Pontifícia Universidade Católica
do Rio de Janeiro

Tiane Marcarini Pinto

Generation of spines in porcupine-like horseshoes

Tese de Doutorado

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC–Rio, as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Lorenzo Justiniano Díaz Casado

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Abstract

Marcarini, Tiane; Díaz, Lorenzo J.. **Generation of spines in porcupine-like horseshoes**. Rio de Janeiro, 2014. 74p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We study one parameter families of skew-product maps $(F_t)_{t\in(0,1]}$, $F_t\colon \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$, such that the map F_t has a "partially hyperbolic" transitive set Λ_t called a *porcupine-like horseshoe* for every $t \in (t_0, 1)$ where $t_0 \in [0, 1)$. The porcupine-like horseshoes Λ_t exhibit a very rich fiber structure: the set Σ_2 is the disjoint union of two dense and uncountable subsets $\Sigma_2^{\mathcal{N}}$ and $\Sigma_2^{\mathcal{T}}$ with opposite behaviour:

- $\Sigma_2^{\mathcal{N}}$ consists of sequences ξ such that there is a non-trivial interval $I_{\xi,t}$ such that $\xi \times I_{\xi,t}$ (called the non-trivial spine of ξ) is contained in Λ_t,
- $\Sigma_2^{\mathcal{T}}$ consists of sequences ξ such that $(\xi \times \mathbb{R}) \cap \Lambda_t$ is just a point x (in this case we say that the spine (ξ, x) of ξ is trivial).

We study two types of porcupine-like horseshoes that we call central contracting and central expanding. Our goal is to analyze how the spines of the porcupine-like horseshoe are created and destroyed as t evolves and goes to 1. Concerning this creation/destruction process, we show that there is a large subset (Hausdorff dimension greater than one) of Σ_2 consisting of sequences whose spines remain trivial for every $t \in (0,1)$ (i.e., $\xi \in \Sigma_2^{\mathcal{T}}$ for all $t \in (0,1)$). We also prove that there is an uncountable dense subset of Σ_2 of sequences whose spines remain non-trivial for every $t \in (0,1]$ (i.e., $\xi \in \Sigma_2^{\mathcal{N}}$ for all $t \in (0,1)]$. This implies that there are non-trivial spine which are instantly created after $t = 0$ and never disappear afterwards.

We also prove that this creation process is not monotone in general and that a non-trivial spine can be destroyed (it becomes trivial) as the parameter t increases.

We also study the predominance of the size of the set $\Sigma_2^{\mathcal{T}}$ corresponding to sequences with trivial spines. We prove that $\mathfrak{b}_{1/2}(\Sigma_2^{\mathcal{T}}) = 1$, where $\mathfrak{b}_{1/2}$ is the equidistributed Bernoulli measure defined on Σ_2.

Keywords

Concavity; iterated function system; Hausdorff dimension; heterodimensional cicle; homoclinic class; porcupine-like horseshoes; skew-product; spines; transitivity.

Resumo

Marcarini, Tiane; Díaz, Lorenzo J.. **Formação de espinhas em ferraduras tipo porco-epinho**. Rio de Janeiro, 2014. 74p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Estudaremos famílias a um parâmetro de skew-products $(F_t)_{t\in(0,1]}$, $F_t\colon \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$, na qual o mapa F_t tem um conjunto transitivo "parcialmente hiperbólico" Λ_t chamado *ferradura tipo porco-espinho* para todo $t \in (t_0, 1)$ onde $t_0 \in [0, 1)$. A ferradura tipo porco-espinho Λ_t exibe uma estrutura bastante rica nas fibras: o conjunto Σ_2 é a união disjunta de dois não-enumeráveis e densos subconjuntos $\Sigma_2^{\mathcal{N}}$ e $\Sigma_2^{\mathcal{T}}$ com comportamentos opostos:

– $\Sigma_2^{\mathcal{N}}$ consistindo de sequências ξ para as quais existe um segmento não degenerado $I_{\xi,t}$ tal que $\xi \times I_{\xi,t}$ (chamado o espinho não-trivial of ξ) está contido em Λ_t,

– $\Sigma_2^{\mathcal{T}}$ consistindo de sequências ξ tal que $(\xi \times \mathbb{R}) \cap \Lambda_t$ é apenas um ponto x (nesse caso dizemos que o espinho (ξ, x) de ξ Ã© trivial).

Estudaremos dois tipos the ferraduras tipo porco-espinho que chamamos central contrativo e central expansor. Analizaremos como os espinhos de um porco-espinho são criados e destruídos quando t aumenta e vai para 1. Com respeito a esse processo de criação/destruição, mostraremos que existe um subconjunto grande (dimensão de Hausdorff maior que 1) de Σ_2 consistindo de sequências cujos espinhos permanecem trivial para todo $t \in (0, 1)$ (i.e., $\xi \in \Sigma_2^{\mathcal{T}}$ para todo $t \in (0, 1)$). Também provaremos que existe um subconjunto denso e não-enumerável de Σ_2 de sequências cujos espinhos permanecem não trivial para todo $t \in (0, 1]$ (i.e., $\xi \in \Sigma_2^{\mathcal{N}}$ para todo $t \in (0, 1)$]. Isto implica que existem espinhos não triviais que são instantâneamente cridaos após $t = 0$ e nunca desaparecem.

Temos ainda que este processo de criação em geral não é monótono e que um espinho não-trivial pode ser destruído (tronando-se trivial) quando o parâmetro t cresce.

Também estudaremos a predominância do conjunto $\Sigma_2^{\mathcal{T}}$ correspondendo às sequências com espinho trivial. Provaremos que $\mathfrak{b}_{1/2}(\Sigma_2^{\mathcal{T}}) = 1$, onde $\mathfrak{b}_{1/2}$ é a medida de Bernoulli equidistribuída definida em Σ_2.

Palavras–chave

Concavidade; sistema iterado de funções; dimensão de hausdorff; ciclo heterodimensional;

classe homoclínica; ferradura tipo porco-espinho; produto-cruzado; espinhas; transitividade.

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Introduction

1.1
General setting

We consider a dynamical system of skew-product type defined over a full shift of two symbols (Σ_2, σ) and with one-dimensional fibers,

$$F\colon \Sigma_2 \times \mathbb{R} \longrightarrow \Sigma_2 \times \mathbb{R}, \quad F(\xi, x) = (\sigma(\xi), f_\xi(x)).$$

This dynamics is partially hyperbolic, it has a uniformly hyperbolic part inherited from the shift dynamics and a central part corresponding to the fibers. The central part is given by two fiber maps defined on the interval, the maps have no critical points, one of them preserves and the other reverses orientation. Many of the interesting properties of the dynamics originated from this reversion of the orientation, for example, this property is essential to generate a heterodimensional cycle and also to ensure the existence of many periodic points.

The resulting dynamics of the skew-product on a maximal invariant set Λ is topologically transitive (existence of dense orbits) and genuinely non-hyperbolic (existence of hyperbolic periodic points of different indices) with points which are contracting in the fiber dynamics and other points which are expanding in this direction. The topological structure of the set Λ is very rich and it contains uncountable many continua tangent to the fibre (central) direction.

More precisely, the dynamics in Λ is semi-conjugate to the full shift of two symbols, that is, there is a continuous and surjective map $\Pi\colon \Lambda \to \Sigma_2$ such that $\Pi \circ F = \sigma \circ \Pi$, this map is called a semi-conjugation. This means that the dynamics of F is richer than the one of the shift (although in this case, due to the fact that the dynamics in the fibers is non-critical, the topological entropy of $F_{|\Lambda}$ is equal to the one of the shift, see [2] and the construction in [?] section 3).

A *spine* of the set Λ is a non-trivial continuum contained in the pre-image $\Pi^{-1}(\xi)$ of some sequence ξ of Σ_2. The relevant property of Λ is that Σ_2

splits into two uncountable dense sets Σ_2^t and Σ_2^n, the first one consisting of sequences ξ such that $\Pi^{-1}(\xi)\cap\Lambda$ is just a point and the second one consisting of sequences ξ such that $\Pi^{-1}(\xi)\cap\Lambda$ is a non-trivial closed interval. If $\xi\in\Sigma_2^t$ we say that its spine is trivial.

A transitive set Λ as above is called a *porcupine.* The very naive and rough geometrical idea of a porcupine is the following: Consider a horseshoe in the plane and select two uncountable dense subsets of Λ, for each point in the first set one glues a segment and for the second one one just glue a point.

Here we consider a one-parameter family of maps $(F_t)_{t\in[0,1]}$ as above with porcupines Λ_t for every $t\in(0,1)$.

The fiber dynamics consists of two maps: an increasing concave map with two fixed points at 0 and 1 which is independent of t and the affine orientation reversing contraction $x\mapsto t(1-x)$ mapping 1 to 0. The dynamics for $t=0$ and $t=1$ correspond to two degenerate cases of different nature: for $t=0$ we have a degenerate dynamics with no spines at all while for $t=1$ there is a porcupine having only non-trivial spines (thus formally it is not a porcupine). Thus there is a transition from a dynamics without spines (for $t=0$) to a dynamics with a full set of spines ($t=1$). Let us observe that the family $(F_t)_{t\in[0,1]}$ we consider is in some sense the simplest model describing a transition from a dynamics without spines to a completely spiny dynamics.

First let us note that the the semi-conjugation Π_t above is independent of t: we always have $\Lambda_t\subset\Sigma_2\times\mathbb{R}$ and, defining the projection

$$\widetilde{\Pi}\colon \Sigma_2\times\mathbb{R}\longrightarrow\Sigma_2\colon (\xi,x)\mapsto\xi$$

for every $t\in[0,1]$ the semi-conjugation $\Pi_t\colon\Lambda_t\longrightarrow\Sigma_2$ coincides with $\widetilde{\Pi}|_{\Lambda_t}$.

The very naive idea is that the dynamics of the porcupine Λ_t gains complexity as t increases and new non-trivial spines appear. The goal is to understand the generation (appearance/disappearance) of non-trivial spines. A first natural question that illustrates our goal is the following: Can a spine disappear, that is, if $\Pi_{t_0}^{-1}(\xi)$ is a continuum then $\Pi_t^{-1}(\xi)$ is also a continuum for all $t\in[t_0,1]$? This sort of question has the same flavor as the one about the monotonicity of the complexity of the dynamics for the quadratic family

$$F_t\colon[0,1]\longrightarrow[0,1],\quad x\mapsto 4\,t\,x(1-x),\quad t\in[0,1].$$

As in our case, for $t=0$ the dynamics is degenerate and for $t=1$ there is a completely chaotic dynamics conjugate to the full shift. In this case, the complexity of the dynamics is increasing: the entropy is (non-strictly) increasing and once one periodic point of some period is created for it persists as

t increases (see [3, 14]). In our setting, the spines play the role of the periodic points above. Let us observe that in our setting the topological entropy is constant for $t \in (0, 1]$.

Let us say a few words about previous results about porcupines. These sets appeared (without a name) for the first time in the work [9] about the destruction of hyperbolic sets via heterodimensional cycles. In this bifurcation scenario the hyperbolic set explodes and engulfs a hyperbolic fixed point of different stable index. In the above terminology, the resulting set is a porcupine. An interesting property is that this set is, in a certain sense, essentially hyperbolic and that the spectrum of the central Lyapunov exponent exhibits a gap. In [12] the authors explore this gap of the spectrum from the point of view of thermodynamic formalism. The notion of porcupine was introduced in [4], where genuinely non-hyperbolic porcupines are considered. Finally, [6, 8, 7] explore ergodic aspects of these porcupines. A survey on this topic can be found in [5].

Let us finally observe that a porcupine has the same flavor as the so-called *bony attractor*, that is, an attractor which is the union of the graph of a continuous function (over the shift space) and an uncountable set of vertical segments (so-called *bones*) belonging to the closure of this graph. The bones correspond to the non-trivial spines of the porcupine, see [11].

We now go to the detailed description of our results.

1.2 Statement of results and precise definitions

In this paper we consider one-parameter families of step skew-product maps of the form

$$F_t \colon \Sigma_2 \times \mathbb{R} \longrightarrow \Sigma_2 \times \mathbb{R}, \qquad (\xi, x) \;\mapsto\; (\sigma(\xi), f_{\xi_0,t}(x)), \tag{1.2.1}$$

where $f_{0,t} = f_0$ is an increasing concave map fixing 0 and 1 that does not depend on t and where $f_{1,t}$ is the affine map $f_{1,t}(x) = t\,(1-x)$. The two key properties of $f_{1,t}$ are that it reversed ordered and satisfies the cycle condition $f_{1,t}(1) = 0$.

Under some mild conditions on f_0, the maximal invariant set Λ_t of F_t in $\Sigma_2 \times [0, 1]$ is a porcupine for all $t \in (0, 1)$. For $t = 1$ the set Λ_1 is a completely spiny porcupine.

We now give the definition of a porcupine in our context:

Definition 1.2.1 (Porcupine)**.** Given a skew product map $F \colon \Sigma_2 \times [0, 1] \to \Sigma_2 \times [0, 1]$ we say that a compact F-invariant set Λ is a *porcupine* is it is transitive and there is a semi-conjugation $\Pi \colon \Lambda \to \Sigma_2$, $\Pi \circ F = \sigma \circ \Pi$, such that $\Pi^{-1}(\xi)$ is a continuum for uncountably many $\xi \in \Sigma$ and a singleton for uncountably many $\xi \in \Sigma$.

The set $\Pi^{-1}(\xi)$ is the *spine* of ξ. If this set is not a singleton we say that it is a *non-trivial spine*. If $\Pi^{-1}(\xi)$ is non-trivial for all $\xi \in \Sigma_2$ we say that Λ is a *completely spiny porcupine.*

As mentioned above, for the family $(F_t)_{t\in[0,1]}$ we consider the set Λ_t that is a porcupine for all $t \in (0,1]$. The goal of this paper is to understand the appearance and disappearance of the non-trivial spines as t evolves. As we will see this is a rather complicated process. Naively, one may guess that if a sequence has a non-trivial spine for some t_0 this spine will remain non-trivial for all $t > t_0$. There are examples of families of porcupines similar to the ones we consider (the choice of f_0 is slightly different, but $f_{1,t}$ is the same) where a spine is non-trivial for t_0 but becomes trivial for some $t_1 \in (t_0, 1)$, see the discussion in Section 7.

The dynamics of F_t is determined by the underlying iterated function system (i.f.s.) generated by the maps f_0 and $f_{1,t}$. We require the following conditions: $f_0, f_{1,t}\colon \mathbb{R} \to \mathbb{R}$ are C^2 injective maps satisfying the following properties:

(P0.i) The map f_0 is increasing, concave, and has exactly the two fixed points 0 (repelling) and 1 (attracting).

(P0.ii) Given $f_0'(0) = \beta > 1$ and $f_0'(1) = \lambda < 1$, it holds

$$\frac{\lambda^2}{\beta}\frac{(1-\lambda)}{1-\beta^{-1}} > 1.$$

(P1) $f_{1,t}(x) = t\,(1-x)$.

Note that $f_{1,t}(1) = 0$, that is, $f_{1,t}$ maps the attracting fixed point of f_0 into the repelling fixed point of f_0. This condition is called a *cycle condition.* Note that $f_0([0,1]) = [0,1]$ and $f_{1,t}([0,1]) \subset [0,1]$ for all $t \in [0,1]$, where the latter inclusion is strict if $t \neq 1$. We denote the maximal invariant set of F_t in $\Sigma_2 \times [0,1]$ by Λ_t,

$$\Lambda_t \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{Z}} F_t^i(\Sigma_2 \times [0,1]).$$

We fix some notation and state some simple remarks about the spines of a porcupine.

Remark 1.2.2. Given

$$\xi = \dots \xi_{-i} \dots \xi_{-2}\, \xi_{-1}\, \xi_0\, \xi_1\, \xi_2\, \dots \xi_i \dots \in \Sigma_2$$

we write

$$\xi = \xi^{-}.\xi^{+},$$

where

$$\xi^{-} \stackrel{\text{def}}{=} \dots \xi_{-i} \dots \xi_{-2}\, \xi_{-1} \qquad \text{and} \qquad \xi^{+} \stackrel{\text{def}}{=} \xi_0\, \xi_1\, \xi_2\, \dots \xi_i \dots .$$

Let Σ_2^+ and Σ_2^- be the set of sequences ξ^+ and ξ^-, respectively.

Since $f_{i,t}([0,1]) \subset [0,1]$ for every $t \in [0,1]$ and $i = 0,1$, it follows that the property of a spine being a singleton or not is determined only by the negative part ξ^- of the sequence $\xi = \xi^-.\xi^+$. In other words, if the spine of $\xi = \xi^-.\xi^+$ is a singleton then $\alpha = \xi^-.\alpha^+$ is also a singleton for every choice of $\alpha^+ \in \Sigma_2^+$. This justifies the following definition. We will say that the sequence $\xi^- \in \Sigma_2^-$ has a *trivial spine* if, and only if, the spine of any sequence of the form $\alpha = \xi^-.\alpha^+$ is a singleton. Otherwise, we say that $\xi^- \in \Sigma_2^-$ has a non-trivial spine.

We now define the subset $\Sigma_{2,t}^{-,\mathcal{N}}$ of Σ_2^- of sequences with a non-trivial spine and the subset $\Sigma_{2,t}^{-,\mathcal{T}}$ of Σ_2^- of sequences with a trivial spine.

Remark 1.2.3. The geometry of the maps F_t implies that $\Pi_t^{-1}(\xi)$ is either a point (singleton) or a non-trivial closed segment. In the first case, we write $\Pi_t^{-1}(\xi) = \{(\xi, s_{\xi,t})\}$ and say that the spine is trivial. In the second case, we write $\Pi_t^{-1}(\xi) = \{\xi\} \times I_{\xi,t}$ where $I_{\xi,t} = [s_{\xi^-,t}, s_{\xi^+,t}]$, $0 \le s_{\xi^-,t} < s_{\xi^+,t} \le 1$, and say that the spine is non-trivial. By definition, we have that

$$I_{\xi,t} = \left\{ x \in [0,1]\colon \text{such that } (f^{-1}_{\xi_{-i},t} \circ \dots \circ f^{-1}_{\xi_{-1},t})(x) \in [0,1] \text{ for every } i \in \mathbb{N} \right\}.$$

Using the cylinder notation for the composition of maps we let

$$f_{[\xi_0 \dots \xi_n],t} \stackrel{\text{def}}{=} f_{\xi_n,t} \circ f_{\xi_{n-1},t} \circ \dots \circ f_{\xi_0,t}.$$

Note that if (ξ, x) is a periodic point of F_t of period $n+1$ then $\xi = (\xi_0 \dots \xi_n)^{\mathbb{Z}}$ and $f_{[\xi_0 \dots \xi_n],t}(x) = x$. The periodic point (ξ, x) is said to be *central contracting* if $|f'_{[\xi_0 \dots \xi_n],t}(x)| < 1$ and *central expanding* if $|f'_{[\xi_0 \dots \xi_n],t}(x)| > 1$.

We now borrow the following result from [4].

Theorem 1.2.4. *Consider F_t as in* (1.2.1) *and assume that f_0 and $f_{1,t}$ satisfies (P0.i), (P0,ii), and (P1). Then for every $t \in (0,1)$ the set Λ_t is a porcupine and has the following properties:*

i) There is a continuous semi-conjugation $\Pi_t\colon \Lambda_t \to \Sigma_2$ *of the form* $\Pi_t(\xi, x) = \xi$ *such that:*

(a) The subset $\Sigma_{2,t}^{-,\mathcal{N}}$ *of* Σ_2^- *of sequences with a non-trivial spine is uncountable and dense.*

(b) The subset $\Sigma_{2,t}^{-,\mathcal{T}}$ *of* Σ_2^- *of sequences with a trivial spine is residual in* Σ_2^-*.*

(c) There are two dense (uncountable) subsets $\Lambda_t^{-,\mathcal{T}}$ *and* $\Lambda_t^{-,\mathcal{N}}$ *of* Λ_t *such that*

- *if* $(\xi, x) \in \Lambda_t^{-,\mathcal{T}}$ *then* $\Pi_t^{-1}(\xi) = \{(\xi, x)\}$ *and*
- *if* $(\xi, x) \in \Lambda_t^{-,\mathcal{N}}$ *then* $\Pi_t^{-1}(\xi) = \{(\xi, I_{\xi,t}\}$*, where* $I_{\xi,t}$ *is a non-trivial and closed continuum containing* x*.*

ii) The sets of central contracting periodic points and of central expanding periodic points of Λ_t *are both dense subsets of* Λ_t*.*

Remark 1.2.5. We will see, (Remark 9.1.8), that the proof in [4] implies that for $t = 1$ the set Λ_1 is a completely spiny porcupine.

In view of the theorem above, it is natural to ask how the subsets $\Sigma_{2,t}^{-,\mathcal{N}}$ and $\Sigma_{2,t}^{-,\mathcal{T}}$ of Σ_2^- vary as the parameter t varies. Note that for $t < 1$ the set $\Sigma_{2,t}^{-,\mathcal{T}}$ is residual in Σ_2^- and for $t = 1$ the set $\Sigma_{2,1}^{-,\mathcal{T}}$ is empty. For $t_0 \in [0, 1)$ define the subsets of Σ_2^-

$$\Sigma_2^{-,\mathcal{N}}(t_0) \stackrel{\text{def}}{=} \bigcap_{t\in(t_0,1)} \Sigma_{2,t}^{-,\mathcal{N}} \quad \text{and} \quad \Sigma_2^{-,\mathcal{T}}(t_0) \stackrel{\text{def}}{=} \bigcap_{t\in(t_0,1)} \Sigma_{2,t}^{-,\mathcal{T}}$$

consisting of the sequences in Σ_2^- whose spines are non-trivial, respectively trivial, for all $t \in (t_0, 1)$.

A first natural question is about the existence of sequences ξ whose spine has the same type (trivial or not) after some parameter t_0, that is, such that $\Sigma_2^{-,\mathcal{N}}(t_0)$ is non-empty. A second natural problem is to determined the size of these sets. Here, the concept of "size" may depend on your point of view and could be, for example, fractal dimension, entropy, measure, etc. This sort of problem is addressed in the next theorem.

