to a Hilbert space, hence contractible. Indeed, the long exact sequence in homotopy

$$\cdots \longrightarrow \pi_j(F) \longrightarrow \pi_j(M) \longrightarrow \pi_j(B) \longrightarrow \pi_{j-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(M) \longrightarrow 0$$

implies that if $\pi_j\left(F\right)$ and $\pi_j\left(B\right)$ are trivial for all j then so is $\pi_j\left(M\right)$, hence M is weakly homotopy equivalent to a Hilbert space. The conclusion follows from the fact that two (infinite-dimensional) Hilbert manifolds are diffeomorphic if and only if they are weakly homotopy equivalent (15).

Since we do not want to multiply symbols for ancillary constructions, let us make the convention that all the fibration maps that are to make an appearance in what follows are to be assigned the same generic letter f. The context will make it clear which fibration we are talking about.

We restate theorem 4.1 for sake of reader's convenience:

Theorem 4.1 *Given a monodromy matrix $Q \in \mathrm{SO}_{n+1}$ and an acceptable itinerary*

$$\mathcal{I} = (Q_0, Q_1, \cdots, Q_k) \in \mathcal{A}\left(Q\right),$$

the set $\mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right)$ is a contractible submanifold of $\mathcal{L}\mathbb{S}^n\left(Q\right)$ of codimension

$$\operatorname{codim} \mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right) = \sum_{j=1}^{k-1}\left(\frac{n\left(n+1\right)}{2} - \operatorname{inv}\left(Q_j\right) - 1\right)$$

First of all, we observe that $\mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right)$ fibres over $\mathcal{L}\mathbb{S}^n\left(Q_k|\mathcal{I}\right)$ with contractible fibres. We define the fibration $f : \mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right) \to \mathcal{L}\mathbb{S}^n\left(Q_k|\mathcal{I}\right)$ as follows. Given $\gamma \in \mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right)$, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k \leq 1$ be the increasing sequence of times for which $\mathfrak{F}_\gamma\left(\tau_j\right)$ belongs to the Bruhat cell Bru_{Q_j} of positive codimension in SO_{n+1} for all $j \in [k]_0$ and let $U_\gamma \in \mathrm{Up}^1_{n+1}$ be such that its Bruhat action takes $\mathfrak{F}_\gamma\left(\tau_k\right)$ to the capital Q_k of its Bruhat cell:

$$\mathcal{B}\left(U_\gamma, \mathfrak{F}_\gamma\left(\tau_k\right)\right) = Q_k$$

We can let U_γ act over the whole curve, and it does so in a way that preserves the itinerary of γ. Then we define $f\left[\gamma\right] \in \mathcal{L}\mathbb{S}^n\left(Q_k|\mathcal{I}\right)$ by a reparameterization over the unit interval $[0,1]$ of the restriction to $[0,\tau_k]$ of the U_γ*-modified curve*

$$\pi\left[\mathcal{B}\left(U_\gamma,\mathfrak{F}_\gamma\left(t\right)\right)\right]=\mathcal{B}\left(U_\gamma,\mathfrak{F}_\gamma\left(t\right)\right)\cdot e_1$$

Now, the fibres of $f:\mathcal{L}\mathbb{S}^n\left(Q|\mathcal{I}\right)\to\mathcal{L}\mathbb{S}^n\left(Q_k|\mathcal{I}\right)$ are modeled after the space of endpieces of the U_γ-modified curves

$$\widetilde{\mathrm{Ac}}\left(\mathrm{adv}\left(Q_k\right)|Q_k\right)=\left\{\begin{array}{c|l}\Gamma:[0,1]\to\overline{\mathrm{Bru}_{\mathrm{adv}(Q_k)}} & \Gamma\left(0\right)=Q_k \text{ and}\\ \text{holonomic} & \forall t\in(0,1]\\ & \left(\Gamma\left(t\right)\in\mathrm{Bru}_{\mathrm{adv}(Q_k)}\right)\end{array}\right\}$$

which in turn fibres over the set of frames in $\mathrm{Bru}_{\mathrm{adv}(Q_k)}$ convexly accessible from Q_k, defined as

$$\mathrm{Ac}\left(\mathrm{adv}\left(Q_k\right)|Q_k\right)=\left\{X\in\mathrm{Bru}_{\mathrm{adv}(Q_k)}\;\middle|\;\begin{array}{l}\exists\;\Gamma:[0,1]\to\overline{\mathrm{Bru}_{\mathrm{adv}(Q_k)}}\text{ holonomic}\\ \Gamma\left(0\right)=Q_k,\\ \forall t\in(0,1]\left(\Gamma\left(t\right)\in\mathrm{Bru}_{\mathrm{adv}(Q_k)}\right)\\ \text{and }\Gamma\left(1\right)=X\end{array}\right\}$$

with bundle map given by $f:\Gamma\mapsto\Gamma\left(1\right)$. These are instances of the following general definition.

Given $P,Q\in\ \mathrm{B}_{n+1}^{+}$, we define the subset of Bru_Q of *frames convexly accessible* from P as the set

$$\mathrm{Ac}\left(Q|P\right)=\left\{X\in\mathrm{Bru}_Q\;\middle|\;\begin{array}{l}\exists\;\Gamma:[0,1]\to\overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}\text{ holonomic}\\ \Gamma\left(0\right)=P,\ \forall t\in(0,1)\left(\Gamma\left(t\right)\in\mathrm{Bru}_{\mathrm{chop}(Q)}\right)\\ \text{and }\Gamma\left(1\right)=X\end{array}\right\}$$

Provided it is not empty, it is the base space of the fibred space

$$\widetilde{\mathrm{Ac}}\left(Q|P\right)=\left\{\begin{array}{c|l}\Gamma:[0,1]\to\overline{\mathrm{Bru}_{\mathrm{chop}(Q)}} & \Gamma\left(0\right)=P,\\ \text{holonomic} & \forall t\in(0,1)\left(\Gamma\left(t\right)\in\mathrm{Bru}_{\mathrm{chop}(Q)}\right)\\ & \text{and }\Gamma\left(1\right)\in\mathrm{Bru}_Q\end{array}\right\}$$

under the fibration map $f:\Gamma\mapsto\Gamma\left(1\right)$.