In the set Σ_2 we consider the following canonical distance d. Given $\varpi, \theta \in \Sigma_2$ denote by $n_{\varpi,\theta}$ the smallest value of $|n|$ with $\varpi_n \neq \theta_n$, then

$$d(\varpi, \theta) = 2^{1/2}\, 2^{-n_{\varpi,\theta}}.$$

With this distance one has that the Hausdorff dimensions of Σ_2 and Σ_2^- are two and one, respectively. For the definition and properties of the Hausdorff dimension see Section 2.

Theorem 1. *There exists an uncountable subset of* Σ_2^- *consisting of sequences with non-trivial spines for every* $t \in (0,1]$*. That is, the set* $\Sigma_2^{-,\mathcal{N}}(0)$ *is uncountable.*

The Hausdorff dimension of $\Sigma_2^{-,\mathcal{N}}(t_0)$ *is strictly positive for every* $t_0 > 0$*.*

Let us observe that there is strong indications that the uncountable set $\Sigma_2^{-,\mathcal{N}}(0)$ has zero Hausdorff dimension. We observe that in Corollary 3.2.3 we prove that for every $\xi \in \Sigma_2^{-,\mathcal{N}}(t_0)$ the frequency of $1's$ of ξ^- is necessarily zero. Note that the second part of the theorem answers to a slightly different question: replacing the parameter 0 by parameter $t_0 > 0$ we obtain that the sets $\Sigma_2^{-,\mathcal{N}}(t_0)$, consisting of the sequences that has non-trivial have positive dimension.

Our next result claims that there is a large subset (positive Hausdorff Dimension) of Σ_2 whose spines are trivial for all $t \in (0,1)$. This means that the transition to a completely spiny porcupine at $t = 1$ happens suddenly and a lot of spines are created instantaneously.

Theorem 2 (Abrupt appareance of spines)**.** *The set* $\Sigma_2^{-,\mathcal{T}}(0)$ *has positive Hausdorff dimension.*

A key ingredient of the proofs of this theorem is the concavity of the map f_0. This concavity will ensure that (finite) compositions of the maps f_0 and $f_{1,t}$ such that the maps $f_{1,t}$ only appear in groups with an even number of elements (in particular these maps are ordering preserving and concave) have exactly a unique fixed point. And so, considering only sequences $\xi = \xi^-.\xi^+$ for which ξ^- is a especial sequence where the digit 1 only appear in groups with even number of elements, we will have that the spine of this sequences is trivial for every $t \in (0,1)$.

Another useful property, that is explored in the next results, is the contracting property (P0.ii) $\beta\,\lambda < 1$ (a consequence of the condition $\frac{\lambda^2}{\beta}\frac{(1-\lambda)}{1-\beta^{-1}} > 1$). This property is especially used in Theorem 3 that study the stability of spines.

The property $\beta\,\lambda < 1$ implies that if we consider a (finite) composition of the maps f_0 and $f_{1,t}$ such that the maps $f_{1,t}$ appear in odd quantities (in particular this map is ordering reverse) and then take its square, this last map g_t^2 satisfy $(g_t^2)'(0) = (g_t^2)'(1) = (\beta\,\lambda)^n$ for same $n \in \mathbb{N}$, and by hypothesis $\beta\,\lambda < 1$. This will implies that if we consider only sequences $\xi = \xi^-.\xi^+$ for which ξ^- is a sequence obtained from *concatenation* of tow (or any finite number) of *words* that are the square of a word with odd numbers of i's, we will have that this sequence ξ has a *stable* spine at $t = 1$.

Definition 1.2.6. Consider $t_0 \in (0,1]$. We say that a spine of $\xi \in \Sigma_2$ is *stable at* t_0 if there is a neighborhood $V(t_0)$ of t_0 in $(0,1]$ such that $\Pi_t^{-1}(\xi)$ is close to $\Pi_{t_0}^{-1}(\xi)$ (in the Hausdorff distance) for all $t \in V(t_0)$. In particular, this means that if ξ has a non-trivial spine for t_0 the same holds for all t close to t_0.

Let us observe that the stabilization of some trivial spines is a relatively simple problem. For instance, fixed any $t_0 \in (0,1)$ there is $\delta(t_0)$ such that every sequence with a "proportion" of 1's bigger than $\delta(t_0)$ has a trivial spine for every $t \in (0, t_0]$, see Proposition 3.2.2 and Corollary 3.2.3 for details. A much more difficult problem concerns the stabilization of non-trivial spines. Theorem 2 implies in particular that for $t = 1$ there are many spines which are not stable.

Theorem 3. *There is a subset* $\Sigma_2^{\mathfrak{c}} \subset \Sigma_2$ *with Hausdorff dimension bigger than one with the following property: for any given* $\epsilon > 0$ *there exists* $t_\epsilon \in (0,1)$ *such that* $|\Pi_t^{-1}(\xi)| > 1 - \epsilon$ *for every* $t \in [t_\epsilon, 1]$ *and* $\xi \in \Sigma_2^{\mathfrak{c}}$. *Therefore the spine of every* $\xi \in \Sigma_2^{\mathfrak{c}}$ *is stable at* $t = 1$.

In the opposite direction, we show that "almost every" sequence has a trivial spine. The space Σ_2 is endowed with a natural family of Borel sets $\mathfrak{B}$ that is generated by the cylinders of Σ_2. On this measure space we consider the Bernoulli probability measures $\mathfrak{b}_p$ indexed by the probability $p \in [0,1]$ this measure gives to the position 0. In some sense, the most natural one is the equidistributed measure $\mathfrak{b}_{1/2}$ that assigns the same weight to 0's and 1's.

As above, for each t consider the subsets $\Sigma_{2,t}^{\mathcal{T}}$ and $\Sigma_{2,t}^{\mathcal{N}}$ of sequences of Σ_2 with trivial and non-trivial spines for F_t.

Theorem 4. *For every* $t \in [0,1)$ *the Hausdorff dimension of* $\Sigma_{2,t}^{\mathcal{N}}$ *is less than* 2.

This theorem also implies that the topological entropy of the shift in Σ_2 is concentrated in $\Sigma_{2,t}^{\mathcal{T}}$. So the topological entropy of the map F_t is concentrated in the trivial spines.

Recall that by Theorem 1.2.4 the set $\Sigma_{2,t}^{\mathcal{T}}$ is a residual subset of Σ_2 for all $t \in (0,1)$. Next result states that this set has full $\mathfrak{b}_{1/2}$ measure and has some persistence.

Theorem 5. *Consider the probability space* $(\Sigma_2, \mathcal{B}, \mathfrak{b}_{1/2})$. *Then* $\mathfrak{b}_{1/2}(\Sigma_{2,t}^{\mathcal{T}}) = 1$ *for all* $t \in (0,1)$ *and* $\mathfrak{b}_{1/2}\big(\bigcap_{t\in(0,\beta^{-1})} \Sigma_{2,t}^{\mathcal{T}}\big) = 1$.

In the previous results we have considered a one-parameter family of skew-product $(F_t)_{t\in(0,1]}$ where the fiber maps f_0 and $f_{1,t}$ satisfy conditions

(P0.i), (P0.ii), and (P1). Theorem 1.2.4 claims that every t the elements of this family have a porcupine.

We see that condition (P0.ii) can be (in some sense) eliminated. In such a case, we obtain a larger class of skew-products $(F_t)_{t\in(0,1]}$ that display porcupines for every parameter larger than some $t_0 \in (0,1)$ depending of the family (the existence of porcupines is in principle not guaranteed for small t). The value t_0 is given by the condition:

(C) Consider the parameter t_0 defined by $f_0'(t_0) = 1$. Then $f_0'(x) < 1$ for every $x > t_0$, that is, f_0 is a contraction in $(t_0, 1]$.

Note that as $f_0'(1) < 1$ we have that $t_0 \in (0,1)$. This means that we can consider problems similar to the ones considered above for this more general family, in particular, studying the appearence and dissapearance of spines for $t \in [t_0, 1]$.

We say that a sequence ξ has a *evanescent spine* $I_{\xi,t}$ if $I_{\xi,t}$ if there are $0 < t_1 < t_2 < 1$ such that I_{ξ,t_1} is non-trivial and I_{ξ,t_2} is trivial for all $t \in [t_2, 1)$.

For this family we have the following result that claims that the appearance of non-trivial spines is not monotone.

Theorem 6. *There is an skew product family $(F_t)_{t\in(0,1]}$ satisfying (P0.i) and (P1) with an evanescent spine.*

1.3
Organization of the thesis

This thesis is organized as follows. In Chapter 2 we state some basic results about Hausdorff dimension that we will use.

In Chapter 3 we first characterize and state some general properties of spines of porcupines. We also give sufficient conditions for a sequence having a trivial spine. Finally we study some properties of spines associated to sequences whose symbol 1 appears only in groups of even size. With these results we prove Theorem 2 guaranteing the existence of a subset of Σ_2^- with positive Hausdorff dimension consisting of sequences with trivial spines for every $t \in (0,1)$.

In Chapter 4 we prove Theorem 3 about the "stability" of the spines at $t = 1$ for a large subset (positive Hausdorff dimension) of Σ_2^-.

Theorems 4 and 5 are proved in Section 5. The aim of this section is to compare the subsets of Σ_2^+ corresponding to trivial and non-trivial spines and see that the first dominates.

In Chapter 6 we study the persistence of non-trivial spines and prove Theorem 1.

In Chapter 7, we present an example of family of skew product maps with porcupines exhibiting an evanescent spine and prove Theorem 6.

In Appendix A we adapt some standard notions of differentiable dynamics (as homoclinic points and homoclinic classes) to the skew-product setting. In Apendix B we review some constructions in [8, 7] and prove the transitivity of the porcupines. Finally, in Appendix C, we discuss how to obtain a one-parameter in $\mathbb{R}^3$ exhibiting porcupine-like horseshoes.

2
Hausdorff dimension

In this section we define Hausdorff measure and dimension and state some of the properties of it that will be used throughout the paper.

2.1
Definition and general properties of Hausdorff dimension

Definition 2.1.1 (Hausdorff Dimension)**.** Let (M, d) be a compact metric space and K a subset of M. Given a finite covering $\mathcal{U} = (U_i)_{i\in J}$ of K we define its *diameter* by

$$\operatorname{diam}(\mathcal{U}) \stackrel{\text{def}}{=} \sup\{\operatorname{diam}(U_i),\, i \in J\},$$

where $\operatorname{diam}(U)$ denotes the diameter of the set U (these numbers may be infinite). We let

$$m_s(\mathcal{U}) \stackrel{\text{def}}{=} \sum_{i\in J} \big(\operatorname{diam}(U_i)\big)^s$$

and, for $s, \epsilon > 0$,

$$m_{s,\epsilon}(K) \stackrel{\text{def}}{=} \inf\{m_s(\mathcal{U}) \text{ where } \mathcal{U} \text{ is a finite covering of } K \text{ with } \operatorname{diam}(\mathcal{U}) < \epsilon\}.$$

Note that $m_{s,\epsilon}(K)$ decreases when ϵ increases. The *s-dimensional Hausdorff measure* of K is defined by

$$m_s(K) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+} m_{s,\epsilon}(K).$$

Remark 2.1.2. Let $\mathfrak{P}(K)$ be the set of parts of K. Then for any $s \in \mathbb{R}$

$$m_s \colon \mathfrak{P}(K) \longrightarrow \mathbb{R}, \qquad U \mapsto m_s(U),$$

is a measure of K.

The map $m_s(K)$ decreases when s increases and there is a value $\operatorname{HD}(K)$, called the *Hausdorff Dimension* of K such that

$$\operatorname{HD}(K) \stackrel{\text{def}}{=} \inf\{s \in \mathbb{R} : m_s(K) = 0\} = \sup\{s \in \mathbb{R} : m_s(K) = \infty\}.$$

For further details see, for instance, [10].

Note that the definitions of $\text{HD}(K)$ and $m_s(K)$ (and so $m_{\text{HD}(K)}(K)$) are dependent of the metric considered in K. Thus, when necessary, we write $\text{HD}_d(K)$ and $m_{s,d}(K)$ for emphasizing the metric considered.

Definition 2.1.3 (Hausdorff Measure)**.** We define *Hausdorff Measure* of K the number $m_{\text{HD}(K)}(K)$, that is calculated like above taking $s = \text{HD}(K)$.

We have that the Hausdorff dimension is the same for equivalent metrics (see [10], p. 32-33 for details).

Proposition 2.1.4. *Consider two equivalent metrics d and $\bar{d}$ in K. Then* $\text{HD}_d(K) = \text{HD}_{\bar{d}}(K)$.

Another classical result about Hausdorff dimension is the following, see [13] p.1041.

Proposition 2.1.5. *Let $K = \bigcup_{n\in\mathbb{N}} K_n$. Then* $\text{HD}(K) = \sup_{n\in\mathbb{N}}\{\text{HD}(K_n)\}$.

2.2 Hausdorff dimension and measure of subsets of Σ_2

In what follows, we will study the set Σ_2 endowed with its canonical metric defined as follows. Given a pair of elements ϖ and θ of Σ_2 we denote by $n_{\varpi,\theta}$ the smallest value of $|n|$ with $\varpi_n \neq \theta_n$. We consider the canonical distance d in Σ_2 defined by

$$d(\varpi, \theta) = 2^{1/2}\, 2^{-n_{\varpi,\theta}}.$$

It is well known that with this distance one has (we omit the dependence on the metric)

$$\text{HD}(\Sigma_2) = 2, \quad \text{and} \quad m_2(\Sigma_2) = 1.$$

Similarly, one has that

$$\text{HD}(\Sigma_2^-) = 1, \quad \text{and} \quad m_1(\Sigma_2^-) = 2^{1/2}/2.$$

In the set Σ_2, given $m \in \mathbb{N}$, integers $i_0 < i_1 < \cdots < i_m$, and $k_0, k_1 \dots, k_m \in \{0, 1\}$, we consider the *cylinder* defined

$$C(i_0, i_1, \dots, i_m; k_0, k_1 \dots, k_m) \stackrel{\text{def}}{=} \{\theta \in \Sigma_2 \text{ such that } \theta_{i_\ell} = k_\ell,\ 0 \leq \ell \leq m\},$$

and the σ-algebra $\mathcal{B}$ generated by these cylinders.

We have that the topology associated to the metric d is the same as the one generated by σ-algebra $\mathcal{B}$.

Consider the probability space $(\Sigma_2, \mathcal{B}, \mathfrak{b}_{1/2})$ where $\mathfrak{b}_{1/2}$ is the Bernoulli probability given by

$$\mathfrak{b}_{1/2}\big(C(i_0, \ldots, i_m; k_0, \ldots, k_m)\big) = 2^{-(m+1)}.$$

Proposition 2.2.1. $m_2 = \mathfrak{b}_{1/2}$.

Proof. We just need to see that for every cylinder

$$\mathfrak{b}_{1/2}\big(C(i_0, \ldots, i_m; k_0, \ldots, k_m)\big) = 2^{-(m+1)} = m_2\big(C(i_0, \ldots, i_m; , k_0, \ldots, k_m)\big).$$

Lemma 2.2.2. *Let* $C = C(i_0, \ldots, i_m; k_0, \ldots, k_m)$ *and* $C' = C(i'_0, \ldots, i'_m; k'_0, \ldots, k'_m)$ *be two cylinders having the same number* m *of fixed entries. Then* $m_2(C) = m_2(C')$.

Proof. This lemma is a directly consequence of the following claim:

Claim 2.2.3. *Given* $\epsilon > 0$ *small enough, for each covering* $\mathcal{U} = \cup U_i$ *of* C, *diam*$(\mathcal{U}) < \epsilon$, *we associate a covering* $\mathcal{U}' = \cup U'_i$ *of* C' *with the same number of elements and satisfying diam*$(U_i) =$ *diam*(U'_i) *for every* i.

Proof. Note that if $\epsilon > 0$ is close to 0 and $\mathcal{U} = \cup U_i$ is a covering of C with diam$(\mathcal{U}) < \epsilon$, then there exist cylinders $C_{n_i} = C(-n_i, \ldots, n_i, j_{-n_i} \ldots, j_{n_i})$, $n_i > max\{|i_m|, |i'_m|\}$, such that $U_i \subset C_{n_i}$, for every i. Without lost generality we suppose $C_{n_i} \subset C$.

In the case that $\{i_0, i_1, \ldots, i_m\} \cap \{i'_0, i'_1, \ldots, i'_m\} \neq \emptyset$ we can ordered the indices to obtain $i_0 = i'_0 < i_1 = i'_1 < \ldots i_{m'} = i'_{m'}$, $m' \leq m$ and $\{i_{m'+1}, \ldots, i_m\} \cap \{i'_{m'+1}, \ldots, i'_m\} = \emptyset$. We will define a map $h : \mathcal{U} \to \Sigma_2$ such that $h(\mathcal{U})$ is a covering of C' and diam$(h(U_i)) =$ diam(U_i) for every i.

Let $\xi \in \mathcal{U}$, $\xi = \ldots \xi_{-n} \ldots \xi_0 \ldots \xi_n \ldots$, then, $h(\xi) = \varpi = \ldots w_n \ldots w_0 \ldots w_n \ldots$ and to define w_r we need to consider three cases:

i) $r = i_j = i'_j$: in this case we take $w_r \overset{\text{def}}{=} k'_j$,

ii) $r = i_j$, $j > m'$: in this case we take $w_r \overset{\text{def}}{=} \xi_{i'_j}$ and $w_{i'_j} \overset{\text{def}}{=} k'_j$.

Note that w_r is defined for every $r \in \{i_0, \ldots, i_m\} \cup \{i'_0, \ldots, i'_m\}$, then remains to see the last case:

iii) $r \notin \{i_0, \ldots, i_m\} \cup \{i'_0, \ldots, i'_m\}$: in this case we take $w_r \overset{\text{def}}{=} \xi_r$.

□

Lemma 2.2.2 is now proved. □

Now observe that we can write $\Sigma_2 = \cup_{1\leq i\leq 2^{m+1}} C_1$, where $C_i, \ldots, C_{2^{m+1}}$ are cylinder with $m+1$ fixed entries and this union is disjoint then

$$1 = m_2(\sigma_2) = \sum_{1\leq i\leq 2^{m+1}} m_2(C_i) = 2^{2m+1}\, m_2(C),$$

where the last equality follows by Lemma 2.2.2. This implies the proposition. □

In this work we consider a special case of subsets of Σ_2 defined using *independent words* we proceed to describe. First, we say that $a = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0,1\}^n$ is a word of length $|a| = n$.

Definition 2.2.4 (Concatenation of words)**.** Given a finite set of words $W = \{w_1, \ldots, w_m\}$ we say that a one-sided sequence $\xi = (\xi_i)_{i\geq 1} \in \Sigma_2^+$ is a concatenation of words in W if there is an increasing infinite sequence of indices $(i_k)_{k\in\mathbb{N}}$ with $i_1 = 1$ such that for every k the word $\xi_{i_k} \ldots \xi_{i_{k+1}-1}$ is in W.

Given a sequence $\xi \in \Sigma_2$, write $\xi = \xi^-.\xi^+$, $\xi^\pm \in \Sigma_2^\pm$. Associated to ξ^- we define its *conjugate sequence* $\widehat{\xi^-} = (\widehat{\xi_j^-})$ by

$$\widehat{\xi_j^-} \stackrel{\text{def}}{=} \xi^-_{-j+1}.$$

Given a finite set W of words we define the following sets

$$\mathbb{E}_W \stackrel{\text{def}}{=} \{\xi^- \in \Sigma_2^- \quad \text{such that} \quad \widehat{\xi^-} \text{ is a concatenation of words in } W\} \subset \Sigma_2^-,$$
$$\mathbb{S}_W \stackrel{\text{def}}{=} \{\xi = \xi^-.\xi^+ \quad \text{such that} \quad \widehat{\xi^-} \in \mathbb{E}_W\} \subset \Sigma_2.$$

Remark 2.2.5. $\mathrm{HD}(\mathbb{S}_W) = 1 + \mathrm{HD}(\mathbb{E}_W)$.

Definition 2.2.6 (Independent words)**.** We say that that two words w and u are *independents* if for every pair of natural numbers $\{n', m'\} \neq \{n, m\}$ we have that $w^n\, u^m \notin \{w^{n'} u^{m'}, w^{m'} u^{n'}, u^{n'} w^{m'}, u^{m'} w^{n'}\}$.

We have the following result.

Proposition 2.2.7. *Let $W = \{w, u\}$, where w and u are independent words. Then* $0 < \mathrm{HD}(\mathbb{E}_W)$.

Proof. Without loss of generality we can assume that $|w| = n \leq m = |u|$. Consider the real number s satisfying

$$1 = \frac{1}{2^{n\,s}} + \frac{1}{2^{m\,s}}.$$

Consider the "conjugate" set of $\mathbb{E}_W$ defined by $\mathbb{E}'_W = \{w, u\}^{\mathbb{N}}$ and note that $\mathrm{HD}(\mathbb{E}'_W) = \mathrm{HD}(\mathbb{E}_W)$ (to each covering of $\mathbb{E}'_W$ there is a associated a conjugate covering of $\mathbb{E}_W$). Thus to prove the proposition it is enough to see that $0 < \mathrm{HD}(\mathbb{E}'_W)$.

To prove the proposition we use the following lemma whose simple proof can be found in [10], p. 32.

Lemma 2.2.8. *Let E and F be two metric spaces. Assume that there is a map $h : E \to F$ that is α-Hölder. Then*

$$\mathrm{HD}(h(E)) \leq \frac{1}{\alpha}\mathrm{HD}(E).$$

In view of this lemma to prove the proposition it is enough to construct a surjective α-Hölder map $h \colon \mathbb{E}'_W \to [0, 1]$.

For each $\xi \in \mathbb{E}'_W$, we define, a sequence $(c_{\xi,k})_{k\in\mathbb{N}}$ of embedded compact intervals as follows: Given $\xi \in \mathbb{E}'_W$, $\xi = \xi_1\xi_2 \dots$ with $\xi_k \in \{w, u\}$ let

$$c_{\xi,1} = \big[0, \frac{1}{2^{n\,s}}\big] \text{ , if } \xi_1 = w \quad \text{or} \quad c_{\xi,1} = \big[\frac{1}{2^{n\,s}}, 1\big] \text{ , if } \xi_1 = u.$$

Supposing defined the intervals $c_i = [a_i, b_i]$, for $i = 1, \dots, k$, we define

$$c_{\xi,k+1} = \big[a_k, a_k + \frac{b_k - a_k}{2^{n\,s}}\big], \quad \text{if } w_{k+1} = w \quad \text{or}$$
$$c_{\xi,k+1} = \big[a_k + \frac{b_k - a_k}{2^{n\,s}}, b_k\big], \quad \text{if } w_{k+1} = u.$$

Finally we take

$$h(\xi) = \lim_{n\to\infty} c_{\xi,k}.$$

This number is well defined since the intervals $c_{\xi,k}$ form a nested sequence whose sizes go to zero as $k \to \infty$. More precisely, recalling the choice of s, we have that the size $|c_{\xi,k+1}|$ of the interval $c_{\xi,k+1}$ is

$$|c_{\xi,k+1}| = 2^{-ns}\,|c_{\xi,k}|, \quad \text{if } \xi_{k+1} = w,$$
$$|c_{\xi,k+1}| = (1 - 2^{-ns})\,|c_{\xi,k}| = 2^{-ms}\,|c_{\xi,k}|, \quad \text{if } \xi_{k+1} = u.$$

Therefore, since $n \leq m$,

$$|c_{\xi,k+1}| \leq 2^{-ns}\,|c_{\xi,k}|. \tag{2.2.1}$$

Lemma 2.2.9. *The map h is surjective and $\frac{s\,n}{m}$-Hölder.*

Proof. We first check that h is $\frac{s\,n}{m}$-Hölder. Take $\xi, \theta \in \mathbb{E}'_W$ with $\xi \neq \theta$. Write $\xi = \xi_1\xi_2 \dots$ and $\theta = \theta_1\theta_2 \dots$ where $\xi_j, \theta_j \in \{w, u\}$. Take i such that $\xi_j = \theta_j$

for every $1 \leq j < i$ and $\xi_i \neq \theta_i$. Then, by construction,

$$d(\xi, \theta) \geq \frac{1}{2^{(i-1)\,m+n}}.$$

On the other hand, equation (2.2.1) immediately implies that

$$|h(\xi) - h(\theta)| \leq \frac{1}{2^{sn(i-1)}}.$$

Then

$$|h(\xi) - h(\theta)| \leq \frac{1}{2^{s\,n(i-1)}} = 2^{n\,s}\left(\frac{1}{2^{i\,m}}\right)^{\frac{s\,n}{m}} \leq 2^{n\,s}\left(\frac{1}{2^{(i-1)\,m+n}}\right)^{\frac{s\,n}{m}} \leq 2^{n\,s}\, d(\xi, \theta)^{\frac{s\,n}{m}}.$$

We now check that h is surjective. Given $x \in [0,1]$ we want to define a sequence $\xi_x = \xi_1 \xi_2 \cdots \in \mathbb{E}'_W$ such that $h(\xi_x) = x$. We proceed inductively: If $x \in \left[0, \frac{1}{2^{n\,s}}\right]$, we let $\xi_1 = w$, otherwise we let $\xi_1 = u$. Suppose defined $\xi_1, \xi_2, \ldots, \xi_k$ and their associated intervals $c_1, c_2, \ldots, c_k$ defined as above and satisfying $x \in c_i$ for all $i = 1, \ldots, k$. We let $\xi_{k+1} = w$ if $x \in \left[a_k, a_k + \frac{b_k - a_k}{2^{n\,s}}\right]$ or $\xi_{k+1} = u$ otherwise. By construction, $h(\xi_x) = x$, This completes the proof of the claim. □

By Lemmas 2.2.8 and 2.2.9 we have that $1 = \mathrm{HD}([0,1]) \leq \frac{m}{s\,n}\mathrm{HD}(\mathbb{E}'_W)$, thus $0 < \frac{s\,n}{m} \leq \mathrm{HD}(\mathbb{E}'_W)$, proving the proposition. □

3
Trivial spines

In this section we first characterize and state properties of spines. In Section 3.2, we state simple conditions guaranteeing the triviality of the spine of a sequence. Section 3.3 deals with a special kind of sequences, the 11-sequences (sequences where the symbol 1 only appears in groups of even size). The main result in this section is Lemma 3.3.5 that localizes the spines of 11-sequences. Finally, in Sections 3.4 and 3.4.1 we prove Theorem 2 .

3.1
Characterization and properties of spines

Before going to our constructions let us introduce some notation. Given a word $a = w_1 w_2 \dots w_k$, $w_i = 0, 1$, we let

$$g_{a,t} \stackrel{\text{def}}{=} f_{w_1,t} \circ \cdots f_{w_{r-1},t} \circ f_{w_k,t}, \quad \text{where } f_{0,t} = f_0. \tag{3.1.1}$$

We have the following characterization of a spine $I_{\xi,t}$.

Lemma 3.1.1 (Characterization of spines)**.** *For every $t \in [0,1]$ and $\xi = \xi^-.\xi^+ \in \Sigma_2$, if we write $\widehat{\xi^-} = a_1 a_2 \dots a_n \dots$ where a_i are words in $\{0,1\}$, we have*

$$I_{\xi,t} = \lim_{n\to\infty} g_{a_1,t} \circ g_{a_2,t} \circ \cdots \circ g_{a_n,t}([0,1]).$$

Proof. By definition of spine we have that $x \in I_{\xi,t}$ if, and only if, $g^{-1}_{a_n,t} \circ \cdots \circ g^{-1}_{a_2,t} \circ g^{-1}_{a_1,t}(x) \in [0,1]$ for every $n \in \mathbb{N}$, proving the lemma. □

We have the following corollary.

Corollary 3.1.2. *Consider a set $W = \{u, w\}$ consisting of two words. Suppose that there is an interval $[a,b] \subset [0,1]$ and a parameter $t \in (0,1)$ such that*

$$[a,b] \subset g_{u,t}([a,b]) \quad \textit{and} \quad [a,b] \subset g_{w,t}([a,b]).$$

Then

$$[a,b] \subset I_{\xi,t} \quad \textit{for every } \xi \in \mathbb{S}_W.$$

Proof. Take a sequence $\xi \in \mathbb{S}_W$ and write $\widehat{\xi^-} = u^{h_1}w^{n_1}\dots u^{h_j}w^{n_j}\dots$, with $h_i, n_i \geq 0$. By Lemma 3.1.1 we have that

$$I_{\xi,t} = \lim_{r\to\infty} g_{u,t}^{h_1} \circ g_{w,t}^{n_1} \circ \cdots \circ g_{u,t}^{h_r} \circ g_{w,t}^{n_r}([0,1]).$$

Now observe that by hypothesis

$$[a,b] \subset g_{u,t}^{h_j} \circ g_{w,t}^{n_j}([a,b]), \quad \text{for every } j \in \mathbb{N}.$$

Therefore by Lemma 3.1.1

$$[a,b] \subset \lim_{r\to\infty} g_{u,t}^{h_1} \circ g_{w,t}^{n_1} \circ \cdots \circ g_{u,t}^{h_r} \circ g_{w,t}^{n_r}([a,b]) \subset I_{\xi,t},$$

which implies the corollary. □

Lemma 3.1.3. *Consider a finite word ξ_0 and the periodic sequence $\xi_0^{\mathbb{Z}}$. If $g_{\xi_0,t}$, $t \in (0,1)$, has a repelling fixed point then the spine $I_{\xi,t}$ is non-trivial.*

Proof. Denote by p the repelling fixed point for $g_{\xi_0,t}$, then there exist $\epsilon > 0$ such that $[p-\epsilon, p+\epsilon] \subset g_{\xi_0,t}([p-\epsilon, p+\epsilon])$. By Corollary 3.1.2 we have the lemma. □

3.2 A sufficient condition for trivial spines

Recall the definitions of the conjugate $\widehat{\xi^-}$ of a one-sided sequence ξ^- and of a spines $I_{\xi,t}$.

Lemma 3.2.1 (A sufficient condition for trivial spines)**.** *Let $\xi = \xi^-.\xi^+ \in \Sigma_2$ be a sequence such that*

$$\widehat{\xi^-} = a_1\, a_2\, \dots a_r\, \dots, \qquad \text{for some sequence of finite words } a_i.$$

Suppose that there is $\rho > 1$ and $C > 0$ such that

$$\left(g_{a_r,t}^{-1} \circ g_{a_{r-1},t}^{-1} \circ \cdots \circ g_{a_1,t}^{-1}\right)'(x) \geq C\,\rho^r$$

for a sequence $r \to \infty$ and $x \in I_{\xi,t}$. Then $I_{\xi,t}$ is trivial.

Proof. Consider the spine $I_{\xi,t}$ of ξ for F_t. Then

$$1 \geq \left|\left(g_{a_r,t}^{-1} \circ g_{a_{r-1},t}^{-1} \circ \cdots \circ g_{a_1,t}^{-1}\right)(I_{\xi,t})\right| \geq C\,\rho^r |I_{\xi,t}|.$$

Since that $r \to \infty$ this implies that $|I_{\xi,t}| = 0$ and thus $I_{\xi,t}$ is trivial. □

Roughly speaking, this lemma says that if the pass of the sequence ξ is contractive (corresponding to a concatenation of contractive functions) then the spine ξ is trivial. A simple form of to guarantee this contraction is to consider sequences ξ^- with a lot of 1's. More precisely, consider for $k = 0, 1$ the limit frequency of k's of ξ^- given by

$$\phi_k(\xi) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\#\{i \in [1,n] \colon \xi_{-i} = k\}}{n}. \tag{3.2.1}$$

Proposition 3.2.2. *Consider $\delta \in (0,1)$. Then $I_{\xi,t}$ is trivial for all ξ with $\phi_1(\xi) > \delta$ and $t \in (0, (\beta)^{-\frac{1-\delta}{\delta}})$.*

Proof. Take $\xi = \xi^-.\xi^+ \in \Sigma_2$ as in the proposition and write $\widehat{\xi^-} = \alpha_1\alpha_2\dots\alpha_n\dots$. The hypothesis implies that for every $n \in \mathbb{N}$ there exist $m \geq n$ such that

$$\frac{\#\{i \in [1,m] \colon \xi_{-i} = 1\}}{m} > \delta.$$

Recalling that $f_{1,t} = t\,(1-x)$ and $\beta \geq f_0'(x) > 0$ if $x \in (0,1)$ this implies that for all $x \in I_{\xi,t}$ we have

$$(f^{-1}_{\alpha_m,t} \circ \cdots \circ f^{-1}_{\alpha_2,t} \circ f^{-1}_{\alpha_1,t})'(x) \geq t^{-\lfloor \delta\, m \rfloor}\, \beta^{-m+\lfloor \delta\, m \rfloor},$$

where $\lfloor x \rfloor$ stands for the entire part of a number $x \in \mathbb{R}$. Now taking $t_\delta = (\beta)^{-\frac{1-\delta}{\delta}}$ and applying Lemma 3.2.1 the proposition follows. $\square$

Corollary 3.2.3. *Consider ξ such that $\phi_1(\xi) > 0$. Then there is $t_\xi > 0$ such that $I_{\xi,t}$ is trivial for all $t \in (0, t_\xi]$.*

Intuitively, the previous results imply that the set of sequences with trivial spines "growths" as t goes to 0^+. Note that the spines of sequences of the form $0^{-\mathbb{N}}.\xi^+$ are non-trivial for all $t \in (0,1]$. Now a natural question is to determine whether the spines with asymptotic frequency of 1's equal to zero have non-trivial spines for every $t \in (0,1]$. The answer to this question is in general negative.

Proposition 3.2.4. *There is a sequence $\xi \in \Sigma_2$ and $t \in (0,1)$ such that $\phi_1(\xi) = 0$ and $I_{\xi,t}$ is trivial.*

Let us give a heuristic naive explanation of this fact. Given a sequence ξ with $\phi_1(\xi) = 0$ it may occur that most of the pre-images of $[0,1]$ by the corresponding maps fall in regions where the map f_0^{-1} is expanding. This implies that the distance between two different points of $I_{\xi,t}$ uniformly increases after backward iterations. This will force the spine $I_{\xi,t}$ to be trivial.

The sequences we consider in what follows are of a special type where the 1 digits appear in blocks of two elements. This will imply that the maps consider for defining the spines are concave. Having this in mind, we say that a word a is a 11-*word* if the 1's of the word a appear in blocks of even size, that is,

$$a = 0^{m_0}\, 1^{2\,n_1}\, 0^{m_1}\, \ldots\, 1^{2\,n_r}\, 0^{m_r}, \quad m_0 \geq 0 \text{ and } n_i, m_i \geq 1 \text{ for } i = 1, \ldots, r.$$

We say that $\xi = \xi^-.\xi^+$ is a 11-*sequence* if $\widehat{\xi^-}$ can written as an infinite concatenation of 11-words and every block of 1's have finite size (or equivalently, it contains infinitely many 0's). The map $g_{a,t}$ associated to a is called a 11-*map.*

Proposition 3.2.5. *Consider $\xi = \xi^-.\xi^+$ such that ξ^- is a periodic* 11-*sequence. Then $I_{\xi,t}$ is trivial for every $t \in (0, 1)$.*

Consider now the subset $\mathcal{P} \subset \Sigma_2^-$:

$$\mathcal{P} = \{\xi^- = a^{\mathbb{N}} \text{ where } a \text{ is a 11-word}\}.$$

Proposition 3.2.5 implies that $I_{\xi,t}$ is trivial for every $t \in [0, 1)$ and $\xi = \xi^-.\xi^+$ with $\xi^- \in \mathcal{P}$. Note that the set $\mathcal{P}$ is countable and therefore has zero Hausdorff dimension. Thus to prove Theorem 2 (existence of a subset of Σ_2^- with positive Hausdorff dimension and only having elements with trivial spines for every $t \in (0, 1)$) we need to concatenate different 11-words in order to get a set positive Hausdorff dimension.

We prove these propositions in the next subsection after give some properties of 11-maps.

3.3 Localization of spines associated to 11-sequences

We begin with a simple property of 11-maps.

Remark 3.3.1. Given a pair of concave maps with positive derivatives, g_1, $g_2\colon [0, 1] \to [0, 1]$, its composition is also concave and has positive derivative. To check the concavity it is enough to see that $(g_1 \circ g_2)''(x) \leq 0$ for every $x \in [0, 1]$. Noting that $g_1''(x) \leq 0$, $g_2''(x) \leq 0$, and $g_1'(x) \geq 0$ it follows

$$(g_1 \circ g_2)''(x) = (g_1'(g_2(x))\, g_2'(x))' = g_1''(g_2(x))\, (g_2'(x))^2 + g_1'(g_2(x))\, g_2''(x) \leq 0.$$

Lemma 3.3.2. *Let $t \in (0, 1)$. If a is a* 11-*word then $g_{a,t}$ is concave and $0 < g_{a,t}(0) < g_{a,t}(1) < 1$. In particular, $g_{a,t}$ has a unique fixed point $p_{a,t}$ that is attracting.*

Proof. Observe that, since a is a 11-word the map $g_{a,t}$ is a composition of the maps $f_{1,t}^2$ and f_0 that are concave and orientation preserving. Thus the concavity of $g_{a,t}$ follows from Remark 3.3.1.

Note that by definition $g_{a,t}(0), g_{a,t}(1) \in (0,1)$. As $g_{a,t}\colon [0,1] \to [0,1]$ it has at least one fixed point. We see that the fixed points of $g_{a,t}$ are attracting points, thus $g_{a,t}$ has exactly one fixed point. Pick any fixed point z of $g_{a,t}$ By the mean value theorem, there is a $y \in (0,z)$ such that

$$g'_{a,t}(y) = \frac{g_{a,t}(z) - g_{a,t}(0)}{z-0} = \frac{z - g_{a,t}(0)}{z} < \frac{z}{z} = 1.$$

Hence, by concavity, $g'_{a,t}(z) \le g'_{a,t}(y) < 1$ and thus z is attracting. □

We now are ready to prove Proposition 3.2.4

3.3.1
Proof of Proposition 3.2.4

Let $\xi = \xi^-.\xi^+$ such that

$$\widehat{\xi^-} = 110110011000110000\ldots110^i110^{i+1}11\ldots.$$

That is ξ is a 11-sequence.

Let $v \overset{\text{def}}{=} 110$ and denote by $p_{v,t}$ the (attracting) fixed point of $g_{v,t}$. Note that $p_{v,1} = 1$, $p_{v,t}$ depends continuously on t, and $p_{v,t}$ is close to 1^- is t is close to 1^-. Thus there are $\lambda \in (0,1)$ and $t_0 \in (0,1)$ such that

$$f'_0(x) < \lambda, \quad \text{for every } x \in [p_{v,t}, 1] \text{ and } t \in [t_0, 1]. \tag{3.3.1}$$

Take

$$a_0 \overset{\text{def}}{=} v, \qquad a_i \overset{\text{def}}{=} 0^i\, v \quad \text{for every } i \ge 1.$$

Lemma 3.3.3. *It holds $g'_{a_i,t}(x) < \lambda$ for all $x \in [p_{v,t}, 1]$, $i \ge 0$, and $t \in [t_0, 1]$.*

Proof. Observe first that $g_{v,t}$ is a contraction in $[p_{v,t}, 1]$. The lemma now follows from (3.3.1) and $g_{v,t}([p_{v,t}, 1]) \subset [p_{v,t}, 1]$. □

Lemma 3.3.4. *There exists $n_0 \in \mathbb{N}$ such that*

$$g_{a_n}^{-1} \circ g_{a_{n-1}}^{-1} \circ \cdots \circ g_{a_1}^{-1} \circ g_{a_0}^{-1} \circ g_{a_0}^{-1}(I_{\xi,t}) \subset [p_{v,t}, 1], \quad \textit{for every } n \ge n_0.$$

Proof. Note that there is $n_0 \in \mathbb{N}$ such that $f_0^n(g_{v,t}(0)) > p_{v,t}$ for every $n \ge n_0$. Therefore for every $x \in [0,1]$ we have

$$g_{a_n,t}(x) = f_0^n \circ g_{v,t}(x) > f_0^n \circ g_{v,t}(0) > p_{v,t}.$$

Therefore

$$g_{a_n,t}([0,1]) \subset [p_{v,t},1], \quad \text{for every } n \geq n_0.. \tag{3.3.2}$$

Note that $g_{v,t}([p_{v,t},1]) \subset [p_{v,t},1]$ and f_0 is increasing, therefore

$$g_{a_i,t}([p_{v,t},1]) \subset [p_{v,t},1], \quad \text{for every } i \geq 0. \tag{3.3.3}$$

Equations (3.3.2) and (3.3.3) imply that for $n \geq n_0$ one has that

$$\begin{aligned} &g_{a_0,t} \circ g_{a_0,t} \circ g_{a_1,t} \circ \cdots \circ g_{a_{n-1},t} \circ g_{a_n,t}([0,1]) \subset \\ &\subset g_{a_0,t} \circ g_{a_0,t} \circ g_{a_1,t} \circ \cdots \circ g_{a_{n-1},t}([p_{v,t},1]). \end{aligned} \tag{3.3.4}$$

By Lemma 3.1.1, we have

$$I_{\xi,t} = \lim_{n\to\infty} g_{a_0,t} \circ g_{a_0,t} \circ g_{a_1,t} \circ \cdots \circ g_{a_{n-1},t} \circ g_{a_n,t}([0,1]).$$

Since this limit is decreasing it follows that for every $n \geq n_0$

$$I_{\xi,t} \subset g_{a_0,t} \circ g_{a_0,t} \circ g_{a_1,t} \circ \cdots \circ g_{a_{n-1},t} \circ g_{a_n,t}([0,1]).$$

Finally, from (3.3.4) one gets

$$I_{\xi,t} \subset g_{a_0,t} \circ g_{a_0,t} \circ g_{a_1,t} \circ \cdots \circ g_{a_{n-1},t} \circ g_{a_n,t}([p_{v,t},1]), \text{ for every } n \geq n_0.$$

This immediately implies the lemma. □

Take $\rho = \lambda^{-1} > 1$ and note that Lemmas 3.3.3 and 3.3.4 imply that for every n and $t \in [t_0,1]$ one has

$$\left(g^{-1}_{a_n,t} \circ g^{-1}_{a_{n-1},t} \circ \cdots \circ g^{-1}_{a_1,t} \circ g^{-1}_{a_0,t} \circ g^{-1}_{a_0,t}\right)'(x) \geq C\,\lambda^{-r}, \quad \text{for every } x \in I_{\xi,t}.$$

Lemma 3.2.1 now implies that $I_{\xi,t}$ is a singleton, proving the proposition. □

3.3.2 Proof of Proposition 3.2.5

In what follows, for a given 11-word a recall that $p_{a,t}$ is the only (attracting) fixed point of $g_{a,t}$, see Lemma 3.3.2. We have the following lemma which will be used also in Subsection 3.4.

Lemma 3.3.5 (Localization of spines)**.** *Consider two 11-words a and c and the set $W = \{a,c\}$. For $t \in (0,1)$ let $J_{W,t}$ be the interval bounded by the fixed points $p_{a,t}$ and $p_{c,t}$ of $g_{a,t}$ and $g_{c,t}$. Then $I_{\xi,t} \subset J_{W,t}$ for every $\xi \in \mathbb{S}_W$ and $t \in (0,1)$.*

From this lemma one gets immediately the following corollary.

Corollary 3.3.6. *Consider a set $W = \{a_1, \ldots, a_r\}$ consisting of 11-words. For each $t \in (0,1)$ let $J_{W,t} = [p^-_{W,t}, p^+_{W,t}]$ where*

$$p^-_{W,t} = \min\{p_{a_i,t},\, a_i \in W\} \quad \textit{and} \quad p^+_{W,t} = \max\{p_{a_i,t},\, a_i \in W\}.$$

Then $I_{\xi,t} \subset J_{W,t}$ for every $\xi \in W$ and $t \in (0,1)$.