For reader's convenience, we postpone to the next section the proof of the two following rather technical results, which are to be invoked once again shortly.

Lemma 4.3 $\mathrm{Ac}(Q|P)$ *is a non-empty, open and contractible subset of* Bru_Q *if and only if* $\mathrm{chop}(Q)=\ \mathrm{adv}(P)$.

Lemma 4.4 *If* $\operatorname{chop}(Q) = \operatorname{adv}(P)$*, then the fibres* $f^{-1}[X]$ *of the fibration*

$$\begin{array}{rccc} f: & \widetilde{\mathrm{Ac}}(Q|P) & \longrightarrow & \mathrm{Ac}(Q|P) \\ & \Gamma & \mapsto & \Gamma(1) \end{array}$$

are non-empty and contractible.

Since we can endow $\widetilde{\mathrm{Ac}}(Q|P)$ with a Hilbert manifold structure, it follows from the two lemmas above and the remark in the first paragraph of this section that $\widetilde{\mathrm{Ac}}(Q|P)$ is contractible. This concludes the proof that the fibres of $f : \mathcal{L}\mathbb{S}^n(Q|\mathcal{I}) \to \mathcal{L}\mathbb{S}^n(Q_k|\mathcal{I})$ are contractible and we have thus reduced the problem of establishing the non-emptiness and contractibility of $\mathcal{L}\mathbb{S}^n(Q|\mathcal{I})$ to the same problem now regarding $\mathcal{L}\mathbb{S}^n(Q_k|(Q_0, Q_1, \cdots, Q_k))$. This will be done by an induction argument in k, as we pointed above.

For $k = 1$, let $Q \in \mathrm{B}^+_{n+1}$ satisfy $(I, Q) \in \mathcal{A}(Q)$, *i.e.*, $\operatorname{chop}(Q) = \operatorname{adv}(I) = A^T$. In other words, Q is the capital of an exit cell of Bru_{A^T}. The space $\mathcal{L}\mathbb{S}^n(Q|(I, Q))$ fibres over the contractible base $\mathcal{L}\mathbb{S}^n\left(A^T|(I)\right)$ under the map $f : \mathcal{L}\mathbb{S}^n(Q|(I, Q)) \to \mathcal{L}\mathbb{S}^n\left(A^T|(I)\right)$ defined as follows. Given $\gamma \in \mathcal{L}\mathbb{S}^n(Q|(I, Q))$, pick the upper matrix $U_{1/2} \in \mathrm{Up}^1_{n+1}$ such that

$$\mathcal{B}\left(U_{1/2}, \mathfrak{F}_\gamma\left(\frac{1}{2}\right)\right) = A^T$$

and define $f[\gamma] \in \mathcal{L}\mathbb{S}^n\left(A^T|(I)\right)$ by

$$\begin{aligned} f[\gamma](t) &= \pi\left[\mathcal{B}\left(U_{1/2}, \mathfrak{F}_\gamma\left(\frac{t}{2}\right)\right)\right] \\ &= \mathcal{B}\left(U_{1/2}, \mathfrak{F}_\gamma\left(\frac{t}{2}\right)\right) \cdot e_1 \end{aligned}$$

The fibres of $f : \mathcal{L}\mathbb{S}^n(Q|(I, Q)) \to \mathcal{L}\mathbb{S}^n\left(A^T|(I)\right)$ are clearly modeled after the contractible set $\widetilde{\mathrm{Ac}}\left(Q|A^T\right)$ of endpieces of $U_{1/2}$-modified curves (herein we invoke lemmas 4.3 and 4.4 stated above once again for non-emptiness and contractibility). Therefore, the fibred space $\mathcal{L}\mathbb{S}^n(Q|(I, Q))$ is non-empty and contractible and the induction basis is verified.

As for the inductive step, consider the fibration

$$f : \mathcal{L}\mathbb{S}^n(Q_k|(Q_0, Q_1, \cdots, Q_k)) \longrightarrow \mathcal{L}\mathbb{S}^n(Q_{k-1}|(Q_0, Q_1, \cdots, Q_{k-1}))$$

defined as follows. Given $\gamma \in \mathcal{L}\mathbb{S}^n(Q_k|(Q_0, Q_1, \cdots, Q_k))$, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k = 1$ be the increasing sequence of times for which $\mathfrak{F}_\gamma(\tau_j)$ belongs to the Bruhat cell Bru_{Q_j} of positive codimension in SO_{n+1} for all $j \in [k]_0$ and let $U_\gamma \in \mathrm{Up}^1_{n+1}$ be such that its Bruhat action takes $\mathfrak{F}_\gamma(\tau_{k-1})$ to the capital

Q_{k-1} of its Bruhat cell:

$$\mathcal{B}\left(U_\gamma, \mathfrak{F}_\gamma\left(\tau_{k-1}\right)\right) = Q_{k-1}$$

Then we define $f[\gamma] \in \mathcal{LS}^n\left(Q_k|\mathcal{I}\right)$ by a reparameterization over the unit interval $[0,1]$ of the restriction to $[0, \tau_{k-1}]$ of the U_γ-*modified curve*

$$\pi\left[\mathcal{B}\left(U_\gamma, \mathfrak{F}_\gamma(t)\right)\right] = \mathcal{B}\left(U_\gamma, \mathfrak{F}_\gamma(t)\right) \cdot e_1$$

Now, the fibres of f are modeled after the space $\widetilde{\mathrm{Ac}}\left(Q_k|Q_{k-1}\right)$ of endpieces of the U_γ-modified curves, which, by lemmas 4.3 and 4.4 above, is contractible. Since, by induction hypothesis, $\mathcal{LS}^n\left(Q_{k-1}|\left(Q_0, Q_1, \cdots, Q_{k-1}\right)\right)$ is non-empty and contractible, it follows by induction that so is $\mathcal{LS}^n\left(Q_k|\left(Q_0, Q_1, \cdots, Q_k\right)\right)$.