We now prove the proposition. The sequence $\widehat{\xi^-}$ is obtained concatenating a unique 11-word a. Thus we can take $a = c$ in the lemma. In this case we have $W = \{a, c = a\}$ and $J_{W,t} = \{p_{a,t}\}$. This concludes the proof of the proposition.

Proof of Lemma 3.3.5. Let us assume that $J_{W,t} = [p_{a,t}, p_{c,t}]$. The proof is by contradiction, suppose that there is $x \in [0, p_{a,t}) \cap I_{\xi,t}$ for some $\xi = \xi = \xi^-.\xi^+ \in \mathbb{S}_W$. Note that

$$\widehat{\xi^-} = w_1\, w_2\, w_3 \,\cdots w_i \cdots \quad \text{where } w_i = a \text{ or } c.$$

The fact that $x \in I_{\xi,t}$ implies that

$$x_r \stackrel{\text{def}}{=} g^{-1}_{w_r,t} \circ \cdots \circ g^{-1}_{w_2,t} \circ g^{-1}_{w_1,t}(x) \in [0,1], \quad \text{for all } r \geq 1. \tag{3.3.5}$$

Note that $g_{w_i,t} = g_{a,t}$ or $g_{w_i,t} = g_{c,t}$. As $x < p_{a,t} \leq p_{c,t}$ and $p_{a,t}, p_{c,t}$ are the attracting fixed points of $g_{a,t}$ and $g_{c,t}$, the concavity of these maps implies that the sequence $(x_r)_r$ is decreasing and thus it has a limit $x_\infty \in [0,1]$.

Note that $f^2_{1,t}(0) = t - t^2$. This implies that if v is a 11-word then $g_{v,t}(0) \geq t - t^2$. This implies that that $x_r \geq t - t^2$ and hence this implies that $x_\infty \in [t - t^2, x]$.

As $x < p_{a,t}$ there is $\delta_x > 0$ such that for every $y \in [t - t^2, x]$ one has

$$\max\{g^{-1}_{a,t}(y),\, g^{-1}_{c,t}(y)\} < y - \delta_x.$$

Taking $y = x_\infty$ for large r we have

$$x_{r+1} = g^{-1}_{w_{r+1},t}(x_r) < x_\infty$$

which is a contradiction.

A similar argument implies that $(p_{c,t}, 1] \cap I_{\xi,t} = \emptyset$. This completes the proof of the lemma. □

Scholium 3.3.7. *Consider the set $W = \{a_1, \ldots, a_r\}$, where consisting of 11-words. For each $t \in (0,1)$ let $J_{W,t} = [p^-_{W,t}, p^+_{W,t}]$ where*

$$p^-_{W,t} = \min\{p_{a_i,t},\, a_i \in W\} \quad \textit{and} \quad p^+_{W,t} = \max\{p_{a_i,t},\, a_i \in W\}.$$

Then $I_{\xi,t} \subset I_{W,t}$ *for every* $\xi \in \mathbb{S}_W$ *and* $t \in (0,1)$.

3.4 Proof of Theorem 2

For each $\ell \in \mathbb{N}$, $\ell \geq 2$, we consider the set of two 11-words

$$B_\ell \overset{\text{def}}{=} \{e_\ell \overset{\text{def}}{=} 1^{2\ell}0^\ell,\ v = 110\}$$

and its associated sets of sequences $\mathbb{E}_{B_\ell}$ and $\mathbb{S}_{B_\ell}$.

Theorem 3.4.1. *There is* ℓ_0 *such that for every* $\ell \geq \ell_0$ *and every* $t \in (0,1)$ *the spine* $I_{\xi,t}$ *of any* $\xi \in \mathbb{S}_{B_\ell}$ *is trivial.*

This result implies that $\mathbb{E}_{B_\ell} \subset \Sigma_2^{-,\mathcal{T}}(0)$ for all $t \in (0,1)$ and $\ell \geq \ell_0$. Therefore, by Proposition 2.2.7, for each $\ell \geq \ell_0$ we have

$$0 < \mathrm{HD}(\mathbb{E}_{B_\ell}) \leq \mathrm{HD}(\Sigma_2^{-,\mathcal{T}}(0)).$$

This implies Theorem 2.

3.4.1 Proof of Theorem 3.4.1

. Given $\xi = \xi^-.\xi^+ \in \mathbb{S}_{B_\ell}$, consider the conjugate of $\widehat{\xi^-}$

$$\widehat{\xi^-} = e_\ell^{h_0} v^{n_1} e_\ell^{h_1} \dots v^{n_r} e_\ell^{h_r} \dots, \quad \text{where } h_i, n_i \geq 0 \text{ for } i \geq 0.$$

Assume first that is $j \in \mathbb{N}$ such that $h_i = 0$ for every $i \geq j$ or $n_i = 0$ for every $i \geq j$. Let us consider the first possibility (the second one is similar and thus omitted). Note that is the spine of ξ is trivial then the spine of any $\sigma^k(\xi)$ is also trivial. Therefore we can assume without loss of generality that $h_i = 0$ for every $i \geq 0$. In this case $\widehat{\xi^-}$ is generated by the 11-word v and by Proposition 3.2.5 one has $I_{\xi,t}$ is a singleton.

Thus it remains to consider the case where $n_i, h_i \geq 1$ for all $i \geq 1$ (note that h_0 may be 0). This case is considered in the next proposition.

Proposition 3.4.2. *There is* ℓ_0 *such that for every* $\ell \geq \ell_0$ *and for every* $t \in (0,1)$ *there are constants* $C > 0$ *and* $\eta_t > 1$ *with the the following property: Given any* $\xi = \xi^-.\xi^+ \in \mathbb{S}_{B_\ell}$ *consider the conjugate of* $\widehat{\xi^-}$

$$\widehat{\xi^-} = e_\ell^{h_0} v^{n_1} e_\ell^{h_1} \dots v^{n_r} e_\ell^{h_r} \dots, \qquad \text{where } h_0 \geq 0 \text{ and } h_i, n_i \geq 1 \text{ for } i \geq 1. \tag{3.4.1}$$

Then

$$((g_{e_\ell,t}^{h_r})^{-1} \circ (g_{v,t}^{n_r})^{-1} \circ \cdots \circ (g_{e_\ell,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1} \circ (g_{e_\ell,t}^{h_0})^{-1})'(x) \geq C\,\eta_t^r$$

for all $r \geq 1$ *and* $x \in I_{\xi,t}$*.*

Note by Lemma 3.2.1, this proposition implies that the spine $I_{\xi,t}$ is trivial for all $t \in (0,1)$ and $\xi \in \mathbb{S}_{B_\ell}$, ending the proof of Theorem 3.4.1.

Proof of Proposition 3.4.2: first part. Note that to prove the proposition it is enough to see that

$$((g_{e_\ell,t}^{h_r})^{-1} \circ (g_{v,t}^{n_r})^{-1} \circ \cdots \circ (g_{e_\ell,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1})'(x) \geq C\eta_t^r$$

for all $r \geq 1$ and $x \in I_{\sigma^{-\bar{h}_0}(\xi),t}$, where $\bar{h}_0 = 3\,h_0\,\ell$.

Consider the attracting fixed points $p_{v,t}$ and $p_{e_\ell,t}$ of the maps $g_{v,t}$ and $g_{e_\ell,t}$. Note that for t close to 0 these maps are uniformly contracting, while for t close to 1 this is not anymore the case (there are contracting and expanding regions).

Define t_1 by the condition

$$g'_{v,t_1}(0) = 1, \qquad t_1 = \beta^{-1/2}. \tag{3.4.2}$$

The choice of t_1 implies that

$$g'_{v,t}(x) \in (0,1) \quad \text{for all } t \in (0,t_1) \text{ and } x \in [0,1]. \tag{3.4.3}$$

The definition of t_1 also implies that for $t \geq t_1$ there is (exactly) one point $q_{v,t} \in [0,1]$ (depending continuously on t) with

$$g'_{v,t}(q_{v,t}) = 1, \quad q_{v,t_1} = 0.$$

For $t \in [0,t_1]$ we let $q_{v,t} \stackrel{\text{def}}{=} 0$. Consider the fixed point $a_t \stackrel{\text{def}}{=} t/(1+t)$ of $f_{1,t}$ and note that for $t \in (0,1]$

$$g_{v,t}(a_t) = g_{v,t}\left(\frac{t}{1+t}\right) = f_{1,t}^2 \circ f_0\left(\frac{t}{1+t}\right) > f_1^2\left(\frac{t}{t+1}\right) = \frac{t}{t+1} = a_t.$$

This fact and the concavity of $g_{v,t}$ immediately imply that

$$p_{v,t} > \frac{t}{1+t} = a_t \quad \text{for all } t \in (0,1]. \tag{3.4.4}$$

Remark 3.4.3. Consider a 11-word c whose first entry is 0 and recall that $g_{c,t}$ is concave (Lemma 3.3.2). The calculation above implies that the fixed point $p_{c,t}$ of $g_{c,t}$ is larger than a_t for every $t \in (0,1]$.

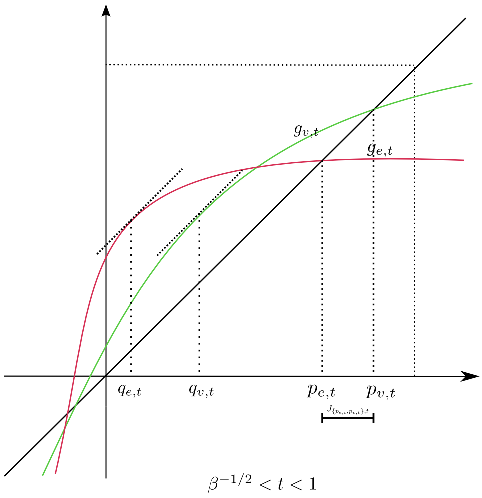

Figure 3.1: Trivial spine

Choice of ℓ_0. To define ℓ_0 we need some preparatory steps. We first define auxiliary constants k_0 and R.

The concavity of $g_{v,t}$ implies that $q_{v,t} < p_{v,t}$ for all $t \in (0,1]$. By (3.4.4) we have that $a_t < p_{v,t}$ for all $t \in (0,1]$. These two facts imply that the number k_0 below is well defined,

$$k_0 \stackrel{\text{def}}{=} \min\{k \geq 0 \text{ such that } g_{v,t}^{-k}(q_{v,t}) < a_t \text{ for all } t \in [t_1, 1]\}. \tag{3.4.5}$$

Let

$$R \stackrel{\text{def}}{=} \max\{g'_{v,t}(a_t),\, t \in [t_1, 1]\}.$$

Lemma 3.4.4. *$R^{k_0} \geq 1$ and*

$$R^{k_0} \geq \max\{(g^n_{v,t})'(x) \colon x \in [a_t, p_{v,t}],\, n \in \mathbb{N},\, \textit{and } t \in [t_1, 1]\} \geq 1.$$

Proof. We first see that $R^{k_0} \geq 1$. This is obvious if $k_0 = 0$. If $k_0 \geq 1$ its definition implies that $k_0 \geq 1$ then $a_t < q_{v,t}$ for some t. The concavity of $g_{v,t}$ implies that $1 = g'_{v,t}(q_{v,t}) < g'_{v,t}(a_t) \leq R$. Thus $R^{k_0} \geq 1$ proving the first assertion.

By the concavity of $g^n_{v,t}$ (recall Lemma 3.3.2),

$$(g^n_{v,t})'(a_t) \geq (g^n_{v,t})'(x) \quad \text{for every} \quad x \in [a_t, p_{v,t}] \text{ and } t \in [t_1, 1].$$

This implies that

$$\max\{(g^n_{v,t})'(x) \colon x \in [a_t, p_{v,t}],\, n \in \mathbb{N},\, t \in [0,1]\} = \\ = \max\{(g^n_{v,t})'(a_t) \colon\, n \in \mathbb{N},\, t \in [t_1, 1]\}.$$

Thus it remains to see that

$$R^{k_0} \geq \max\{(g^n_{v,t})'(a_t)\colon n \in \mathbb{N},\ t \in [t_1, 1]\}.$$

For each $t \in [t_1, 1]$ define k_t as the first k with $g^{-k}_{v,t}(q_{v,t}) < a_t$. Note that $k_t \leq k_0$ and $g^j_{v,t}(a_t) \leq q_{v,t}$ for all $0 \leq j \leq k_t - 1$.

Claim 3.4.5. *It holds $(g^n_{v,t})'(a_t) \leq (g^{k_t}_{v,t})'(a_t)$ for all $n \geq 0$ and $t \in [t_1, 1]$*

Before proving the claim let us deduce the lemma from it. First, if $k_0 = 0$ then $k_t = 0$ and $R = 1$. In this case $a_t > q_{v,t}$ for all $t \in [t_1, 1]$ and thus $(g^n_{v,t})'(a_t) \leq g'_{v,t}(a_t) \leq 1 = R^0$. Finally, if $k_0 \geq 1$, as $R \geq 1$, then from the definition of k_t one has

$$R^{k_0} \geq R^{k_t} \geq 1 \geq (g^{k_t}_{v,t})'(a_t),$$

proving the lemma.

Proof of Claim 3.4.5. In what follows we fix $t \in [t_1, 1]$ There are two cases.
Case 1: $n \geq k_t$. Note that $(g^{k_t}_{v,t})(a_t) > q_{v,t}$ implies that $(g^{n-k_t}_{v,t})'(g^{k_t}_{v,t}(a_t) < 1$. Hence

$$(g^n_{v,t})'(a_t) = (g^{n-k_t}_{v,t})'(g^{k_t}_{v,t}(a_t))\,(g^{k_t}_{v,t})'(a_t) < (g^{k_t}_{v,t})'(a_t).$$

Case 2: $0 \leq n < k_t$. In this case $(g^j_{v,t})(a_t) \leq q_{v,t}$ for every $0 \leq j \leq k_t - 1$. Thus

$$(g_{v,t})'(g^j_{v,t}(a_t)) \geq 1 \quad \text{for every } 0 \leq j \leq k_t - 1.$$

This implies that

$$(g^{k_t}_{v,t})'(a_t) = g'_{v,t}(g^{k_t-1}_{v,t}(a_t))\, g'_{v,t}(g^{k_t-2}_{v,t}(a_t)) \cdots g'_{v,t}(g^n_{v,t}(a_t))\,(g^n_{v,t})'(a_t) \geq (g^n_{v,t})'(a_t).$$

This concludes the proof of the claim. □

The proof of the lemma is now complete. □

We are now ready for defining ℓ_0. We let

$$\ell_0 \stackrel{\text{def}}{=} \min\{\ell \geq 2 \quad \text{such that} \quad (f^\ell_0)'(x) < (2\,R)^{-(k_0+1)} \quad \text{for all } x \in [a_{t_1}, 1]\}. \tag{3.4.6}$$

Note that as $f'_0(1) < 1$ and $f^n_0(x) \to 1$ as $n \to \infty$, for every $x \in [a_{t_1}, 1]$, this number is well defined.

Lemma 3.4.6. *For every $\ell \geq \ell_0$ and $t \in [t_1, 1)$ one has*

$$(f^\ell_0)'(p_{v,t}) < (2\,R)^{-(k_0+1)} < 1.$$

Proof. As $p_{v,t} > a_t \geq a_{t_1}$, recall (3.4.4), the definition of ℓ_0 and $t \geq t_1$ imply that

$$1 > (2\,R)^{-(k_0+1)} > (f_0^{\ell_0})'(p_{v,t}) = f_0'(f_0^{\ell_0-1}(p_{v,t}))\, f_0'(f_0^{\ell_0-2}(p_{v,t})) \cdots f_0'(p_{v,t}).$$

As the product of the derivatives is increasing,

$$f_0'(f_0^{\ell_0-1}(p_{v,t})) < f_0'(f_0^{\ell_0-2}(p_{v,t})) < \cdots < f_0'(p_{v,t}),$$

we have that

$$f_0'(f_0^{\ell_0-1}(p_{v,t})) < 1 \quad \text{for all} \quad t \in [t_1, 1].$$

The concavity of f_0 now implies that $f_0'(f_0^{\ell_0}(p_{v,t})) < 1$ and thus

$$(f_0^{\ell-\ell_0})'(f_0^{\ell_0}(p_{v,t})) < 1 \quad \text{for all} \quad t \in [t_1, 1].$$

This immediately implies that for all $t \in [t_1, 1]$ one has

$$(f^\ell)'(p_{v,t}) = (f_0^{\ell-\ell_0})'(f_0^{\ell_0}(p_{v,t}))\, (f_0^{\ell_0})'(p_{v,t}) < (f_0^{\ell_0})'(p_{v,t}) < (2\,R)^{-(k_0+1)},$$

ending the proof of the lemma. □

3.4.2
End of the proof of Proposition 3.4.2

We now see that the expansion hypothesis the proposition holds for the integer ℓ_0 fixed above. We fix $\ell \geq \ell_0$ and, for simplicity, write $e = e_\ell$. Given a sequence $\xi \in \mathbb{S}_{B_\ell}$ write

$$\widehat{\xi^-} = e^{h_0} v^{n_1} e^{h_1} v^{n_2} e^{h_2} \dots v^{n_r} e^{h_r} \dots, \quad \text{where } h_0 \geq 0 \text{ and } h_i, n_i \geq 1 \text{ for } i \geq 1.$$

For $r \geq 1$ define

$$\bar{n}_r \stackrel{\text{def}}{=} n_1 + \sum_{2 \leq i \leq r} h_{i-1} + n_i \quad , \quad \bar{h}_r \stackrel{\text{def}}{=} \sum_{1 \leq i \leq j} n_i + h_i.$$

Given any $j \in \mathbb{N}$ we can write

$$j = j_r + \bar{n}_r \text{ with } 0 \leq j_r < h_r \text{ if } j \in [\bar{n}_r, \bar{h}_r),$$
$$j = j_r + \bar{h}_r \text{ with } 0 \leq j_r < n_{r+1} \text{ if } j \in [\bar{h}_r, \bar{n}_{r+1}).$$

Consider the spine of $\sigma^{-3\ell h_0}(\xi)$ for F_t that is given by

$$I_{0,t} \stackrel{\text{def}}{=} I_{\sigma^{-3\ell h_0}(\xi),t}$$

and for every $j \geq 1$ define

$$I_{j,t} = g_{e,t}^{-j_r} \circ g_{v,t}^{-n_r} \circ \cdots \circ g_{e,t}^{-h_1} \circ g_{v,t}^{-n_1}(I_{0,t}), \text{ if } j_r \in [\bar{n}_r, \bar{h}_r],$$
$$I_{j,t} = g_{v,t}^{-j_r} \circ g_{e,t}^{-h_r} \cdots \circ g_{e,t}^{-h_1} \circ g_{v,t}^{-n_1}(I_{0,t}) \text{ if } j_r \in [\bar{h}_r, \bar{n}_{r+1}].$$

Note that, by construction, the spines satisfy

$$I_{j,t} = I_{\sigma^{-j-h_0}(\xi),t}.$$

Remark 3.4.7. *By Lemma 3.3.5, the set $I_{j,t}$ is contained in the closed interval $I_{\{e,v\},t}$ bounded by the fixed points $p_{e,t}$ and $p_{v,t}$ of $g_{e,t}$ and $g_{v,t}$, respectively. In particular, $J_{j,t} \subset J_{\{e,v\},t}$ for all $j \geq 1$.*

There are two cases according to the value of $t \in (0,1)$.

Case 1: $t < t_1 = \beta^{-1/2}$. In this case $g_{v,t}$ is a contraction (recall (3.4.3)). We claim that $g_{e,t}$ is also a contraction. Note that for any $x \in [0,1]$ it holds

$$g'_{e,t}(x) \leq g'_{e,t}(0) = (f_{1,t}^{2\ell} \circ f_0^{\ell})'(0) = t^{2\ell}(f_0^{\ell})'(0) = t^{2\ell}\beta^{\ell} <$$
$$< t_1^{2\ell}\beta^{\ell} = \beta^{-\ell}\beta^{\ell} = 1.$$

Thus, in this case $g_{e,t}^{-1}$ and $g_{e,t}^{-1}$, are expanding in $[0,1]$ and thus (3.4.1) holds.

Case 2: $t \geq t_1 = \beta^{-1/2}$. We need to consider two cases according to the relative positions of $p_{e,t}$ and $p_{v,t}$.

Case 2A: $p_{v,t} < p_{e,t}$. By Remark 3.4.7, we have that $I_{j,t} \subset [p_{v,t}, p_{e,t}]$. We now see that $g_{e,t}$ and $g_{v,t}$ are uniformly contracting in $[p_{v,t}, p_{e,t}]$. By concavity of $g_{e,t}$ we have $g_{e,t}(p_{v,t}) > p_{v,t}$. Thus

$$p_{v,t} < p_{e,t} \quad \text{and} \quad g'_{e,t}(p_{v,t}) = (f_{1,t}^{2\ell} \circ f_0^{\ell})'(p_{v,t}) \leq (f_0^{\ell})'(p_{v,t}) < 1,$$

where the last inequality follows from $t \in [t_1, 1)$ and Lemma 3.4.6. By concavity this implies that

$$g'_{e,t}(x) < 1, \quad \text{for all } x \in [p_{v,t}, p_{e,t}].$$

The contraction for $g_{v,t}$ follows noting that

$$g'_{v,t}(x) \leq g'_{v,t}(p_{v,t}) < 1 \quad \text{for all } x \in [p_{v,t}, p_{e,t}].$$

This completes the proof of Case A.

Case 2B: $p_{e,t} \leq p_{v,t}$. By Remark 3.4.7, we have that $I_{r,t} \subset [p_{e,t}, p_{v,t}]$. In this case, the proposition is a consequence of the following lemma.