As for the codimension of $\mathcal{LS}^n\left(Q|\mathcal{I}\right)$ in $\mathcal{LS}^n\left(Q\right)$, we could have built the result in the induction argument above, but refrained from doing so for the sake of clarity. Resuming the induction argument, a simple geometrical reasoning will establish the induction basis, to wit:

$$\operatorname{codim} \mathcal{LS}^n\left(Q|\mathcal{I}\right) = \frac{n(n+1)}{2} - \operatorname{inv}(Q) - 1$$

The reader would have noticed that this is nothing but the codimension of Bru_Q in SO_{n+1} minus one. Indeed, if Bru_Q had codimension one in SO_{n+1}, every generic perturbation of a curve in $\mathcal{LS}^n\left(Q|\mathcal{I}\right)$ would still yield a curve in $\mathcal{LS}^n\left(Q|\mathcal{I}\right)$, making clear that the codimension of $\mathcal{LS}^n\left(Q|\mathcal{I}\right)$ in $\mathcal{LS}^n\left(Q\right)$ should be zero. Besides, each extra independent direction in SO_{n+1} one can evade Bru_Q (not taking into acount the direction which punctures Bru_Q holonomically – coming from $\mathrm{Bru}_{\mathrm{chop}(Q)}$ and going to $\mathrm{Bru}_{\mathrm{adv}(Q)}$) allows for an independent generic pertubation of $\gamma \in \mathcal{LS}^n\left(Q|\mathcal{I}\right)$ into a curve with another itinerary, increasing the codimension of $\mathcal{LS}^n\left(Q|\mathcal{I}\right)$ in one unit. [Notice, *en passant*, that there are only finitely many itineraries available to the perturbed curve]. This establishes the induction basis. Now, since $\mathcal{LS}^n\left(Q_k|\left(Q_0, Q_1, \cdots, Q_k\right)\right)$ fibres over $\mathcal{LS}^n\left(Q_{k-1}|\left(Q_0, Q_1, \cdots, Q_{k-1}\right)\right)$ with fibres modeled after $\widetilde{\mathrm{Ac}}\left(Q_k|Q_{k-1}\right)$, an argument similar to the one used above to justify the induction basis will show that

$$\begin{aligned}\operatorname{codim} \mathcal{LS}^n\left(Q_k|\left(Q_0, Q_1, \cdots, Q_k\right)\right) &= \operatorname{codim} \mathcal{LS}^n\left(Q_{k-1}|\left(Q_0, Q_1, \cdots, Q_{k-1}\right)\right) \\ &\quad + \frac{n(n+1)}{2} - \operatorname{inv}(Q) - 1,\end{aligned}$$

completing the induction step.

4.2
Some technical results

Lemma 4.3 *Given $P, Q \in \mathrm{B}_{n+1}^{+}$, the set*

$$\mathrm{Ac}\left(Q|P\right)=\left\{X\in\mathrm{Bru}_Q\;\middle|\;\begin{array}{l}\exists\,\Gamma:[0,1]\to\overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}\text{ holonomic}\\ \Gamma\left(0\right)=P,\,\forall t\in\left(0,1\right)\left(\Gamma\left(t\right)\in\mathrm{Bru}_{\mathrm{chop}(Q)}\right)\\ \text{and }\Gamma\left(1\right)=X\end{array}\right\}$$

is a non-empty, open and contractible subset of Bru_Q *if and only if* $\mathrm{chop}(Q) = \mathrm{adv}(P)$.

Proof. The "only if" part is obvious. Let us prove that $\mathrm{Ac}(Q|P) \neq \emptyset$ if $\mathrm{chop}(Q) = \mathrm{adv}(P)$.

Fix once and for all a holonomic path $\Gamma_0 : \left[0, \frac{1}{2}\right] \to \overline{\mathrm{Bru}_{\mathrm{adv}(P)}} = \overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}$ such that

$$\Gamma_0\left(0\right)=P\text{ and }\forall t\in\left(0,\frac{1}{2}\right]\;\left(\Gamma_0\left(t\right)\in\mathrm{Bru}_{\mathrm{adv}(P)}\right)$$

and a holonomic path $\Gamma_1 : \left[\frac{1}{2}, 1\right] \to \overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}$ such that

$$\forall t\in\left[\frac{1}{2},1\right)\left(\Gamma_1\left(t\right)\in\mathrm{Bru}_{\mathrm{chop}(Q)}\right)\text{ and }\Gamma_1\left(1\right)=Q$$

Now, pick the upper matrix $U \in \mathrm{Up}_{n+1}^{1}$ such that

$$\mathcal{B}\left(U,\Gamma_1\left(\frac{1}{2}\right)\right)=\Gamma_0\left(\frac{1}{2}\right)$$

and define a clearly holonomic path $\Gamma : [0,1] \to \overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}$ by

$$\Gamma\left(t\right)=\begin{cases}\Gamma_0\left(t\right),\text{ if }0\leq t\leq\frac{1}{2}\\ \mathcal{B}\left(U,\Gamma_1\left(t\right)\right),\text{ if }\frac{1}{2}<t\leq 1\end{cases}$$

Since $\Gamma\left(1\right) = \mathcal{B}\left(U,\Gamma_1\left(1\right)\right) = \mathcal{B}\left(U,Q\right) \in \mathrm{Bru}_Q$, we have $\Gamma\left(1\right) \in \mathrm{Ac}(Q|P)$.

Now for the contractibility. We shall proceed by induction in the dimension of Bru_Q, *i.e.*, in the number $\mathrm{inv}(Q)$.

If $\mathrm{inv}(Q) = 0$, then $\mathrm{Bru}_Q = \{Q\}$ and the result is trivial. If $\mathrm{inv}(Q) \geq 1$, then Bru_Q fibres over one of its boundary cells $\mathrm{Bru}_{\widehat{Q}}$, with $\widehat{Q}$ obtained from Q according to the following rule: take the leftmost two contiguous columns of Q that contain an inversion. These are the j_0-th and the (j_0+1)-th columns such that

$$\sigma\left(j_0\right)>\sigma\left(j_0+1\right)$$

and

$$\sigma\left(j\right) < \sigma\left(k\right) \text{ for all } k > j \geq j_0,$$

where $\Pi\left[Q\right] = P_\sigma$ (in the notation of section 2.2). Now, in order to obtain $\widehat{Q}$, we forget the signs of the columns j and $(j+1)$ only (preserving the signs of the remaining columns), switch the columns j and $(j+1)$ of the resulting matrix (again preserving the remaining columns), and finally multiply the $(j+1)$-th column of the resulting matrix by -1, as in the following illustrative example.

$$Q = \begin{pmatrix} & 1 & & & & \\ & & & -\mathbf{1} & & \\ -1 & & & & & \\ & & & & 1 & \\ & & \mathbf{1} & & & \\ & & & & & 1 \end{pmatrix} \gtrdot \begin{pmatrix} & 1 & & & & \\ & & \mathbf{1} & & & \\ -1 & & & & & \\ & & & & 1 & \\ & & & \mathbf{1} & & \\ & & & & & 1 \end{pmatrix} = \widehat{Q}$$

It is straightforward to see that

$$\operatorname{inv}\left(\widehat{Q}\right) = \operatorname{inv}\left(Q\right) - 1$$

so that

$$\dim \mathrm{Bru}_{\widehat{Q}} = \dim \mathrm{Bru}_Q - 1.$$

Using the system of coordinates for Bru_Q introduced in section 2.2, we see that there is a free coordinate u in the $\sigma\left(j+1\right)$-th row and the j-th column of a generic $\mathcal{B}\left(U,Q\right) \in \mathrm{Bru}_Q$. Now, the limit $\lim\limits_{u\to\infty} \mathcal{B}\left(U,Q\right)$, intended as leaving fixed the remaining U-coordinates, is easily seen to be an element of $\mathrm{Bru}_{\widehat{Q}}$, regardless of the particular $\mathcal{B}\left(U,Q\right) \in \mathrm{Bru}_Q$ we started with. We thus define the bundle map

$$f : \mathrm{Bru}_Q \to \mathrm{Bru}_{\widehat{Q}}$$

by

$$f\left(\mathcal{B}\left(U,Q\right)\right) = \lim_{u\to\infty} \mathcal{B}\left(U,Q\right)$$

Notice that $\widehat{Q} \lessdot Q$ implies $\mathrm{Bru}_{\widehat{Q}} \subset \overline{\mathrm{Bru}_Q}$ indeed (cf. proposition 2.4). Moreover, one has $f\left(Q\right) = \widehat{Q}$.

We are now going to show that this line bundle structure of Bru_Q over $\mathrm{Bru}_{f(Q)}$ restricts to a fibration of $\mathrm{Ac}(Q|P)$ over $\mathrm{Ac}(f\left(Q\right)|P)$ whose fibres are final intervals of the line. More explicitly, let $\Gamma_0 \in \widetilde{\mathrm{Ac}}\left(Q|P\right)$ and $X_0 \equiv \Gamma_0\left(1\right) = \mathcal{B}\left(U_0,Q\right) \in \mathrm{Ac}(Q|P) \subseteq \mathrm{Bru}_Q$ and consider the one-parameter family of upper matrices $U\left(t\right) \in \mathrm{Up}^1_{n+1}$ with the exact same entries as U_0 but the forementioned $(\sigma(j+1), j)$-entry, which is u_0 for U_0, and

$$u(t) = u_0 + \frac{t}{1-t}$$

for $U(t)$, for $0 \le t < 1$, so that $u(0) = u_0$ and $U(0) = U_0$. Now, define the continuous and piecewise smooth curve $\Gamma_1 : [0,1] \to \overline{\mathrm{Bru}_Q}$ by

$$\Gamma_1(t) = \begin{cases} \mathcal{B}(U(t), Q), & \text{if } 0 \le t < 1 \\ f(X_0), & \text{if } t = 1 \end{cases}$$

Since $\Gamma_1(0) = \Gamma_0(1) = X_0$, we are allowed to weld Γ_0 and Γ_1 together obtaining (under a reparameterization) a continuous and piecewise smooth curve

$$\Gamma = \Gamma_0 * \Gamma_1 : [0,1] \to \overline{\mathrm{Bru}_{\mathrm{chop}(Q)}} = \overline{\mathrm{Bru}_{\mathrm{chop}(f(Q))}}$$

such that

$$\Gamma(0) = P,$$
$$\forall t \in \left(0, \frac{1}{2}\right) \left(\Gamma(t) \in \mathrm{Bru}_{\mathrm{chop}(Q)}\right),$$
$$\Gamma\left(\frac{1}{2}\right) = X_0 \in \mathrm{Bru}_Q,$$
$$\forall t \in \left(\frac{1}{2}, 1\right) (\Gamma(t) \in \mathrm{Bru}_Q)$$

and

$$\Gamma(1) = f(X_0) \in \mathrm{Bru}_{f(Q)}.$$

Problem is that Γ is not holonomic, for Γ_1 is not holonomic. Fortunately, this is not a dead end, for Γ is *quasi-holonomic* in a sense to be defined in the following paragraph and can be perturbed slightly to yield a holonomic arc $\widetilde{\Gamma}$ wholly contained in $\mathrm{Bru}_{\mathrm{chop}(f(Q))} = \mathrm{Bru}_{\mathrm{chop}(Q)}$ except for its endpoints $\widetilde{\Gamma}(0) = \Gamma(0) = P$ and $\widetilde{\Gamma}(1) = \Gamma(1) = f(X_0) \in \mathrm{Bru}_{f(Q)}$ which are preserved by this perturbation. Therefore, $\widetilde{\Gamma} \in \widetilde{\mathrm{Ac}}(f(Q)|P)$, hence $f(X_0) = \widetilde{\Gamma}(1) \in \mathrm{Ac}(f(Q)|P)$. This shows that $\mathrm{Ac}(Q|P)$ fibres over $\mathrm{Ac}(f(Q)|P)$. By construction, if $u > u_0$, then $\mathcal{B}(U(u), Q) \in f^{-1}(f(X_0)) \cap \mathrm{Ac}(Q|P)$, what shows that the fibres of $f|\mathrm{Ac}(Q|P)$ are final intervals of the corresponding fibres in Bru_Q.