Lemma 3.4.8. *Let* $x \in I_{0,t} \subset [p_{e,t}, p_{v,t}]$. *Then, for every* $j \geq 1$, *one has*

$$\Big((g_{e,t}^{h_j})^{-1} \circ (g_{v,t}^{n_j})^{-1} \circ \dots (g_{e,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1}\Big)'(x) \geq 2^r.$$

Proof. We first estimate the derivatives of $g_{e,t}^{h_i}$. For that recall that the interval $[p_{e,t}, p_{v,t}]$ is contained in $[a_t, 1]$, see Remark 3.4.3. Since a_t is increasing with t, it follows that $[p_{e,t}, p_{v,t}]$ is contained in $[a_{t_1}, 1]$. Thus Lemma 3.4.6 implies that

$$(f_0^\ell)'(x) < (2R)^{-(k_0+1)}, \quad \text{for all } x \in [p_{e,t}, p_{v,t}].$$

Therefore,

$$g_{e,t}'(x) \leq (f_0^\ell)'(x) < (2R)^{-(k_0+1)}, \quad \text{for all } x \in [p_{e,t}, p_{v,t}]. \tag{3.4.7}$$

Observe that if $x \in I_{2i-1,t}$ for some $i \in \mathbb{N}$, then $g_{e,t}^{-m}(x) \in [p_{e,t}, p_{v,t}]$ for every $0 \leq m \leq h_i$. The concavity of $g_{e,t}$ and (3.4.7) implies that

$$(g_{e,t}^{h_i})'(x) \leq (g_{e,t}')^{h_i}(x) < (2R)^{-h_i\,(k_0+1)}, \quad \text{for every } x \in I_{2i-1,t} \text{ and } i \in \mathbb{N}. \tag{3.4.8}$$

This provides an estimate for the terms with *subscript* e in the product in the lemma. To estimate the *complete* product define

$$c_{i,e,t} = \min_{[p_{e,t},p_{v,t}]} (g_{e,t}^{-h_j})'(x) \qquad \text{and} \qquad c_{i,v,t} = \min_{[p_{e,t},p_{v,t}]} (g_{v,t}^{-n_i})'(x).$$

Note that

$$c_{i,v,t} = \left(\max_{[p_{e,t},p_{v,t}]} (g_{v,t}^{n_i})'(x)\right)^{-1}.$$

From Lemma 3.4.4 and (3.4.8) we have

$$(c_{i,v,t}) \geq R^{-k_0}, \quad \text{and} \quad c_{j,e,t} \geq (2R)^{h_i(k_0+1)}$$

Using these inequalities and $h_i \geq 1$, we obtain that for every $x \in I_{0,t}$

$$\begin{aligned}\Big((g_{e,t}^{h_j})^{-1} \circ (g_{v,t}^{n_j})^{-1} \circ \dots (g_{e,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1}\Big)'(x) &\geq (c_{i,e,t})\,(c_{i,v,t}) \dots (c_{1,e,t})\,(c_{1,v,t}) = \\ &\geq (2\,R)^{h_j(k_0+1)}\, R^{-k_0} \dots (2\,R)^{h_1(k_0+1)}\, R^{-k_0} \geq \\ &\geq 2^{h_j+\dots+h_1}\, R^{(h_j+\dots+h_1)(k_0+1)-r\,k_0} \geq 2^j,\end{aligned}$$

for every $j \in \mathbb{N}$, thus proving the lemma. □

The proof of Proposition 3.4.2 is now completed. □

4
Stabilization of spines. Proof of Theorem 3

In Section 3 we described a large subset of Σ_2 for which the spines are abruptly created at $t = 1$. In this section we prove Theorem 3 that is a result in the opposite direction: there is also a large subset of Σ_2 with Hausdorff dimension bigger than one where the spines are non-trivial and depend continuously on the parameter t for $t = 1$ (in particular, these spines are created before $t = 1$).

In the previous section, we restricted our attention to 11-sequences. This allows us to use properties of concave maps of the interval. In this section, our approach is in some sense the opposite: we consider words with an odd number of ones and their associated maps which reverse the orientation.

Proof of Theorem 3. Consider the set of two words

$$C = \{u = 0101, s = 001001\},$$

its associated maps $g_{u,t}$ and $g_{s,t}$, $t \in (0, 1]$, and the sets $\mathbb{E}_C \subset \Sigma_2^-$ and $\mathbb{S}_C \subset \Sigma_2$.

By Proposition 2.2.7 we have that $0 < \mathrm{HD}(\mathbb{E}_C)$. Remark 2.2.5 now implies that

$1 < \mathrm{HD}(\mathbb{S}_C)$.

Let $\Sigma_2^{\mathfrak{c}} \stackrel{\text{def}}{=} \{\xi \in \Sigma_2$ such that the spine of ξ is stable at $t = 1\}$.

Proposition 4.0.9. $\mathbb{S}_C \subset \Sigma_2^{\mathfrak{c}}$

This proposition immediately implies

$$1 < \mathrm{HD}(\Sigma_2^{\mathfrak{c}}),$$

thus proving Theorem 3

We now prove Proposition 4.0.9. For that first recall that

$$\frac{\lambda^2}{\beta^2}\frac{1-\lambda}{1-\beta^{-1}} > 1 \qquad \Longrightarrow \qquad \beta\,\lambda < 1. \tag{4.0.1}$$

Claim 4.0.10. *The points* 0 *and* 1 *are hyperbolic attracting fixed points of* $g_{u,1}$ *and* $g_{s,1}$.

Proof. We prove the claim for $g_{u,1}$, the proof for $g_{s,1}$ is identical. Since $f_{1,1}(x) = 1 - x$ and by hypothesis $\beta\,\lambda < 1$ (recall (4.0.1)) we have

$$\begin{aligned}
g_{u,1}(0) &= (f_0 \circ f_{1,1} \circ f_0 \circ f_{1,1})(0) = 0,\\
g'_{u,1}(0) &= (f_0 \circ f_{1,1} \circ f_0 \circ f_{1,1})'(0) =\\
&= f'_0(1 - f_0(1))(-1)(f'_0(1))(-1) = f'_0(0)(f'_0(1)) =\\
&= \beta\,\lambda < 1,\\
g_{u,1}(1) &= (f_0 \circ f_{1,1} \circ f_0 \circ f_{1,1})(1) = f_0(1 - f_0(0)) = f_0(1) = 1,\\
g'_{u,1}(1) &= (f_0 \circ f_{1,1} \circ f_0 \circ f_{1,1})'(1) =\\
&= f'_0(1 - f_0(0))(f'_0(0)) = f'_0(1)(f'_0(0)) =\\
&= \beta\,\lambda < 1.
\end{aligned}$$

This implies that 0 and 1 are hyperbolic attracting fixed points for $g_{u,1}$. □

For $j = u, s$ and $t \in [0, 1]$ close to 1, we denote by $p^0_{j,t}$ the continuations for $g_{j,t}$ of the hyperbolic fixed point 0 of $g_{j,1}$. Similarly, $p^1_{j,t}$ denotes the continuations of 1 for $g_{j,t}$.

Consider the sets of words

$$\mathcal{U} \stackrel{\text{def}}{=} \{0, 01, 010, 0101\} \quad \text{and} \quad \mathcal{S} \stackrel{\text{def}}{=} \{0, 00, 001, 0010, 00100, 001001\}.$$

Lemma 4.0.11. *There exists $\bar{t} \in (0, 1)$ such that*

$$p^0_{u,t}, p^0_{s,t}, p^1_{u,t}, p^1_{s,t} \in [0, 1], \quad \textit{for every } t \in [\bar{t}, 1].$$

In particular,

$$g_{\bar{u},t}(p^i_{u,t}) \in [0, 1], \quad \textit{for every } \bar{u} \in \mathcal{U} \textit{ and } i = 0, 1$$

and

$$g_{\bar{s},t}(p^i_{s,t}) \in [0, 1] \quad \textit{for every } \bar{s} \in \mathcal{S} \textit{ and } i = 0, 1.$$

Proof. The second part of the lemma follows from $p^i_{j,t} \in [0, 1]$ for $i = 0, 1$ and $j = s, u$, and $f_{i,t}([0, 1]) \subset [0, 1]$, $i = 0, 1$.

Thus to prove the lemma is enough to see that $p^0_{u,t}, p^0_{s,t}, p^1_{u,t}, p^1_{s,t} \in [0, 1]$ for $t < 1$ close to 1. We prove this fact for the continuations $p^0_{u,t}$ and $p^1_{u,t}$.

By Claim 4.0.10 one has $0 < g'_{u,1}(1) < 1$ and $0 < g'_{u,1}(0) < 1$. Thus for every $\delta > 0$ small enough we have

$$g_{u,1}(-\delta) > -\delta, \quad g_{u,1}(\delta) < \delta, \quad g_{u,1}(1 - \delta) > 1 - \delta, \quad g_{u,1}(1 + \delta) < 1 + \delta,$$

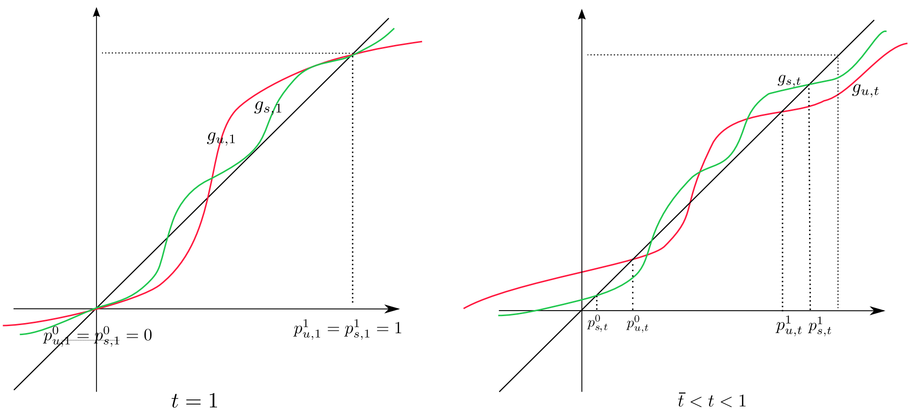

Figure 4.1: A big spine

and $(g_{u,1})'(x) < 1$ for every $x \in [-\delta, \delta] \cup [1-\delta, 1+\delta]$.

Note that for any $t \in (0,1)$ we have $g_{u,t}(0) > 0$ and $g_{u,t}(1) < 1$. This fact, the continuous dependence on t of $g_{u,t}$, and $g'_{u,t}(x) < 1$ in $[-\delta, \delta] \cup [1-\delta, 1+\delta]$ for t close to 1 imply that there exist $t_u \in (0,1)$ and small $\delta > 0$ such that for every $t \in [t_u, 1]$ we have

$$g_{u,t}(\delta) < \delta \ , \ g_{u,t}(x) > x, \ \text{ for all } x \in [-\delta, 0),$$
$$g_{u,t}(1-\delta) > 1-\delta \ , \ g_{u,t}(x) < x, \ \text{ for all } x \in [1, 1+\delta].$$

These inequalities that $p^0_{u,t} \in (0,\delta)$ and $p^1_{u,t} \in (1-\delta, 1)$ for all $t \in [t_u, 1)$, proving the lemma for the continuations $p^1_{u,t}$ and $p^0_{u,t}$. Arguing similarly, we get t_s such that $p^0_{s,t} \in (0,\delta)$ and $p^1_{s,t} \in (1-\delta, 1)$ for all $t \in [t_s, 1)$. The lemma follows taking $\bar{t} = \max\{t_s, t_u\}$. □

Let $\bar{t}$ as in Lemma 4.0.11 and for each $t \in [\bar{t}, 1]$ define

$$p^0_t \stackrel{\text{def}}{=} \max\{p^0_{u,t}, p^0_{s,t}\} \quad \text{and} \quad p^1_t \stackrel{\text{def}}{=} \min\{p^1_{u,t}, p^1_{s,t}\}.$$

Note that $p^0_t < p^1_t$. We have the following lemma.

Lemma 4.0.12. *There is $\hat{t} \in [\bar{t}, 1)$ such that $[p^0_t, p^1_t] \subset I_{\xi,t}$ for all $\xi \in \mathbb{S}_C$ and $t \in [\hat{t}, 1]$.*

Proof. For each parameter $t \in [\bar{t}, 1]$ and $i = s, u$ consider the sets

$$F_{i,t} \stackrel{\text{def}}{=} \{r \in [0,1] \colon g_{i,t}(r) = r \text{ and } r \text{ is not an attractor}\}.$$

Select the following subset of parameters

$$L \stackrel{\text{def}}{=} \{t \in [\bar{t}, 1] \colon \big(F_{u,t} \cup F_{s,t}\big) \cap \big([0, p_t^0] \cup [p_t^1, 1]\big) = \emptyset\}.$$

Claim 4.0.13. *There is $\hat{t} \in [\bar{t}, 1)$ such that $[\hat{t}, 1] \subset L$.*

Proof. Note that for $t = 1$ one has

$$0 = p_1^0 = p_{u,1}^0 = p_{s,1}^0 \quad \text{and} \quad 1 = p_1^1 = p_{u,1}^1 = p_{s,1}^1$$

and these points are hyperbolic attractors of $g_{u,1}$ and $g_{s,1}$. Thus there is small $\epsilon > 0$ such that for t close to 1, $t < 1$, the only fixed point of $g_{u,t}$ in $[1-\epsilon, 1+\epsilon]$ (resp. $g_{s,t}$) is $p_{u,t}^1$ (resp. $p_{s,t}^1$) which is the continuation of 1 and is attracting). Similarly, the only fixed point of $g_{u,t}$ (resp. $g_{s,t}$) in $[-\epsilon, \epsilon])$ is $p_{u,t}^0$ (resp. $p_{s,t}^0$). This completes the proof of the claim. □

To prove the lemma we see that $[p_t^0, p_t^1] \subset I_{\xi,t}$ for every $\xi \in \mathbb{S}_C$ and $t \in L$. For that, given $\xi \in \mathbb{S}_C$ write

$$\widehat{\xi^-} = u^{h_1}\, s^{n_1}\, u^{h_2}\, s^{n_2} \dots,$$

where $h_i, n_i \geq 0$ for $i \geq 1$. By the characterization of the spines in Lemma 3.1.1,

$$I_{\xi,t} = \lim_{r\to\infty} g_{u,t}^{h_1} \circ g_{s,t}^{n_1} \circ \dots \circ g_{u,t}^{h_r} \circ g_{s,t}^{n_r}\big([0,1]\big).$$

Therefore to prove the assertion it is enough to see that

$$g_{u,t}^{h_1} \circ g_{s,t}^{n_1} \circ \dots \circ g_{u,t}^{h_r} \circ g_{s,t}^{n_r}(1) \geq p_t^1 \quad \text{and} \quad g_{u,t}^{h_1} \circ g_{s,t}^{n_1} \circ \dots \circ g_{u,t}^{h_r} \circ g_{s,t}^{n_r}(0) \leq p_t^0. \tag{4.0.2}$$

In fact, we prove a bit more general property that immediately implies the inequalities above

Claim 4.0.14. *Let $t \in L$. Then*

- $g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(x) \in [p_t^1, 1]$ *for every* $x \in [p_t^1, 1]$ *and*
- $g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(x) \in [0, p_t^0]$ *for every* $x \in [0, p_t^0]$.

Proof. We just prove the first item, the second follows analogously. There are two cases:

Case 1: $p_t^1 = p_{s,t}^1$. Note that in this case, by the definition of the set of parameters L, we have

$$g_{u,t}^k(p_{s,t}^1) \geq p_{s,t}^1 \quad \text{for every} \quad k \geq 0. \tag{4.0.3}$$

As $g_{s,t}$ is orientation preserving this implies that for every $x \in [p_t^1, 1]$ we have

$$g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(x) \geq g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(p_{s,t}^1) = g_{u,t}^{h_j}(p_{s,t}^1) \geq p_{s,t}^1$$

where the last inequality follows by (4.0.3).

Case 2: $p_t^1 = p_{u,t}^1$. In this case we have

$$\lim_{k\to\infty} g_{s,t}^k(p_{u,t}^1) = p_{s,t}^1 \geq p_{u,t}^1$$

and since $g_{s,t}$ is orientation preserving, this implies

$$g_{s,t}^k(p_{u,t}^1) \geq p_{u,t}^1 \quad \text{for every} \quad k \geq 0 \tag{4.0.4}$$

As that $g_{s,t}$ preserves the orientation we have for for every $x \in [p_t^1, 1]$ that

$$g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(x) \geq g_{u,t}^{h_j} \circ g_{s,t}^{n_j}(p_{u,t}^1) = g_{u,t}^{h_j}(p_{u,t}^1) \geq p_{u,t}^1,$$

where the last inequality follows by (4.0.4). This ends the proof of the claim. □

The proof of the lemma is now complete. □

We are now ready to finish the proof of Proposition 4.0.9. Fix small $\epsilon > 0$. As the points p_t^0 and p_t^1 depends continuously on t, there is $t_\epsilon \in [\bar{t}, 1)$ such that:

$$p_t^0 < \frac{\epsilon}{2} < 1 - \frac{\epsilon}{2} < p_t^1 \quad \text{for every} \quad t \in [t_\epsilon, 1].$$

By Lemma 4.0.12 this implies $1 - \epsilon < \left|[p_t^0, p_t^1]\right| \leq |I_{\xi,t}|$, which ends the proof of the theorem. □

5
Trivial versus non-trivial spines: Preponderance of trivial spines

In this section we compare the subset of sequences with no-trivial spines and the subset with trivial ones. Theorems 4 and 5 say that, for every $t \in (0,1)$ the subset with trivial spines is, in some sense, dominant.

5.1
Proof of Theorem 4

Proof. First we note that by Proposition 2.1.4 we can work with the metric $\bar{d}(\varpi, \theta) = 2^{-n_{\varpi,\theta}}$, where $n_{\varpi,\theta}$ is the smallest $|n|$ such that $\varpi_n \neq \theta_n$, which is equivalent to the metric d.

Fix $t \in (0,1]$ and let

$$\Sigma_{x,t} \stackrel{\text{def}}{=} \{\xi \in \Sigma_2 \text{ such that } x \in I_{\xi,t}\}.$$

The following proposition is the key step of the proof of Theorem 4.

Proposition 5.1.1. *Given $t \in (0,1)$ there is $\rho_t < 2$ such that*

$$HD(\Sigma_{x,t}) < \rho_t < 2$$

for every $t \in (0,1)$ and $x \in [0,1]$.

We first see how Theorem 4 follows from this proposition. Note that if the spine $I_{\xi,t}$ is non-trivial for F_t then it contains some rational number $x \in (0,1)$. Therefore

$$\Sigma_{2,t}^{\mathcal{N}} \subset \bigcup_{x \in \mathbb{Q} \cap (0,1)} \Sigma_{x,t},$$

and, by Proposition 2.1.5,

$$\mathrm{HD}(\Sigma_{2,t}^{\mathcal{N}}) \leq \sup_{x \in \mathbb{Q} \cap (0,1)} \mathrm{HD}(\Sigma_{x,t}) \leq \rho_t < 2,$$

proving the theorem. □

Remark 5.1.2. Consider $t \in (0,1)$. Recall that if ξ is a sequence such that ξ^- has at least two $1's$ then $I_{\xi,t}$ is disjoint from $\Sigma \times \{0,1\}$. This implies that for $t \in (0,1)$ one has

$$\mathrm{HD}(\Sigma_{0,t}) = \mathrm{HD}(\Sigma_{1,t}) = 1.$$

Proof of Proposition 5.1.1. Fix $t \in (0,1)$. Note that $f_{1,t}^2([0,1]) \subset (0,1)$. Thus $f_0^n(f_{1,t}^2([0,1])) \to 1$ as $n \to \infty$. On the other hand, $f_{1,t}^n(f_0^2([0,1]))$ converges to the fixed point of $f_{1,t}$ as $n \to \infty$. This implies that there is large $N_t \in \mathbb{N}$ ($N_t \to \infty$ as $t \to 1$) such that

$$\left(f_0^{N_t-2} \circ f_{1,t}^2([0,1])\right) \bigcap \left(f_{1,t}^{N_t-2} \circ f_0^2([0,1])\right) = \emptyset. \tag{5.1.1}$$

Fix $x \in (0,1)$ and define

$$\Sigma_{x,t}^K \stackrel{\text{def}}{=} \{w \in \{0,1\}^{KN_t} \text{ such that } x \in g_{w,t}([0,1])\}.$$

We now give an upper bound (independent of the point $x \in (0,1)$) of the cardinality of this set

Lemma 5.1.3. $\#\,\Sigma_{x,t}^K \le (2^{N_t} - 1)^K$.

Proof. Write $N = N_t$. Consider $w \in \Sigma_{x,t}^{K+1}$ and write $w = w_1 w_2$ where $|w_1| = KN$ and $|w_2| = N$. Note that

$$x \in g_{w_1,t} \circ g_{w_2,t}([0,1]) \subset g_{w_1,t}([0,1]).$$

Therefore $w_1 \in \Sigma_{x,t}^K$.