All that remains to prove is the claim we made in the former paragraph stating the existence of well-behaved holonomic perturbations of the quasi-holonomic path Γ, in the following sense:

Claim: There is a holonomic arc $\widetilde{\Gamma} : [0,1] \to \overline{\mathrm{Bru}_{\mathrm{chop}(f(Q))}} = \overline{\mathrm{Bru}_{\mathrm{adv}(P)}}$ satisfying the following conditions:

(i) $\widetilde{\Gamma}(0) = P$
(ii) $\widetilde{\Gamma}(1) = f(X_0)$
(iii) $\forall t \in (0,1)\left(\widetilde{\Gamma}(t) \in \mathrm{Bru}_{\mathrm{chop}(f(Q))}\right)$

Proof of the claim. Up to a multiplication by a diagonal element $D \in \mathrm{Diag}^+_{n+1} < \mathrm{B}^+_{n+1}$, there is no loss of generality in assuming

$$\mathrm{adv}(P) = \mathrm{chop}(Q) = \mathrm{chop}(f(Q)) = A^T$$

Consider the translated open cell $A^T{\cdot}\mathrm{Bru}_{A^T}$. It clearly contains the element $\Omega = (-1)^n I = \left(A^T\right)^2$, but even more is true: it intersects each exit cell of Bru_{A^T}. In fact, given $S \in \mathrm{B}^+_{n+1}$ such that $\mathrm{chop}(S) = A^T$, take a short holonomic path beginning at some point interior to Bru_{A^T} and ending up in S and act on this short holonomic path through the Bruhat action $\mathcal{B}$ in order to obtain a strictly convex curve γ with initial frame A^T and final frame $\mathcal{B}(U.,S) \in \mathrm{Bru}_S$ for a certain $U. \in \mathrm{Up}^1_{n+1}$. By definition of strict convexity and theorem 3.2, we have got

$$\mathcal{B}(U.,S) \in \mathrm{Bru}_S \cap A^T \cdot \mathrm{Bru}_{A^T}$$

In particular, we have

$$\mathrm{Bru}_Q \cap A^T \cdot \mathrm{Bru}_{A^T} \neq \emptyset \quad \text{and} \quad \mathrm{Bru}_{f(Q)} \cap A^T \cdot \mathrm{Bru}_{A^T} \neq \emptyset$$

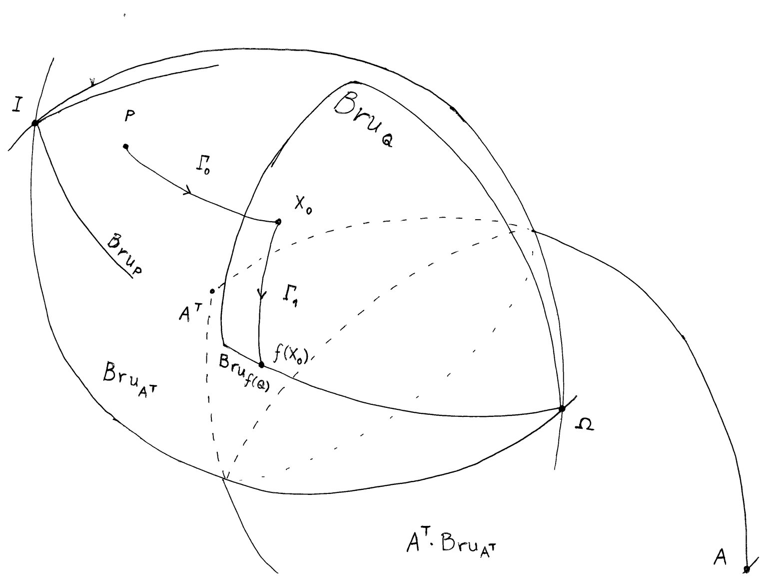

Figure 4.4: quasi-holonomic path $\Gamma = \Gamma_0 * \Gamma_1$ and translated cell $A^T \cdot \mathrm{Bru}_{A^T}$

There is a very handy system of coordinates in $A^T \cdot \mathrm{Bru}_{A^T}$ given by

$$\begin{array}{rccc} \varphi : & \mathrm{Lo}^1_{n+1} & \overset{\text{diffeo}}{\longrightarrow} & A^T \cdot \mathrm{Bru}_{A^T} \\ & L & \mapsto & (-1)^n \mathbf{Q}(L) \end{array}$$

and satisfying $\varphi(I) = \Omega$, and one can use it to see that, for $S \in \mathrm{B}^+_{n+1}$,

$$\mathrm{chop}(S) = A^T \longrightarrow \Omega \in \partial \mathrm{Bru}_S$$

In fact, allowing each one of the $\mathrm{inv}(S)$ free U-variables in $\mathcal{B}(U, S) \in \mathrm{Bru}_S \cap A^T \cdot \mathrm{Bru}_{A^T}$ to tend towards infinity at a time, in a continuous and increasing manner, we get a continuous path

$$\begin{gathered} \mathcal{B}(U_0, S_0) \overset{1}{\longrightarrow} \mathcal{B}(U_1, S_1) \overset{2}{\longrightarrow} \cdots \overset{j}{\longrightarrow} \mathcal{B}(U_j, S_j) \overset{j+1}{\longrightarrow} \cdots \\ \cdots \overset{\mathrm{inv}(S)}{\longrightarrow} \mathcal{B}\left(I, S_{\mathrm{inv}(S)}\right) = \Omega, \end{gathered} \tag{4-1}$$

where, abusing f-notation as usual, we have

$$S_0 = S,$$

$$f(S_j) = S_{j+1}$$

and

$$S_{\mathrm{inv}(S)} = \Omega.$$

In the system of coordinates φ, this same path is obtained by making all the $\mathrm{inv}(S)$ free L-variables of the submanifold $\varphi^{-1}(\mathrm{Bru}_S)$ of Lo^1_{n+1} go to zero, one at a time:

$$\begin{gathered} (-1)^n \mathbf{Q}(L_0) \overset{1}{\longrightarrow} (-1)^n \mathbf{Q}(L_1) \overset{2}{\longrightarrow} \cdots \overset{j}{\longrightarrow} (-1)^n \mathbf{Q}(L_j) \overset{j+1}{\longrightarrow} \cdots \\ \cdots \overset{\mathrm{inv}(S)}{\longrightarrow} (-1)^n \mathbf{Q}(I) = \Omega \end{gathered}$$

In the sequences above, we have

$$\mathcal{B}\left(U_j, f^j(S)\right) = (-1)^n \mathbf{Q}(L_j)$$

for all $j \in [\mathrm{inv}(S)]_0$, and the j-th piece of path between the endpoints $\mathcal{B}(U_{j-1}, f^{j-1}(S))$ and $\mathcal{B}(U_j, f^j(S))$ is wholly contained in $\mathrm{Bru}_{f^{j-1}(S)}$ except for its endpoints.

Use the natural diffeomorphism

$$\mathrm{Lo}^1_{n+1} \rightleftarrows \mathbb{R}^{n(n+1)/2}$$

$$\begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n1} & x_{n2} & \cdots & x_{n,n-1} & 1 \end{pmatrix} \rightleftarrows (x_{21}, x_{31}, x_{32}, \cdots, x_{n1}, \cdots x_{n,n-1})$$

to pull back the Euclidean norm from $\mathbb{R}^{n(n+1)/2}$ to Lo_{n+1}^{1}, and take a small open neighbourhood $B(\Omega, r)$ of $\Omega = \varphi(I)$ in $A^T \cdot \mathrm{Bru}_{A^T}$ of radius $r > 0$. Therefore, if the initial point $\mathcal{B}(U_0, S) = (-1)^n \mathbf{Q}(L_0) \in \mathrm{Bru}_S \cap A^T \cdot \mathrm{Bru}_{A^T}$ in the continuous path (4-1) is taken with small enough free L-variables, we can assume that the whole path in contained in $B(\Omega, r)$, what shows at once that

$$\mathrm{chop}(S) = A^T \longrightarrow \Omega \in \partial \mathrm{Bru}_S$$

Let us get back to our quasi-holonomic curve Γ with endpoints P and $f(X_0)$. We call it *quasi-holonomic* because an easy computation shows that its logarithmic derivative is of the form

$$\Lambda_\Gamma(t) = \begin{pmatrix} 0 & -c_1(t) & & & \\ c_1(t) & 0 & -c_2(t) & & \\ & c_2(t) & \ddots & \ddots & \\ & & \ddots & 0 & -c_n(t) \\ & & & c_n(t) & 0 \end{pmatrix}$$

with non-negative piecewise smooth functions $c_1, \cdots, c_n : [0,1] \to [0, +\infty)$. It is worth emphasizing that, unlike the holonomic case, herein the functions $c_1, \cdots, c_n$ are allowed to attain zero value. Let $U_0 \in \mathrm{Up}_{n+1}^{1}$ be such that $\mathcal{B}(U_0, X_0) = (-1)^n \mathbf{Q}(L_0) \in \mathrm{Bru}_Q \cap A^T \cdot \mathrm{Bru}_{A^T}$ and have small enough L-variables so as to ensure that $\mathcal{B}(U_0, X_0) \in B(\Omega, r)$. Consider the *projective transformation*

$$\begin{array}{rccl} T: & \mathrm{SO}_{n+1} & \to & \mathrm{SO}_{n+1} \\ & X & \mapsto & T(X) = \mathcal{B}(U_0, X) \end{array}$$

and the projective transform of Γ given by

$$T[\Gamma](t) = T(\Gamma(t)).$$

It satisfies the following conditions:

$$T[\Gamma](0) \in \mathrm{Bru}_P,$$

$$\forall t \in \left(0, \frac{1}{2}\right) \left(T\left[\Gamma\right](t) \in \ \mathrm{Bru}_{A^T}\right),$$

$$T\left[\Gamma\right]\left(\frac{1}{2}\right) = T\left(X_0\right) \in \ \mathrm{Bru}_Q \cap B\left(\Omega, r\right),$$

$$\forall t \in \left(\frac{1}{2}, 1\right) \left(T\left[\Gamma\right](t) \in \ \mathrm{Bru}_Q \cap B\left(\Omega, r\right)\right)$$

and

$$T\left[\Gamma\right](1) = T\left(f\left(X_0\right)\right) \in \ \mathrm{Bru}_{f(Q)} \cap B\left(\Omega, r\right).$$

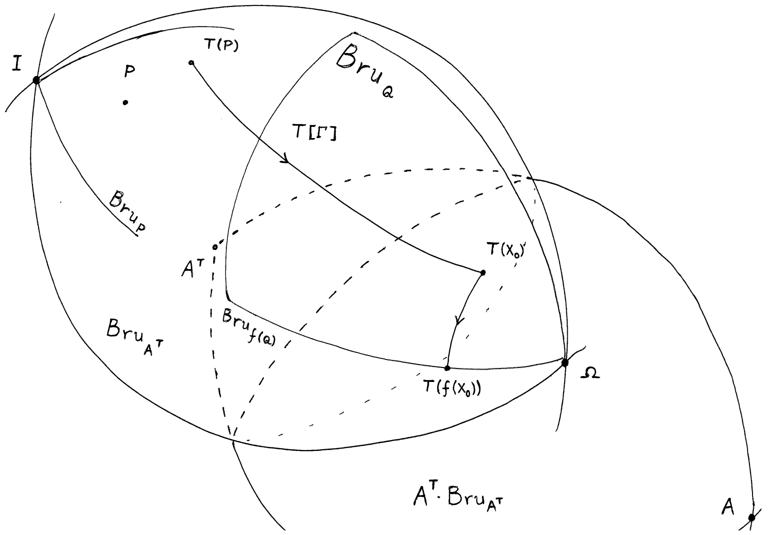

Figure 4.5: a projective transformation T takes the quasi-holonomic piece Γ_1 close enough to Ω