By (5.1.1), if $g_{w_1,t}^{-1}(x) \in f_0^{N-2} \circ f_{1,t}^2([0,1])$ then $g_{w_1,t}^{-1}(x) \notin f_{1,t}^{N-2} \circ f_0^2([0,1])$. This implies that there is at least one element $u \in \{0,1\}^N$ (a word with $|u| = N$) for which $g_{u,t}^{-1} \circ g_{w_1,t}^{-1}(x) \notin [0,1]$ Thus, necessarily, $w \neq w_1 u$. This implies that

$$\#\,\Sigma_{x,t}^{K+1} \le \#\,\Sigma_{x,t}^K\,(2^N - 1).$$

Arguing inductively we get

$$\#\,\Sigma_{x,t}^{KN} \le (2^N - 1)^K,$$

proving the lemma. □

Claim 5.1.4. *The set $\Sigma_{x,t}$ has a covering $\mathcal{U}_K$ consisting cylinders of diameter 2^{-KN_t}. with (at most) $2^{KN_t+1}(2^{N_t} - 1)^K$ elements.*

Proof. Take $\xi \in \Sigma_{x,t}$ and write $\widehat{\xi^-} = a_1 a_2 \dots a_r \dots, \quad a_i \in \{0,1\}$. Observe that for every K one has that $a_1 a_2 \dots a_{KN_t} \in \Sigma_{x,t}^K$. By Lemma 5.1.3, this implies that the set $\Sigma_{x,t}$ has a covering $\mathcal{U}_K$ by cylinders of diameter 2^{-KN_t} with at most $2^{KN_t+1}(2^{N_t}-1)^K$ elements. □

Let $\mathcal{U}_K$ be a covering of $\Sigma_{x,t}$ as in the claim. The claim implies that for any $s \in \mathbb{R}$

$$m_s(\mathcal{U}_K) \le 2^{KN_t+1}\,(2^{N_t}-1)^K\,\left(2^{-KN_t}\right)^s. \tag{5.1.2}$$

Recall that

$$m_{s,\epsilon}(\Sigma_{x,t}) = \inf\{m_s(\mathcal{U}) \;\text{ where } \mathcal{U} \text{ is a covering of } \Sigma_{x,t} \text{ with } \operatorname{diam}(\mathcal{U}) < \epsilon\}.$$

Therefore if $2^{-KN_t} < \epsilon$ then

$$m_{s,\epsilon}(\Sigma_{x,t}) \le m_s(\mathcal{U}_K).$$

Thus from equation (5.1.2) it follows that

$$\begin{aligned} m_s(\Sigma_{x,t}) = \lim_{\epsilon\to 0^+} m_{s,\epsilon}(\Sigma_{x,t}) &\le \lim_{K\to+\infty} 2^{KN_t+1}\,(2^{N_t}-1)^K\,(2^{-KN_t})^s = \\ &= \lim_{K\to+\infty} 2\left(2^{N_t}\,(2^{N_t}-1)\,(2^{-N_t s})\right)^K. \end{aligned}$$

Hence the Hausdorff dimension of $\Sigma_{x,t}$ is upper bounded by the number $s \in \mathbb{R}$ satisfying

$$2^{N_t}\,(2^{N_t}-1)\,(2^{-N_t s}) = 1.$$

Therefore

$$\operatorname{HD}(\Sigma_{x,t}) \le 1 + \frac{\log(2^{N_t}-1)}{N_t \log 2} = \rho_t < 2,$$

which ends the proof of the proposition. □

We have so the following result that is a directly consequence of Theorem 4:

Corollary 5.1.5. *The Hausdorff dimension of the set of sequences with trivial spine for some $t \in (0,1)$ is 2.*

Proof. Define

$$\mathcal{T} \overset{\text{def}}{=} \{\xi \in \Sigma_2,\; \text{such that } I_{\xi,t} \text{ is trivial for some } t \in (0,1)\}.$$

We have that

$$2 = \operatorname{HD}(\Sigma_2) = max\{\operatorname{HD}(\cap_{t\in(0,1)}\Sigma^{\mathcal{T}}_{2,t})\,,\,\operatorname{HD}(\mathcal{T})\}.$$

Theorem 4 implies that

$$\mathrm{HD}(\cap_{t\in(0,1)}\Sigma_{2,t}^{\mathcal{T}}) < 2,$$

then we have $\mathrm{HD}(\mathcal{T}) = 2$, proving the corollary. □

5.2 Proof of Theorem 5

In view of Proposition 2.2.1 Theorem 5 is in fact a consequence of Theorem 4.

Proof. Recall that Proposition 2.2.1 implies that $\mathfrak{b}_{1/2} = m_2$. Thus the theorem is an immediate consequence of the following lemma.

Lemma 5.2.1. *$m_2(\Sigma_{2,t}^{\mathcal{T}}) = 1$ and $m_2\big(\bigcap_{t\in(0,\beta^{-1})}\Sigma_{2,t}^{\mathcal{T}}\big) = 1$.*

Proof. By Proposition 2.1.5,

$$2 = \mathrm{HD}(\Sigma_2) = \max\{\mathrm{HD}(\Sigma_{2,t}^{\mathcal{N}}), \mathrm{HD}(\Sigma_{2,t}^{\mathcal{T}}\}$$

and by Theorem 4 $\mathrm{HD}(\Sigma_{2,t}^{\mathcal{N}}) < 2$. Thus it follows that

$$\mathrm{HD}(\Sigma_{2,t}^{\mathcal{T}}) = 2.$$

On the one hand, Theorem 4 implies that $m_2(\Sigma_{2,t}^{\mathcal{N}}) = 0$. Thus

$$1 = m_2(\Sigma_2) = m_2(\Sigma_{2,t}^{\mathcal{N}}) + m_2(\Sigma_{2,t}^{\mathcal{T}}) = m_2(\Sigma_{2,t}^{\mathcal{T}}), \tag{5.2.1}$$

completing the proof of the first part of the lemma.

To prove the second part of the lemma take $E_{[1]}$ the characteristic function of the cylinder $C(0;1) = \{\xi \in \Sigma_2 \colon \xi_0 = 1\}$ and note that (recall equation (3.2.1))

$$\phi_1(\xi) = -\liminf_{n\to\infty} \frac{\#\{i \in [1,n] \colon \xi_{-i} = k\}}{n} = \liminf_{n\to\infty} \frac{\sum_{j=0}^{n-1} E_{[1]}\big(\sigma^j(\xi)\big)}{n}.$$

Since σ is $\mathfrak{b}_{1/2}$-ergodic, the Birkhoff Ergodic Theorem implies that there is a set $\widehat{\Sigma}_2$ satisfying $\mathfrak{b}_{1/2}(\widehat{\Sigma}_2) = 1$ such that for every $\xi \in \widehat{\Sigma}_2$ this limit is exactly

$$\Phi_1(\xi) = \int E_{[1]}\, d\mathfrak{b}_{1/2} = \frac{1}{2}.$$

Now take an increasing sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $\lim_{n\to\infty} x_n = 1/2$. Note that if $\xi \in \hat{\Sigma}_2$ then $\Phi_1(\xi) > x_n$ for every $n \in \mathbb{N}$, thus Proposition 3.2.2 implies that $I_{\xi,t}$ is trivial for every $\xi \in \widehat{\Sigma}_2$ and

$t \in (0, \beta^{-\frac{1-x_n}{x_n}}) \subset (0, \beta^{-1})$. Taking $n \to \infty$, we have $I_{\xi,t}$ trivial for every $\xi \in \widehat{\Sigma}_2$ and $t \in (0, \beta^{-1})$. Thus $\widehat{\Sigma}_2 \subset \bigcap_{t \in (0,\beta^{-1})} \Sigma_{2,t}^{\mathcal{T}}$ ending the proof of the lemma. □

The proof of the theorem is now concluded. □

6
Persistence of non-trivial spines: Proof of Theorem 1

In this section, we prove Theorem 1 about the occurrence of persistent non-trivial spines and the Hausdorff dimension of this set of sequences.

To prove the theorem, fix a decreasing sequence $(t_n)_{n\in\mathbb{N}}$ converging to zero, $t_n \in (0,1]$. Since $\lambda < \beta\,\lambda < 1 < \beta$ there are non-decreasing sequences $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}$ and $(r_n)_{n\in\mathbb{N}} \in \mathbb{N}$ such that

$$t_n > \beta^{-k_n} > \beta^{-(k_n+1)} > (\beta\,\lambda)^{r_n+1}. \tag{6.0.1}$$

In particular, the previous equation implies that

$$t_n > \beta^{-k_n} > \beta^{-(k_n+1)} > \beta^{-(k_n+r_n+1)} > \beta^{-(k_n+r_n+2)} > \lambda^{r_n+1}. \tag{6.0.2}$$

For each small $\gamma \in (0,1)$, consider the fundamental domains of f_0 in $[0,1]$ given by

$$D^0_\gamma \stackrel{\text{def}}{=} [f_0^{-1}(\gamma),\gamma] \quad \text{and} \quad D^1_\gamma \stackrel{\text{def}}{=} [1-\gamma, f_0(1-\gamma)].$$

Define $\iota(\gamma)$ as the first number with

$$f_0^{\iota(\gamma)}(D^0_\gamma)\cap D^1_\gamma \neq \emptyset.$$

It is not difficult to see that there is a strictly decreasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\iota(\gamma) = n-1$ for all $\gamma \in [\gamma_n, \gamma_{n-1})$ and $f_0^{\iota(\gamma_n)}(D^0_{\gamma_n}) \cap D_{\gamma^1_n} = \{1-\gamma_n\}$. This implies that

$$f_0^{\iota(\gamma_n)+1}(D^0_{\gamma_n}) = f_0^n(D^0_{\gamma_n}) = D^1_{\gamma_n}, \tag{6.0.3}$$

for details see [8].

Fix $\gamma \stackrel{\text{def}}{=} \gamma_N$ for a large N (thus γ is small) and using the sequence $(r_n)_{n\in\mathbb{N}}$ above define the set

$$E_n = E_n(\gamma) \stackrel{\text{def}}{=} \bigcup_{i=0}^{r_n} f_0^i(D^1_\gamma) = [1-\gamma, f_0^{r_n+1}(1-\gamma)].$$

Note that if $m < n$ then $r_m \le r_n$ and thus $E_m \subset E_n$.

The key of the proof of the theorem is the following proposition:

Proposition 6.0.2. *Consider sequences of natural numbers $(r_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ and of parameters $(t_n)_{n\in\mathbb{N}}$ as above. Then*

$$E_n \subset \left(f_0^{k_n+r_n+N} \circ f_{1,t}(E_n)\right) \cap \left(f_0^{k_n+r_n+N+1} \circ f_{1,t}(E_n)\right), \tag{6.0.4}$$

for every $n \in \mathbb{N}$ *and* $t \in [t_n, 1]$.

Proof. To prove the inclusion let us assume (for notational simplicity) that f_0 is affine in $[0, \gamma]$ and $[1-\gamma, 1]$,

$$f_0(x) = \beta\, x \ \text{ and } \ f_0(1-x) = 1 - \lambda x \ \text{ for every } \ x \in [0, \gamma],$$

(the proof in the general case is analogous). In this case,

$$E_n = [1-\gamma, 1-\lambda^{r_n+1}\gamma].$$

By definition of N we have that

$$f_0^{-(k_n+r_n+N)}(E_n) = f_0^{-(k_n+r_n+N)}([1-\gamma, 1-\lambda^{r_n+1}\gamma]) = [\beta^{-(k_n+r_n+1)}\gamma, \beta^{-k_n}\gamma]$$

and

$$f_0^{-(k_n+r_n+N+1)}(E_n) = [\beta^{-(k_n+r_n+2)}\gamma, \beta^{-k_n-1}\gamma].$$

This implies that

$$\begin{aligned} f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N)}(E_n) &= [1 - t^{-1}\beta^{-k_n}\gamma, 1 - t^{-1}\beta^{-(k_n+r_n+1)}\gamma], \\ f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N+1)}(E_n) &= [1 - t^{-1}\beta^{-k_n-1}\gamma, 1 - t^{-1}\beta^{-(k_n+r_n+2)}\gamma]. \end{aligned}$$

By (6.0.2), for the parameter t_n one has that

$$\begin{aligned} [1 - t_n^{-1}\beta^{-k_n}\gamma, 1 - t_n^{-1}\beta^{-(k_n+r_n+1)}\gamma] &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] = E_n, \\ [1 - t_n^{-1}\beta^{-k_n-1}\gamma, 1 - t_n^{-1}\beta^{-(k_n+r_n+2)}\gamma] &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] = E_n. \end{aligned}$$

Since these inclusions also hold for $t \in [t_n, 1]$ it follows that

$$\begin{aligned} f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N)}(E_n) &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] \subset E_n, \\ f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N+1)}(E_n) &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] \subset E_n. \end{aligned} \tag{6.0.5}$$

These inclusions immediately imply the proposition. □

Consider now the sequence

$$s_j \stackrel{\text{def}}{=} k_j + r_j + N.$$

Denote by v_n the word $v_n \stackrel{\text{def}}{=} 0^n 1$ and define the following subset Γ of Σ_2^-,

$$\Gamma \stackrel{\text{def}}{=} \left\{\xi^- \in \Sigma_2^- \text{ such that } \bar{\xi}^- = v_{s_1+i_1} v_{s_2+i_2} \dots v_{s_j+i_j} \dots \ i_j \in \{0,1\}\right\}.$$

It is immediate to check that the set Γ is non-enumerable. Therefore the lemma below implies the first part of Theorem 1.

Lemma 6.0.3. $\Gamma \subset \Sigma_2^{-,\mathcal{N}}(0)$.

Proof. Note that the maps $g_{v_{s_n+i_n},t}$ are of the form

$$g_{v_{s_n+i_n},t} = f_0^{k_n+r_n+N+i_n} \circ f_{1,t}, \quad i = 0, 1.$$

Applying Proposition 6.0.2 to $g_{v_{s_n},t}$ $(i_n = 0)$ and $g_{v_{s_n+1},t}$ $(i_n = 1)$ and the sets $E_m \subset E_n$, $n \geq m$, one has that

$$E_m \subset E_n \subset g_{v_{s_n+i_n},t}(E_n) \quad \text{for every } n \geq m \text{ and } i_n \in \{0, 1\}. \tag{6.0.6}$$

This implies that, for any choice of $i_m, i_{m+1} \in \{0, 1\}$,

$$E_m \subset g_{v_{s_m+i_m},t}(E_m) \subset g_{v_{s_m+i_m},t}(E_{m+1}) \subset g_{v_{s_m+i_m},t} \circ g_{v_{s_{m+1}+i_{m+1}},t}(E_{m+1}).$$

Arguing recursively, for $n \geq m$ and fany choice of $i_m, i_{m+1}, \ldots, i_n \in \{0, 1\}$, we get that

$$E_m \subset g_{v_{s_m+i_m},t}(E_m) \subset g_{v_{s_m+i_m},t} \circ g_{v_{s_{m+1}+i_{m+1}},t} \circ \cdots \circ g_{v_{s_n+i_n},t}(E_n).$$

Therefore

$$E_m \subset \lim_{n\to\infty} g_{v_{s_m+i_m},t} \circ g_{v_{s_{m+1}+i_{m+1}},t} \cdots \circ g_{v_{s_n}+i_n,t}([0, 1]).$$

It follows that (for any choice of $i_1, \ldots, i_n, \cdots \in \{0, 1\}$)

$$g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_{m-1}+i_{m-1}},t}(E_m) \subset \lim_{n\to\infty} g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_n+i_n},t}([0, 1]) = I_{\xi,t},$$

where the last identity follows from Lemma 3.1.1. As E_m is a non-trivial interval, we have that $I_{\xi,t}$ is a non-trivial spine, ending the proof of the lemma. □

To prove the second part of the theorem, fix a small $\bar{t}_0 > 0$ and consider the sequences (t_n), (k_n), and (r_n) in (6.0.1). Given $t \geq \bar{t}_0$, equation (6.0.4) in Proposition 6.0.2 holds for all $t \geq t_n$. Note that $\bar{t}_0 > t_{n_0}$ for some n_0, thus for every $t \geq \bar{t}_0$ and $n \geq n_0$ it holds

$$E_n \subset \Big(f_0^{k_n+r_n+N} \circ f_{1,t}(E_n)\Big) \cap \Big(f_0^{k_n+r_n+N+1} \circ f_{1,t}(E_n)\Big). \tag{6.0.7}$$

Consider the words

$$u \stackrel{\text{def}}{=} 0^{k_{n_0}+r_{n_0}+N}1 \quad \text{and} \quad v \stackrel{\text{def}}{=} 0^{k_{n_0}+r_{n_0}+N+1}1,$$

the set $A = \{u, v\}$, and its associated set of backward sequences $\mathbb{E}_A$. By Proposition 2.2.7 the Hausdorff dimension of $\mathbb{E}_A$ is strirctly positive. Thus the second part of Theorem 1 follows from the next lemma.

Lemma 6.0.4. $\mathbb{E}_A \subset \Sigma_2^{-,\mathcal{N}}(t_0)$.

Proof. Take $\xi^- \in \mathbb{E}_A$ and any sequence ξ of the form $\xi = (\xi^-, \xi^+)$. By (6.0.7) we have

$$E_n \subset g_{u,t}(E_n) \quad \text{and} \quad E_n \subset g_{v,t}(E_n), \quad \text{for all } t \in [t_0, 1).$$

Corollary 3.1.2 now implies that $E_n \subset I_{\xi,t}$. Since E_n is non-trivial this proves the lemma. □

The proof of Theorem 1 is now complete. □

7
Persistence of non-trivial spines after their generation

The subject of this section is the persistence of non-trivial spines after their generation. More precisely, we consider the following naive question: Consider a parameter $t_0 \in (0,1)$ and a sequence ξ such that I_{ξ,t_0} is non-trivial. Is $I_{\xi,t}$ non-trivial for all $t \in [t_0, 1]$?.

In this section we prove Theorem 6 and so answer negatively to question. For this we construct a examples of porcupines with a evanescent spine (there is a sequence $\xi \in \Sigma_2$ and parameters $0 < t_1 < t_2 < 1$ such that I_{ξ,t_1} is non-trivial and $I_{\xi,t}$ is trivial for every $t \in [t_2, 1)$).

7.1
A porcupine with an evanescent spine

We first construct a special model of porcupine, where f_0 is piecewise affine and there is an evanescent spine. Thereafter we will modify this construction to obtain a map f_0 that is C^2.

7.1.1
A evanescent spine: a piecewise affine model

Consider the skew product map F_t defined as in (1.2.1) whose fiber maps are

$$f_{0,t}(x) = f_0(x) = \begin{cases} \frac{5}{2}x, & \text{if } x \le \frac{1}{4}, \\ \\ \frac{1}{2} + \frac{x}{2}, & \text{if } \frac{1}{4} < x \le 1, \end{cases}$$

$$f_{1,t}(x) = t\,(1-x).$$

Note that $f_0'(x) < 1$ for every $x > 1/4$ and that $f_{1,t}(0) = t$, thus f_0 satisfies the contracting property (C) for all $t > 1/4$.

Proposition 7.1.1. *Consider the one-parameter family of skew product maps F_t above. The spine of the periodic sequence $\varpi = \overline{10}^Z$ is non-trivial for $t = \frac{1}{2}$ and is trivial for $t \in (\frac{2}{3}, 1)$.*

Proof. We split the proof of the proposition into two lemmas. First let $u \stackrel{\text{def}}{=} 10$.

Lemma 7.1.2 (Non-trivial spine)**.** *The spine of* $\varpi = \overline{10}^Z$ *is non-trivial for* $t = \frac{1}{2}$.

Proof. The restriction of $g_{u,1/2}$ to $[0, 1/4)$ is of the form

$$\frac{1}{2}\left(1 - \frac{5}{2}x\right).$$

Thus $2/9 < 1/4$ is a fixed point of $g_{u,1/2}$. Since $g'_{u,1/2}(2/9) = -5/4$, this point is repelling and by Lemma 3.1.3 the spine $I_{\varpi,1/2}$ is non-trivial. This proves the lemma. □

Lemma 7.1.3 (Trivial spine)**.** *The spine of* $\varpi = \overline{10}^Z$ *is trivial for* $t \in (\frac{2}{3}, 1)$, $I_{\varpi,3/4} = \{\frac{t}{2+t}\}$.

Proof. The proof is by contradiction. The restriction of $g_{u,t}$ to $[0, 1/4)$ is of the form

$$t\left(1 - \frac{5}{2}x\right).$$

Then for $x \in [0, 1/4)$ we have

$$g_{u,t}(x) = t\left(1 - \frac{5}{2}x\right) > t\left(1 - \frac{5}{2}\frac{1}{4}\right) = t\left(\frac{3}{8}\right).$$

This implies that $g_{u,t}(x) > x$ for every $x \in (0, 1/4)$ and $t > 2/3$ and thus the fixed points of $g_{u,t}$ are in $(1/4, 1]$. Since $g_{u,t}(x)$ is a contraction in $(1/4, 1]$ (the derivative is $-t/2$) this fixed point is unique. A straighforward calculation gives that this fixed point is $q = \frac{t}{2+t}$.

If the spine is non-trivial then $g_{u,t}$ necessarily has a periodic point $q' \neq q$ of period two. It means that $g^2_{u,t}$ has n periodic points, n odd $n \geq 3$ and the point q is the central one, then there are at least one periodic point smaller than q and one periodic point bigger than q. Since that $g^2_{u,t}(0) > 0$ and $g^2_{u,t}(1) < 1$ and $g^2_{u,t}$ changes the deride at most three times, it is not possible. □

This completes the proof of the proposition. □

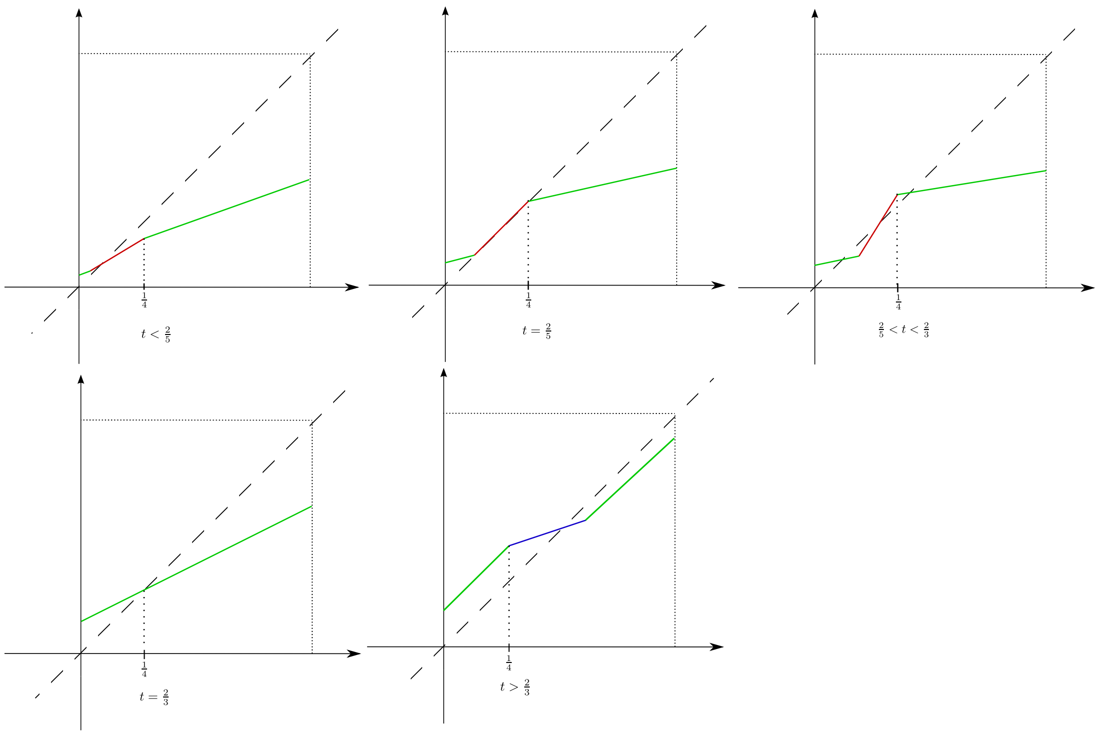

Figure 7.1: A evanescent spine

7.1.2
A evanescent spine: general case

We now go to the construction of an evanescent spine in the general differentiable case. For that we modify f_0 in an small neighborhood of the non-differentiable point $x = 1/4$ and denote by $\widehat{f_0}$ the resulting map and by $\widehat{g}_{u,t}$ the corresponding compositions of $\widehat{f_0}$ and $\widehat{f}_{1,t} = f_{1,t}$. Finally, we let $\widehat{F}_t$ the skew product map associated to $\widehat{f_0}$ and $\widehat{f}_{1,t}$.