Let $t_* \in \left(0, \frac{1}{2}\right)$ be such that $X_* = T\left[\Gamma\right](t_*) \in B\left(\Omega, r\right) \cap \mathrm{Bru}_{A^T}$. The restricted path

$$\Phi(t) \equiv T\left[\Gamma\right](t), \ t \in \left[t_*, 1\right]$$

issuing from X_* and ending up in $T\left(f\left(X_0\right)\right)$ is the solution to the **final value problem**

$$\begin{cases} \Phi'(t) = \Phi(t) \Lambda_\Gamma(t) \\ \Phi(1) = T\left(f\left(X_0\right)\right) \end{cases},$$

It is a quasi-holonomic arc wholly contained in $B\left(\Omega, r\right) \subset A^T \cdot \mathrm{Bru}_{A^T}$, in the sense that its subdiagonals are non-negative (but possibly zero). It can therefore be perturbed into a holonomic arc $\widetilde{\Gamma}_1 : \left[t_*, 1\right] \to B\left(\Omega, r\right)$ ending up in $T\left(f\left(X_0\right)\right)$ and issuing from a point $\widetilde{X}_*$ as close as desired to X_* – and hence still in $B\left(\Omega, r\right) \cap \mathrm{Bru}_{A^T}$ – at the cost of adding some sufficiently small positive constants $\epsilon_1 > 0, \cdots, \epsilon_n > 0$ to its subdiagonal entries (subtracting the same constants from the superdiagonal entries in order to ensure that $\widetilde{\Lambda}_\Gamma(t) \in \mathfrak{so}_{n+1}$).

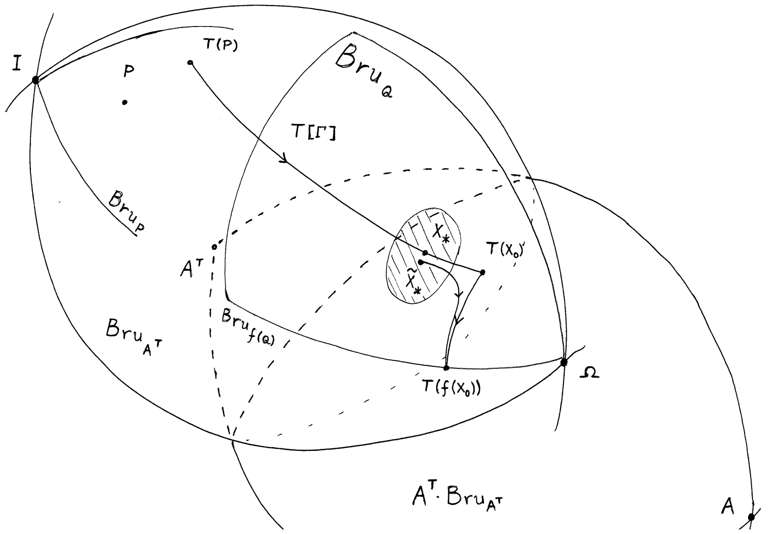

Figure 4.6: a slight holonomic perturbation $\widetilde{\Gamma}_1$ of the quasi-holonomic arc Φ

Accordingly, we can slightly perturb the logarithmic derivative of $T\,[\Gamma]$: $[0, t_*] \to \overline{\mathrm{Bru}_{A^T}}$ in order to produce another holonomic path $\widetilde{\Gamma}_0 : [0, t_*] \to \overline{\mathrm{Bru}_{A^T}}$ issuing from $T\,(P)$ but ending up in $\widetilde{X}_*$, provided the point $\widetilde{X}_*$ was made close enough to X_*. In effect, under suitably defined projective transformations, the end point X_* of the strictly convex arc $T[\Gamma]|[0, t_*]$ can be made to sweep an open subset of $B\,(\Omega, r) \cap \mathrm{Bru}_{A^T}$ around itself while the resulting arcs are still strictly convex arcs wholly contained in Bru_{A^T} with the same initial point $T(P) = \widetilde{\Gamma}_*(0)$ as the original arc. Now, we must weld together $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ at $\widetilde{X}_*$ in order to arrive at the holonomic path $\widetilde{\Gamma}_* = \widetilde{\Gamma}_0 * \widetilde{\Gamma}_1 : [0, 1] \to \mathrm{SO}_{n+1}$, which clearly satisfies the following desired conditions:

$$\widetilde{\Gamma}_*\,(0) = T\,(P)\,,$$

$$\forall t \in (0, t_*] \left(\widetilde{\Gamma}_*\,(t) \in \ \mathrm{Bru}_{A^T}\right),$$

$$\widetilde{\Gamma}_*\,(t_*) = \widetilde{X}_*,$$

$$\forall t \in [t_*, 1] \left(\widetilde{\Gamma}_*\,(t) \in \ B\,(\Omega, r)\right)$$

and

$$\widetilde{\Gamma}_*\,(1) = T\,(f\,(X_0))\,.$$

We claim that $\widetilde{\Gamma}_*$ never gets to leave the open cell Bru_{A^T}. In fact, if it did leave Bru_{A^T}, it would be its final piece $\widetilde{\Gamma}_1 : [t_*, 1] \to B\,(\Omega, r)$ to do so. But since $\mathrm{chop}(f\,(Q)) = A^T$ and $\widetilde{\Gamma}_1\,(1) = T\,(f\,(X_0)) \in \mathrm{Bru}_{f(Q)}$, there is a small $\epsilon > 0$ such that

$$\widetilde{\Gamma}_1\,(t) \in \ \mathrm{Bru}_{A^T} \cap B\,(\Omega, r)$$

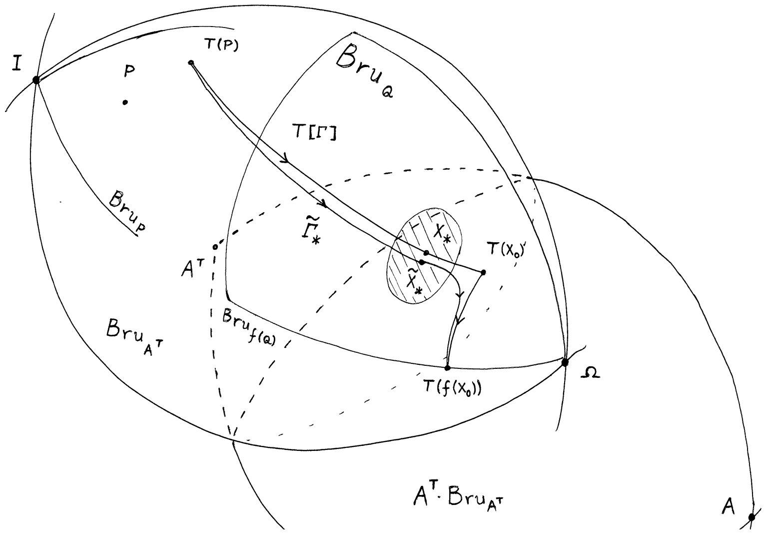

Figure 4.7: welding together slight holonomic perturbations at the joint $\widetilde{X}_*$ producing a holonomic path $\widetilde{\Gamma}_*$

for all $t \in [1-\epsilon, 1)$. Now, take a holonomic path $\Gamma_2 : [1-\epsilon, 1] \to B(\Omega, r)$ such that

$$\Gamma_2(1) = \Omega,$$

$$\forall t \in [1-\epsilon, 1)\,(\Gamma_2(t) \in \text{ Bru}_{A^T})$$

and

$$\Gamma_2(1-\epsilon) = \widetilde{\Gamma}_1(1-\epsilon),$$

which we know to exist due to the properties of projective transformations, since $\text{chop}(\Omega) = A^T$. Now, the concatenation $\Gamma_3 = \widetilde{\Gamma}_1|\,[t_*, 1-\epsilon] * \Gamma_2$ is wholly contained in $B(\Omega, r)$, hence is wholly contained in the translated open cell $A^T{\cdot}\text{Bru}_{A^T}$ and therefore is convex by proposition 3.4. Now, by implication ($a \to d$) of theorem 3.2, we have $\Gamma_3(t) \in \Omega{\cdot}\text{Bru}_A = \text{Bru}_{A^T}$ for all $t \in [t_*, 1)$.and, in particular $\widetilde{\Gamma}_1$ is wholly contained in Bru_{A^T}. It proves that the holonomic perturbation $\widetilde{\Gamma}_*$ above issues from $T(P)$, ends up in $T(f(X_0))$, and stays within Bru_{A^T} in the meanwhile.

Finally, one reverses the projective transformation T used to send X_0 into the neighbourhood $B(\Omega, r)$ and arrive at a holonomic path

$$\widetilde{\Gamma} = T^{-1}\left[\widetilde{\Gamma}_*\right]$$

satisfying all the conditions claimed.

The proof of lemma 4.3 is now finished. ■

Lemma 4.4 *If* $\text{chop}(Q) = \text{adv}(P)$, *then the fibres* $f^{-1}[X]$ *of the*

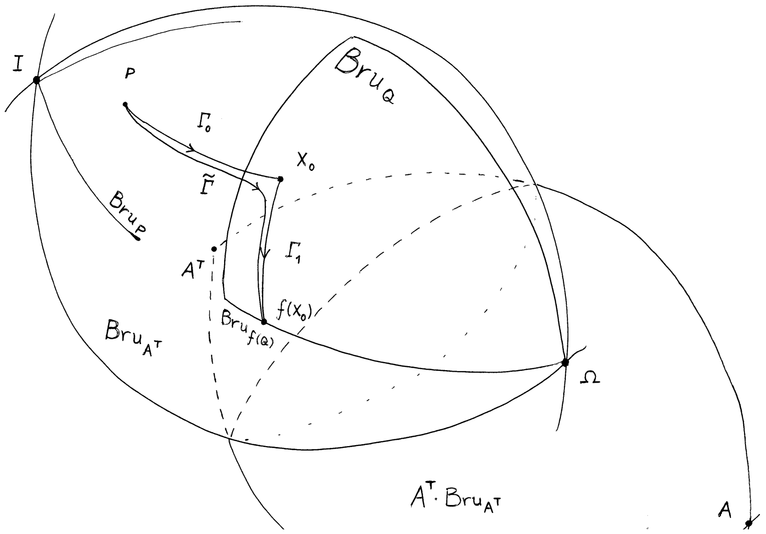

Figure 4.8: projective transformation T^{-1} takes the just constructed holonomic path $\widetilde{\Gamma}_*$ back into the desired holonomic perturbation $\widetilde{\Gamma}$ of the original quasi-holonomic path $\Gamma = \Gamma_0 * \Gamma_1$

fibration

$$\begin{array}{rccc} f: & \widetilde{\mathrm{Ac}}\,(Q|P) & \longrightarrow & \mathrm{Ac}\,(Q|P) \\ & \Gamma & \mapsto & \Gamma\,(1) \end{array}$$

are non-empty and contractible.

Proof. Let $P, Q \in \mathrm{B}^+_{n+1}$ be such that $\mathrm{chop}(Q) = \mathrm{adv}(P)$ and $X \in \mathrm{Ac}(Q|P)$. The fibre $f^{-1}[X]$ is nothing but the space of holonomic curves $\Gamma : [0,1] \to \overline{\mathrm{Bru}_{\mathrm{chop}(Q)}}$ satisfying $\Gamma(0) = P$, $\Gamma(1) = X$ and $\Gamma(t) \in \mathrm{Bru}_{\mathrm{chop}(Q)}$ for all $t \in (0,1)$. Being the limit of paths of frames of strictly convex curves wholly contained in a open cell, each of these holonomic curves is convex, and it follows from implication $(a \to b)$ of theorem 3.2 that $X \in P{\cdot}\mathrm{Bru}_{A^T}$. Now, theorem 3.7 implies that $f^{-1}[X]$ is indeed contractible. ■

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