Recall that $2/9$ is the fixed expanding point for $g_{u,1/2}$ and, for $t > 2/3$, $t/(2+t)$ is the attracting fixed point for $g_{u,t}$. Given $t > 2/3$ consider the "orbits"

$$O_{2/9,1/2} = \{2/9, f_0(2/9)\} \quad \text{and} \quad O_{t/(2+t),t} = \{t/(2+t), f_0(t/(2+t))\}.$$

Claim 7.1.4. Given $\epsilon > 0$ there exist a neighborhood $V = V_\epsilon$ of $1/4$ such that $V \cap (O_{2/9,1/2} \cup O_{t/(2+t),t}) = \emptyset$ for every $t \in [2/3 + \epsilon, 1)$.

Proof. Let $\epsilon > 0$. Observe that $1/4 \notin O_{2/9,1/2}$ and $(2/3+\epsilon)/(2+2/3+\epsilon) > 1/4$ for any $\epsilon > 0$. Since that the fixed point $t/(2+t)$ increase with t and $f_0(x) > x$ for every $x \notin \{0, 1\}$, we have that there exist a neighborhood V of $1/4$ such that $V \cap (O_{2/9,1/2} \cup O_{t/(2+t),t}) = \emptyset$ for every $t \in [2/3+\epsilon, 1)$. Finally we can take, if necessary, a small neighborhood V of $1/4$ that satisfy the claim. $\square$

We consider $\hat{f}_0$ such that $\hat{f}_0$ is C^2 and $\hat{f}_0|_{\mathbb{R}-V} = f_0$. To prove Theorem 6 is enough to prove the following proposition.

Proposition 7.1.5. *Consider the one-parameter family of skew product maps* $\widehat{F}_t$*. The spine of the periodic sequence* $\varpi = \overline{10}^Z$ *is non-trivial for* $t = \frac{1}{2}$ *and is trivial for every* $t \in \left[\frac{2}{3} + \epsilon, 1\right)$.

Proof. Since that $O_{2/9,1/2} \cap V = \emptyset$ one has that

$$\widehat{g}_{u,1/2}(2/9) = \widehat{f}_{1,1/2} \circ \widehat{f}_0(2/9) = f_{1,1/2} \circ f_0(2/9) = 2/9$$

and

$$\widehat{g}'_{u,1/2}(2/9) = \widehat{f}'_{1,1/2}(\widehat{f}_2(2/9))\, \widehat{f}'_0(2/9) = f'_{1,1/2}(f_0(2/9))\, f'_0(2/9) = -5/4.$$

Thus $2/9$ is a fixed expanding point for $\widehat{g}_{u,1/2}$ and by Lemma 3.1.3 $I_{\varpi,1/2}$ is non-trivial.

Analogously, we see that $t/(2+t)$ is a fixed attracting point for $g_{u,t}$ for every $t \in [2/3 + \epsilon, 1)$.

Lemma 7.1.6. $I_{\varpi,t} = \{t/(2+t)\}$ *for every* $t \in [2/3 + \epsilon, 1)$.

Proof. Fix $t \in [2/3 + \epsilon, 1)$, the proof is identical to the one of Lemma 7.1.3 and follows by contradiction. If $I_{\varpi,t} \neq \{t/(2+t)\}$ then $\widehat{g}_{u,t}$ has at least one periodic point q' of period two that not is attracting.

It means that $g^2_{u,t}$ has n periodic points, n odd $n \geq 3$ and the point q is the central one, then there are at least one periodic point smaller than q and one periodic point bigger than q. Since that $g^2_{u,t}(0) > 0$ and $g^2_{u,t}(1) < 1$ and $g^2_{u,t}$ changes the concavity at most three times, it is not possible. □

The proof Proposition 7.1.5 is now complete. □

8 Appendix A: Skew-products and homoclinic and heteroclinic intersections

We now see how the definitions of a heterodimensional cycle and a homoclinic class can be adapted in the case of one-step skew product maps over a shift.

8.1 Indices of periodic points

For differentiable maps the *index* of a hyperbolic periodic point is the dimension of its unstable bundle. In this way one can speak of a pair of saddles of f with different indices.

For a one-step skew-product map $F(\xi, x) = (\sigma(\xi), f_\xi(x))$ defined as in (1.2.1) with one-dimensional fiber maps (differentiable only in the fiber direction) a periodic point $P = \big((\xi_0 \dots \xi_{\pi-1})^{\mathbb{Z}}, p\big)$ of F of period π is *hyperbolic* if

$$f'_{[\xi_0 \dots \xi_{\pi-1}]}(p) = (f_{\sigma^{\pi-1}(\xi)} \circ \cdots \circ f_\xi)'(p) \neq \pm 1.$$

This periodic point is *contracting* if the above derivative has modulus less than one, otherwise the point is *expanding.* In this way, a pair of periodic points have the *same index* if both points are contracting or both points are expanding. Otherwise the points have different index.

Observe a hyperbolic periodic point $P = \big((\xi_0 \dots \xi_{\pi-1})^{\mathbb{Z}}, p\big)$ of period π of F has a uniquely defined *continuation,* that is, every skew-product map G close to F (i.e., $G(\xi, x) = (\sigma(\xi), g_{\xi_0}(x))$ where g_i is close to f_i) has a periodic point P_G of the same period and index as P which is close to P. More precisely, the map $g_{[\xi_0 \dots \xi_{\pi-1}]}$ is close to $f_{[\xi_0 \dots \xi_{\pi-1}]}$ and thus there is p_g close to p with $g_{[\xi_0 \dots \xi_{\pi-1}]}(p_g) = p_g$. Then $P_G = (\xi, p_g)$ is the continuation of P for G. Note that P_G has the same period and index as the ones of P.

For a hyperbolic fixed point p of $f_{[\xi_0 \dots \xi_m]}$ we consider its local invariant manifolds $W^{s,u}_{loc}(p, f_{[\xi_0 \dots \xi_m]})$. If p is contracting (resp. expanding) then $W^u_{loc}(p, f_{[\xi_0 \dots \xi_m]}) = \{p\}$ (resp. $W^s_{loc}(p, f_{[\xi_0 \dots \xi_m]}) = \{p\}$).

8.2 Invariant sets

Let us first introduce some notation. We denote by

$$\left((\rho_{-r}\cdots\rho_{-1})^{\mathbb{N}}\,\eta_{-n}\cdots\eta_{-1}\,.\,\eta_0\cdots\eta_k\,(\alpha_1\cdots\alpha_m)^{\mathbb{N}}\right)$$

the sequence $\xi=(\xi_i)_{i\in\mathbb{Z}}\in\Sigma_\ell$ defined as follows

- $\xi_i=\eta_i$ if $i\in\{-n,\dots,0,\dots,k\}$;
- $\xi_{k+s\,m+i}=\alpha_i$ for every $i\in\{1,\dots,m\}$ and $s\geq 0$;
- $\xi_{-n-s\,r-i}=\rho_{-i}$ for every $i\in\{1,\dots,r\}$ and $s\geq 0$.

Let $R=((\xi_0\dots\xi_{\pi-1})^{\mathbb{Z}},r)$ be a hyperbolic periodic point of F. The set $W^s(R,F)$ is defined as the points $(\eta,x)\in\Sigma_2\times\mathbb{R}$ satisfying:

- $\eta=(\cdots.\eta_0\cdots\eta_k(\xi_0\cdots\xi_{\pi-1})^{\mathbb{N}}),\quad k=s\,\pi-1,$
- $f_{[\eta_0\cdots\eta_k]}(x)\in W^s_{loc}(r,f_{[\xi_0\cdots\xi_{\pi-1}]})$.

Similarly, the unstable set $W^u(R,F)$ is defined as the points (η,x) satisfying:

- $\eta=((\xi_0\cdots\xi_{\pi-1})^{\mathbb{N}}\,\eta_{-k}\cdots\eta_{-1}\,.\,\cdots),\quad k=s\,\pi,$
- $f^{-1}_{[\eta_{-1}\cdots\eta_{-k}]}(x)\in W^u_{loc}(r,f_{[\xi_0\cdots\xi_{\pi-1}]})$.

8.3 Homoclinic and heteroclinic intersections

Given a hyperbolic periodic point $P=\left((\xi_0\dots\xi_{\pi-1})^{\mathbb{Z}},p\right)$ of F a point such that $X\in W^u(P,F)\cap W^s(P,F)$ is called a *homoclinic point* of P. Homoclinic points behave as transverse ones in the differentiable setting and have continuations. More precisely, assume that P is contracting. Then $W^u(p,f_{[\xi_0\dots\xi_{\pi-1}]})=\{p\}$. Since $X=(\eta,x)\in W^u(P,F)\cap W^s(P,F)$, after replacing X by some iterate we can assume that

$$X=(\eta,x)=\left((\xi_0\dots\xi_{\pi-1})^{\mathbb{N}}\,.\eta_0\dots\eta_r\,(\xi_0\dots\xi_{\pi-1})^{\mathbb{N}},x\right).$$

As $W^u(p,f_{[\xi_0\dots\xi_{\pi-1}]})=\{p\}$ this implies that $x=p$ and thus $X=(\eta,p)$. Note that

$$F^{r+1}(X)=\hat{X}=(\hat{\eta},\hat{x})=(\hat{\eta},f_{[\eta_0\dots\eta_r]}(p)),\quad \text{where } \hat{\eta}=\left(\cdots.(\xi_0\dots\xi_{\pi-1})^{\mathbb{N}}\right).$$

This implies that $\hat{x}\in(p-\delta,p+\delta)\subset W^s_{loc}(p,f_{[\xi_0\dots\xi_{\pi-1}]})$.

Consider now G close to F and the continuation $P_G = \big((\xi_0 \dots \xi_{\pi-1})^{\mathbb{N}}, p_g\big)$ of P. If G is close enough to F then $(p-\delta, p+\delta) \subset W^s_{loc}(p_g, g_{[\xi_0 \dots \xi_{\pi-1}]})$. Define the point $X_G = (\eta, p_g) \in W^u(P_G, G)$ and note that $G^{r+1}(X_G) = \hat{X}_G = (\hat{\eta}, g_{[\eta_0 \dots \eta_r]}(p_g))$. Since G is close to F we have that $g_{[\eta_0 \dots \eta_r]}(p_g)$ is close to $f_{[\eta_0 \dots \eta_r]}(p)$ and thus

$$g_{[\eta_0 \dots \eta_r]}(p_g) \in (p-\delta, p+\delta) \subset W^s_{loc}(p_g, g_{[\xi_0 \dots \xi_{\pi-1}]}).$$

This implies that $X_G \in W^s(P_G, G)$ and thus X_G is a homoclinic point of P_G. We say that X_G is the continuation of X.

8.4 Homoclinic classes, homoclinic relations, and heterodimensional cycles

We adapt the definition of homoclinic class of differentiable dynamics to our context of skew product maps F. Note that this definition does not involve transversality.

The *homoclinic class* of a hyperbolic periodic point P of F, denoted by $H(P, F)$, is the closure of the homoclinic points of the orbit of P.

If V is an open neighborhood of the orbit of P, the *relative homoclinic class* of P in V, denoted by $H_V(P, F)$, is the closure of the homoclinic points of the orbit of P whose orbit is contained in V.

Two periodic points P and P' of F of the same index are *homoclinically related* if $W^u(P, F) \cap W^s(P', F) \neq \emptyset$ and $W^s(P, F) \cap W^u(P', F) \neq \emptyset$. These points are *homoclinically related in an open neighborhood* V of the orbits of P and P' if there are points $X \in W^u(P, F) \cap W^s(P', F)$ and $Y \in W^s(P, F) \cap W^u(P', F)$ whose orbits are contained in V.

Remark 8.4.1. As in the case of differentiable dynamics, the homoclinic classes of a periodic point coincides with the closures of the points (of the same index) homoclinically related to it.

A pair of periodic points P and Q of a skew-product map F with different indices have a *heterodimensional cycle* if if their invariant manifolds intersect cyclically, that is $W^u(P, F) \cap W^s(Q, F) \neq \emptyset$ and $W^s(P, F) \cap W^u(Q, F) \neq \emptyset$.

8.5 Existence of homoclinic and heteroclinic intersections

We now state some sufficient conditions for the existence of homoclinic and heteroclinic intersections in terms of the cylinder maps generated by f_0 and f_1.

Lemma 8.5.1. *Let* $A = \big((\xi_0 \dots \xi_{m-1})^{\mathbb{N}}, a\big)$ *and* $B = \big((\eta_0 \dots \eta_{k-1})^{\mathbb{N}}, b\big)$ *be hyperbolic periodic points of* F*. Suppose that there are a point* $c \in W^u_{loc}(a, f_{[\xi_0 \dots \xi_{m-1}]})$ *and a finite sequence* $\beta_0 \dots \beta_r$ *such that*

$$f_{[\beta_0 \dots \beta_r]}(c) = \bar{c} \in W^s_{loc}(b, f_{[\eta_0 \dots \eta_{k-1}]}).$$

Then the point

$$C = \big((\xi_0 \cdots \xi_{m-1})^{\mathbb{N}} \,.\, \beta_0 \cdots \beta_r\, (\eta_0 \cdots \eta_{k-1})^{\mathbb{N}}, c\big) \in W^s(B, F) \cap W^u(A, F).$$

Proof. Note that $C \in W^u(A, F)$. Consider the point

$$\begin{aligned} F_t^{r+1}(C) &= \big((\xi_0 \cdots \xi_{m-1})^{\mathbb{N}}\, \beta_0 \cdots \beta_r .\, (\eta_0 \cdots \eta_{k-1})^{\mathbb{N}}, f_{[\beta_0 \cdots \beta_r],t}(c)\big) = \\ &= \big((\xi_0 \cdots \xi_{m-1})^{\mathbb{N}}\, \beta_0 \cdots \beta_r \,.\, (\eta_0 \cdots \eta_{k-1})^{\mathbb{N}}, \bar{c}\big). \end{aligned}$$

As $\bar{c} \in W^s_{loc}(b, f_{[\eta_0 \dots \eta_{\ell-1}]})$ this implies that $F^{r+1}(C) \in W^s(B, F)$. □

As a consequences of the above lemma we get the following corollary.

Corollary 8.5.2. *Consider two periodic points* $A = ((\xi_0 \dots \xi_{m-1})^{\mathbb{Z}}, a)$ *and* $B = ((\nu_0 \dots \nu_{\ell-1})^{\mathbb{Z}}, b)$ *of contracting and expanding type (respectively) of* F*.*

- *If there are* $\beta_0 \dots \beta_r$ *such that*

  $$f_{[\beta_0 \dots \beta_r]}(a) \in \operatorname{int}\big(W^s_{loc}(a, f_{[\xi_0 \dots \xi_{m-1}]})\big)$$

  then $C = \big((\xi_0 \cdots \xi_{m-1})^{\mathbb{N}} \,.\, \beta_0 \cdots \beta_r\, (\xi_0 \cdots \xi_{m-1})^{\mathbb{N}}, a\big)$ *is a homoclinic point of* A*.*
- *If there are* $\alpha_{-r} \dots \alpha_{-1}$ *and* $\bar{b} \in \operatorname{int}\big(W^u_{loc}(b, f_{[\nu_0 \dots \nu_{\ell-1}]})\big)$ *with*

  $$f_{[\alpha_{-r} \dots \alpha_{-1}]}(\bar{b}) = b.$$

  Then $D = \big((\nu_0 \cdots \nu_{\ell-1})^{\mathbb{N}}\, \alpha_{-r} \cdots \alpha_{-1} \,.\, (\nu_0 \cdots \nu_{\ell-1})^{\mathbb{N}}, b\big)$ *is a homoclinic point of* B*.*

9
Appendix B: Transitivity of porcupine-like horseshoes

In this section we review some ingredients in [4, 7] of the proof of the transitivity of the porcupine-like horseshoes.

9.1
Transitivity of central expanding porcupine-like horsehoes

We first outline the proof the transitivity of Λ_t, $t \in (0,1]$ for families $(F_t)_{t\in(0,1]}$ such that the maps f_0 and $f_{1,t}$ satisfy conditions (P0.i), (P0.ii), and (P1).

We start with the following lemma that is proved in [7, Lemma 3.1]:

Lemma 9.1.1 (Expanding itineraries)**.** *Under conditions (P0.i), (P0.ii) and (P1), for every $t \in (0,1]$ there is $\gamma \in (0,1)$, that can be chosen arbitrarily close to 0, such that for every interval $J \subset [f_0^{-2}(\gamma), (\gamma)]$ there is a finite word of the form $\xi(J) = 0^{n_0}10^{m_0}\dots 0^{n_\ell}10^{m_\ell}$ such that the map*

$$f_{[\xi(J)],t} \stackrel{\text{def}}{=} f_0^{m_\ell} \circ f_{1,t} \circ f_0^{n_\ell} \circ \dots \circ f_0^{m_0} \circ f_{1,t} \circ f_0^{n_0}$$

satisfies the following properties:

1. *$f_{[\xi(J)],t}$ is uniformly expanding in J and*
2. *the interior of $f_{[\xi(J)],t}(J)$ contains $[f_0^{-2}(\gamma), f_0^{-1}(\gamma)]$.*

Sketch of the proof. To simplify the presentation, let us assume that f_0 is linear close to 0 and affine close to 1.

We first note that there is a sequence $\gamma_n \in (0,1)$, $\gamma_n \to 0$, such that for sufficiently large n it holds

$$f_0^n([f_0^{-1}(\gamma_n),\gamma_n]) = f_0^n([\beta^{-1}\gamma_n,\gamma_n]) = [1-t^{-1}\gamma_n, f_0(1-t^{-1}\gamma_n)] = \\ = [1-t^{-1}\gamma_n, 1-t^{-1}\lambda\gamma_n)].$$

We claim that

$$(f_0^n)'(x) \geq \frac{\lambda}{\beta}\frac{t^{-1}(1-\lambda)}{1-\beta^{-1}}, \quad \text{for all } x \in J.$$

To see why this is so, just note that there is a $y \in f_0([\beta^{-1}\gamma_n, \gamma_n])$ such that

$$(f_0^n)'(y) = \frac{|f_0^n(f_0([\beta^{-1}\gamma_n, \gamma_n]))|}{|f_0([\beta^{-1}\gamma_n, \gamma_n])|} = \frac{\lambda}{\beta} \frac{t^{-1}(1-\lambda)}{1-\beta^{-1}}.$$

From the concavity of f_0 and $x \le y$ for all $x \in J$ it follows that

$$(f_0^n)'(x) \ge (f_0^n)'(y) \ge \frac{\lambda}{\beta} \frac{t^{-1}(1-\lambda)}{1-\beta^{-1}}, \quad \text{for every } x \in J. \tag{9.1.1}$$

Take $n(J) = n+1$ and note that since $J \subset [f_0^{-2}(\gamma_n), \gamma_n]$ $n(J) = n+1$ it follows that

$$f_0^{n(J)}(J) \subset [1 - t^{-1}\gamma_n, f_0^2(1 - t^{-1}\gamma_n)] = [1 - t^{-1}\gamma_n, 1 - t^{-1}\lambda^2\gamma_n].$$

By definition of $f_{1,t}$ we have so that

$$f_{1,t} \circ f_0^{n_0}(J) \subset [(1-\lambda^2)\gamma_n, \gamma_n].$$

Finally, let $m(J) \ge 0$ be the smallest natural number such that

$$(f_0^{m(J)} \circ f_{1,t} \circ f_0^{n(J)})(J) \cap (f_0^{-1}(\gamma_n), \gamma_n] \ne \emptyset.$$

and

$$(f_0^{n(J)})'(x) \ge \frac{\lambda^2}{\beta} \frac{t^{-1}(1-\lambda)}{1-\beta^{-1}}, \quad \text{for all } x \in J.$$

By definition of $f_{1,t}$ we have so that

$$f_{1,t} \circ f_0^{n_0}(J) \subset [(1-\lambda^2)\gamma_n, \gamma].$$

Let $m(J)$ be the smallest natural number such that

$$(f_0^{m(J)} \circ f_{1,t} \circ f_0^{n(J)})(J) \cap (f_0^{-1}(\gamma_n), \gamma_n] \ne \emptyset,$$

By equation (9.1.1), $n(J) = n+1$, $m(J) \ge 0$, and the choice of λ and β in condition (P0.ii) it follows

$$(f_0^{m(J)} \circ f_{1,t} \circ f_0^{n(J)})'(x) \ge \frac{\lambda^2}{\beta} \frac{1-\lambda}{1-\beta^{-1}} = \kappa > 1, \quad \text{for every } x \in J. \tag{9.1.2}$$

We now take $J_0 \overset{\text{def}}{=} J$, $n_0 \overset{\text{def}}{=} n(J)$ and $m_0 \overset{\text{def}}{=} m(J)$ and and let

$$(f_0^{m_0} \circ f_{1,t} \circ f_0^{n_0})(J_0) \overset{\text{def}}{=} J_1.$$

Note that by construction the interval J_1 intersects $(f_0^{-1}(\gamma_n), \gamma_n]$. If J_1 contains $[f_0^{-2}(\gamma_n), f_0^{-1}(\gamma_n)]$ we are done (just take $\ell = 0$). Otherwise J_1 is contained in

$[f_0^{-2}(\gamma_n), (\gamma_n)]$ and we can apply the previous construction to J_1 obtaining m_1 and n_1 such that

$$J_2 \stackrel{\text{def}}{=} (f_0^{m_1} \circ f_{1,t} \circ f_0^{n_1})(J_1), \quad J_2 \cap (f_0^{-1}(\gamma_n), \gamma_n] \neq \emptyset, \quad |J_2| \geq \kappa\, |J_1| \geq \kappa^2\, |J_0|.$$

Arguing inductively, we get a sequence of intervals J_i and of numbers n_i, m_i such that

$$J_{i+1} \stackrel{\text{def}}{=} (f_0^{m_i} \circ f_{1,t} \circ f_0^{n_i})(J_i), \quad J_{i+1} \cap (f_0^{-1}(\gamma_n), \gamma_n] \neq \emptyset, \quad |J_{i+1}| \geq \kappa\, |J_i| \geq \kappa^{i+1}\, |J_0|.$$

This implies that there is a first ℓ such that $J_{\ell+1}$ contains $[f_0^{-2}(\gamma_n), f_0^{-1}(\gamma_n)]|$. Now it is enough to take $\xi(J) = 0^{n_0}10^{m_0} \ldots 0^{n_\ell}10^{m_\ell}$ and note that $J_{\ell+1} = f_{[\xi(J)],t}(J)$. The fact that $f_{[\xi(J)],t}$ is uniformly expanding on J follows from (9.1.1). □

As a consequence of Lemma 9.1.1 we get the following:

Corollary 9.1.2. *Consider* $t \in (0,1]$. *Then there is arbitrarily small* γ *such that the map* $f_{[\xi(I)],t}$ *defined on the fundamental domain* $I = [f_0^{-2}(\gamma), f_0^{-1}(\gamma)]$ *has a unique fixed point* $q_t \in I$ *that is repelling and such that the interior of* $W^u(q_t, f_{[\xi(I)],t})$ *contains* I*.*

Let $\xi = \xi(I)$ and $\xi_q = \xi^{\mathbb{Z}} = \xi_q^-.\xi_q^+$. Consider the periodic point of F_t

$$Q_* = (\xi_q, q_t) \in \Lambda_t.$$

Proposition 9.1.3. *For every* $t \in (0,1)$ *we have* $\Lambda_t = H_{\Sigma_2 \times [0,1]}(Q_*, F_t)$.

Sketch of the proof. Given any $X = (\xi^-.\xi^+, x) \in \Lambda_t$ we need to show that $X \in H_{\Sigma_2 \times [0,1]}(Q_*, F_t)$.

Remark 9.1.4. Note that the fundamental domain $I = [f_0^{-2}(\gamma), f_0^{-1}(\gamma)]$ can be choose arbitrarily close to 0, and so we can consider that $f_{1,t}^{-1}(x)$ is at right of I for every x at left of I.

We consider the case where $x \in (0,1)$ and ξ^- contains infinitely many 1's, the general case can be treated as in [7, Section 5, Proposition 5.3].

Consider the set

$$\Delta_n^s = \{(\eta^-.\xi^+, x), \text{where } \eta_{-i} = \xi_{-i} \text{ for } i = 1, \ldots, n\}.$$

Note that the diameter of Δ_n^s goes to 0 as n goes to ∞. Thus to prove the proposition it is enough to see that there is $Y_n \in \Delta_n \cap H_{\Sigma_2 \times [0,1]}(Q_*, F_t)$ for every n sufficiently large.

By Corollary 9.1.2, there is $\epsilon > 0$ such that the ϵ-neigborhood I_ϵ of I, $I_\epsilon = [f_0^{-2}(\gamma) - \epsilon, f_0^{-1}(\gamma) + \epsilon]$, satisfies

$$W^u(Q_*, F_t) \supset (\xi_q^-.) \times I_\epsilon.$$

Since the interior of I_ϵ contains a fundamental domain of f_0 this implies that there is exist m such that

Claim 9.1.5. *There is $m \geq 0$ such that*

$$F_t^m\big((\xi_q^-.) \times I_\epsilon\big) \cap \big((.\xi_{-n} \dots \xi_{-1}\xi^+) \times f^{-1}_{[\xi_{-n}\dots\xi_{-1}],t}(x)\big) \neq \emptyset.$$

Proof. To prove this claim it is enough to find a word $\eta_0 \dots \eta_{m-1}$ such that

$$f^{-1}_{[\xi_{-n}\dots\xi_{-1}],t}(x) \in f_{[\eta_0\dots\eta_{m-1}],t}(I_\epsilon)$$

By Remark 9.1.4 we can suppose that $f^{-1}_{[\xi_{-n}\dots\xi_{-1}],t}(x)$ is at right of I. As the interior of I_ϵ contains a fundamental domains, its enough to take $\eta_0 \dots \eta_{m-1} = 0^m$ for same m. □

Take now any point Y_n in the intersection of the sets in the claim. Note that $F_t^n(Y_n) \in \Delta_n^s$. The following lemma implies proposition.

Lemma 9.1.6. $Y_n \in H_{\Sigma_2\times[0,1]}(Q_*, F_t)$.

Proof. Write $Y_n = \big((\upsilon^-.\xi_{-n} \dots \xi_{-1}\xi^+), y\big)$ and for every large r and small $\delta > 0$ let

$$\Delta_\delta^r \stackrel{\text{def}}{=} (\upsilon^-.\xi_{-n} \dots \xi_{-1}\xi_0 \dots \xi_r) \times [y - \delta, y + \delta] \subset W^u(Q_*, F_t)$$

Claim 9.1.7. *The set Δ_δ^r contains a homoclinic point of the orbit of Q_*.*

Proof. Observe that

$$F_t^{n+r+1}(\Delta_\delta^r) = (\upsilon\,\xi_{-n} \dots \xi_{-1}\xi_0 \dots \xi_r.) \times f_{[\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r],t}([y - \delta, y + \delta]).$$

For large i we have that $f_{[\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r 0^i],t}([y-\delta, y+\delta])$ is close to 1. As $f_{1,t}(1) = 0$ we have that $f_{[\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r 0^i 1],t}([y - \delta, y + \delta])$ is close to 0 and to the left of $[f_0^{-2}(\gamma), \gamma]$ (for that it is enough to take i large enough). Thus, after shrinking δ if necessary, we get $j \geq 0$ such that

$$J \stackrel{\text{def}}{=} f_{[\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r)0^i 1 0^j],t}([y - \delta, y + \delta]) \subset [f_0^{-2}(\gamma), \gamma].$$

If $\xi(J) = \xi_0^J \dots \xi_k^J$ is the expanding itinerary of J (see Lemma 9.1.1). one has that

$$f_{[\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r 0^i 1 0^j \xi_0^J\dots\xi_k^J],t}([y - \delta, y + \delta]) \supset [f_0^{-2}(\gamma), f_0^{-1}(\gamma)].$$

Consider the subset $\Delta' \subset \Delta^r_\delta$ defined by

$$\Delta' \overset{\text{def}}{=} (\upsilon^-.\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r 0^i 1 0^j \xi^j_0 \dots \xi^j_k.) \times [y-\delta, y+\delta] \subset \Delta^r_\delta.$$

Let $\ell = m + r + i + j + k + 3$. Then

$$F^\ell_t(\Delta') \supset (\upsilon^-\xi_{-n}\dots\xi_{-1}\xi_0\dots\xi_r 0^i 1 0^j \xi^j_0 \dots \xi^j_k.) \times [f_0^{-2}(\gamma), f_0^{-1}(\gamma)] \overset{\text{def}}{=} \Delta''$$

Since that $q \in [f_0^{-2}(\gamma), f_0^{-1}(\gamma)]$ we have that Δ'' transversely intersects $W^s(Q_*, F_t)$ and so, Δ'' contain a homoclinic point of Q_*. Now taking $\Delta = F_t^{-\ell}(\Delta'')$, by the injectivity of F_t we have $\Delta \subset \Delta' \subset \Delta^s_\delta$ This completes the proof of the claim. □

As δ can be taken arbitrarily small and r arbitrarily large and a homoclinic class is a closed set, we have that Y_n is accumulated by homoclinic points of Q_* and thus it is in the class. This proves Lemma 9.1.6. □

The proposition follows by noting that Y_n converges to X and that a homoclinic class is closed and invariant. □

Remark 9.1.8. Note that this Lemma 9.1.1 and Corollary 9.1.2 remains true if we take $t = 1$, this fact implies that the proof of Theorem 1.2.4 work if we take $t = 1$, and then $f_{1,t}(x) = 1 - x$. This ensures that for $t = 1$ we have a completely spiny porcupine.

9.2 Transitivity of central contracting porcupine-like horseshoes

In this subsection we verify the transitivity of the set Λ_t for the family $(F_t)_{t\in[t_0,1]}$ in Section 7. Condition (C) is enough to ensure the existence of a *blender* (see [1] section 6.2) that is the main ingredient for obtaining transitivity of sets in many examples. This condition says that one can cover the interval $(0, 1]$ applying the contractive parts of the maps $f_{1,t}$ and f_0. The proof is different for $t \in (t_0, 1)$ and $t = 1$. We also observe that a minor variation of our arguments implies the transitivity of Λ_1.

9.2.1 Transitivity of Λ_t for $t \in (t_0, 1)$

Next lemma plays a role similar to the one Lemma 9.1.1 (in this case we get some uniform expansion and a covering property for backward iterates).

Fix a parameter $t_0 < 1$ such that f_0 is a contraction in $[t_0, 1]$. We have the following result from [7].

Lemma 9.2.1. *Assume that the maps f_0 and $f_{1,t}$ satisfy conditions (P0.i) and (P.1) and $t > t_0$. Then for every non-degenerate closed interval $J \subset [0,1]$ there is a word of the form*

$$(\xi_{-1} \dots \xi_{-n} \dots \xi_{-n-k-2}) = (\xi_{-1} \dots \xi_{-n}\, 1^2\, 0^k)$$

such that

$$(0,1] \subset \bigcup_{k \geq 0} g^{-1}_{1^2\,0^k,t}\big(g^{-1}_{\xi_{-1}\dots\xi_{-n},t}(J)\big).$$

Proof. Observe that $f_{1,t}^{-1}(t) = 1$. This implies that $f_{1,t}^{-1}([t-\epsilon, t]) = [1 - t^{-2}\epsilon, 1]$. Therefore

$$(0,1] = \bigcup_{k \geq 0} f_0^{-k} \circ f_{1,t}^{-1}\big([t-\epsilon, t]\big).$$

Therefore, to prove the lemma given an interval J it is enough to find a word $\xi_{-1} \dots \xi_{-n}$, $n \geq 0$, and $\varepsilon > 0$. such that

$$g^{-1}_{\xi_{-1}\dots\xi_{-n},t}(J) \subset [t-\epsilon, t]. \tag{9.2.1}$$

Using condition (C) we fix small $\epsilon > 0$ such that

$$f_0^{-1}(t-\epsilon) < t_0 < t - \epsilon < t.$$

Then there is $\alpha > 1$ with such that

$$1 < \alpha < \min\{t^{-1}, f_0'(f_0^{-1}(t-\epsilon))\} = \min\{t^{-1}, (f_0^{-1})'(x),\ \mathrm{f}x \in [t-\varepsilon, 1]\}. \tag{9.2.2}$$

If $[t-\epsilon, t] \subset J$ then (9.2.1) holds automatically and there is nothing to prove.

Thus we can suppose that $[t-\epsilon, t]$ is not contained in J. There are two cases:

– if $J \subset [t-\epsilon, 1]$ we take $\xi_{-1} = 0$,

– if $J \subset [0, t]$ we take $\xi_{-1} = 1$.

Let $J = J_0$ and $J_1 = g^{-1}_{\xi_{-1},t}(J_0)$. Note that $|J_1| \geq \alpha\, |J_0|$.

If J_1 contains $[t-\epsilon, t]$ we are done. We argue inductively defining intervals $J_0, J_1, \dots, J_i$ and numbers $\xi_{-1}, \xi_{-2}, \dots, \xi_{-i-1}$ where

– $\xi_{-k-1} = 0$ if $J_k \subset [t-\epsilon, 1]$,

– $\xi_{-k-1} = 1$ if $J_k \subset [0, t]$,

– $J_{k+1} = g^{-1}_{\xi_{-k-1},t}(J_k) = g^{-1}_{\xi_{-1}\dots\xi{-k-1},t}(J_0)$.

Note that by construction,

$$|J_{k+1}| \geq \alpha\,|J_k| \geq \alpha^{k+1}\,|J_0|.$$

This fact immediately implies that there is a first ℓ such that J_ℓ contains $[t-\epsilon, t]$. This completes the proof of the lemma. $\square$

We have the following corollary (that plays a role similar to the one of Corollary 9.1.2).

Corollary 9.2.2. *Consider $t \in (t_0, 1)$. Then there are small $\epsilon > 0$ and a sequence $\xi_{-1}\dots\xi_{-n}$ such that $g^{-1}_{\xi_{-1}\cdots\xi_{-n},t}$ has an expanding point $p_t \in (t-\epsilon, t)$ such that $[t-\epsilon, t] \subset W^u(p_t, g^{-1}_{\xi_{-1}\cdots\xi_{-n},t})$.*

Let $\hat{p} = (\xi_{-1}\dots\xi_{-n})^{\mathbb{Z}}$ and consider the periodic point of F_t

$$P_* = (\hat{p}, p_t) \in \Lambda_t.$$

This point is contracting in the central direction.

Lemma 9.2.1 and Corollary 9.2.2 implies that there are $\rho > 0$ and a sequence of negative tails of sequences ξ_k^- such that

$$W^s(P_*, F_t) \supset \bigcup_{k\geq 0} (\xi_k^-.) \times g^{-1}_{0^k,t}([1-\rho, 1]).$$

This implies the following:

Remark 9.2.3. *$W^s(P_*, F_t)$ intersects any set of the form $(.\xi^+) \times \{x\}$, $x \in (0,1]$.*

As in Section 9.2 this is the main step to prove that Λ_t is the relative homoclinic class of P_*. The argument goes as follows, take $(\xi, x) \in \Lambda_t$ and for simplicity let us assume that $x \in (0,1)$ (the case $x = 0, 1$ follows similarly). Write $\xi = \xi^-.\xi^+$ and note that given the "unstable" disk $(\xi_{-m}\dots\xi_{-1}.\xi^+) \times \{x\}$ we have that

$$F_t^{-m}\big((\xi_{-m}\dots\xi_{-1}.\xi^+) \times \{x\}\big) = (.\xi_{-m}\dots\xi_{-1}\xi^+) \times g^{-1}_{\xi_{-m}\cdots\xi_{-1},t}(x).$$

As $g^{-1}_{\xi_{-m}\cdots\xi_{-1},t}(x) \in (0,1)$, Remark 9.2.3 implies that this set intersects $W^s(P_*, F_t)$.

9.2.2 Transitivity of Λ_1

We begin with an observation about the topological structure of the set Λ_1 This set has disjoint two parts. The subset $\Lambda_1^{\{0,1\}}$ consisting of the points of Λ_1 with central coordinates 0 and 1 corresponds to a two intermingled horseshoes of different indices. Using the notation in [7] this part of Λ_1 is its exposed part. The subset $\Lambda_1^{(0,1)}$ of Λ_1 consisting of the points with central coordinate in $(0,1)$. These points are dense in Λ_1. This is the reason why it is enough to see that $\Lambda_1^{(0,1)}$ is contained in a relative homoclinic class. We now go to the details of the constructions.

As in the previous cases, to prove the transitivity of Λ_1 we need to produce some expanding itineraries for the inverse maps of the systems satisfying some appropriate covering property (note that the proof of the previous section does not work for $t = 1$, recall equation (9.2.2) and remark that in this case $\alpha \leq 1$).

There are two cases. Recall that t_0 is defined by $f_0'(t_0) = 1$.

Let us assume first that $t_0 < 1/2$. Consider $c \in (t_0, 1/2)$ close to t_0 (recall that t_0 is defined by $f_0'(t_0) = 1$) and note that by hypothesis $f_{1,t}^{-1}(c) > c$. Consider small ϵ and the interval $[c, c+\epsilon]$.

Given an interval $I \subset (0,1)$ that does not contain $[c, c+\epsilon]$ we let

- $\xi_{-1} = \xi_{-1}(I) = 0$ if $I \subset [c, 1]]$,
- $\xi_{-1} = \xi_{-1}(I) = 1$ if otherwise (in this case, $I \subset [0, c+\epsilon]$).

In the first case, $|g_{\xi_{-1},t}^{-1}(I)| \geq \kappa\, |I|$. In the second case $g_{\xi_{-1},t}^{-1}(I)| = |I|$ and $|g_{\xi_{-1},t}^{-1}(I) \subset [c, 1]$.

We now argue inductively, let $[c, c+\epsilon] = J_0$ and let $J_1 = g_{\xi_{-1}(J_0)}^{-1}(J_0)$. Inductively, $J_{\ell+1} = g_{\xi_{-1}(J_\ell)}^{-1}(J_\ell)$. By construction, there are no two consecutive applications of $f_{1,t}^{-1}$, therefore

$$|J_{\ell+1}| \geq \kappa^{\ell/2}\, |J_0|.$$

As in Lemma 9.2.1 there is a first ℓ such that J_ℓ contains $[c, c+\epsilon]$.

Let $\xi_{-1} \dots \xi_{-\ell-1} = \xi_{-1}(J_0) \dots \xi_{-1}(J_\ell)$. The previous construction implies that the $g_{\xi_{-1} \dots \xi_{-\ell-1},1}^{-1}$ has a expanding fixed point p in in $[c, c+\epsilon]$ such that $[c, c+\epsilon] \subset W^u(p, g_{\xi_{-1} \dots \xi_{-\ell-1},1}^{-1})$. Then arguing as in the previous cases we have that Λ_1 is the relative homoclinic class of the contracting saddle $P_* = \big((\xi_{-1} \dots \xi_{-\ell-1}), p\big)$. This proves the transitivity of Λ_1. This concludes the proof of the case $t_0 < 1/2$.

If $t_0 > 1/2$ we argue as before but considering forward iterates f_0 and $f_{1,1}$. In this case, we get a periodic point that is expanding in the central direction whose relative homoclinic class is Λ_1.

10
Appendix C: A differentiable model

We worked always with a one parameter family of skew-product defined in $\Sigma_2 \times \mathbb{R}$, in this section we realize a equivalent differentiable model in $\mathbb{R}^2 \times \mathbb{R}$.

We start with the definition of porcupine in a more general case, but before we present two definitions.

Definition 10.0.4 (Dominated splitting)**.** Given a diffeomorphism $F \in Diff^1(M)$ and a f-invariant set Λ, a DF-invariant splitting with two bundles $E \oplus F$ of TM over Λ is *dominated* is there are constants $m > 0$ and $k < 1$ such that

$$||DF^m_{|E}(x)|| \cdot ||DF^m_{|E}(x)|| < k \quad \text{for every} \quad x \in \Lambda,$$

Where $||\,.\,||$ is norm in M. An DF-invariant splitting with three bundles $E \oplus F \oplus G$ is dominated if $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are dominated.

Definition 10.0.5 (Partial hyperbolicity)**.** An F-invariant compact set Λ is said to be *partially hyperbolic* if there is a dF-dominated splitting $E^s \oplus E^c \oplus E^u$ where $dF_{|E^s}$ is uniformly contracting, $dF_{|E^u}$ is uniformly expanding, and E^c is non-trivial and non-hyperbolic. We say that E^c is the *central bundle*. The set Λ is called *strongly partially hyperbolic* if the three bundles E^s, E^c and E^u are non-trivial. ([1])

Definition 10.0.6 (Porcupine)**.** A compact F-invariant set Λ of a (local) diffeomorphism F is a *porcupine* if

- Λ is a maximal invariant set in a compact set $\mathbf{C}$, *transitive* (existence of a dense orbit), and strongly partially hyperbolic with one-dimensional central bundle,
- there is a subshift of finite type $\sigma : \Sigma \to \Sigma$ an a semiconjugation $\Pi : \Lambda \to \Sigma$ such that $\sigma \circ \Pi = \Pi \circ F$, $\Pi^{-1}(\xi)$ contain a non-degenerated interval for uncountable many $\xi \in \Sigma$ and a single point for uncountable many$\xi \in \Sigma$.

Consider the cube $\hat{\mathbf{C}} = [0,1]^2$ and a diffeomorphism Φ defined in $\mathbb{R}^2$ for which the maximal invariant set in a neighborhood of $\hat{\mathbf{C}} = [0,1]^2$ is the two length horseshoes Γ, that is conjugated with the full shift σ of two symbols. Denote by $\pi : \Gamma \to \Sigma_2$ the conjugation map, $\pi \circ \Phi = \sigma \circ \pi$. we consider the subcubes $\hat{\mathbf{C}}_0$ and $\hat{\mathbf{C}}_1$ of $\hat{\mathbf{C}}$ such that $\Phi(\hat{\mathbf{C}}_i) \subset \hat{\mathbf{C}}$ and $\hat{\mathbf{C}}_i$ contain all the points $X \in \Gamma$ whose 0-coordinate $(\pi(X))_0$ is i.

Let $\mathbf{C} = \hat{\mathbf{C}} \times [0,1]$, and the real maps f_0, $f_{1,t}$. Given a point $X \in \mathbf{C}$, we write $X = (\hat{x}, x)$, where $\hat{x} \in \hat{\mathbf{C}}$ and $x \in [0,1]$. For each $t \in [0,1]$ we consider the map

$$\widehat{F}_t \colon \hat{C} \times [0,1] \to \mathbb{R}^2 \times \mathbb{R}$$

given by

$$\widehat{F}_t(\hat{x}, x) = \begin{cases} (\Phi(\hat{x}), f_0(x)), & \text{if } X \in \hat{C}_0 \times [0,1], \\ \\ (\Phi(\hat{x}), f_{1,t}(x)), & \text{if } X \in \hat{C}_1 \times [0,1]. \end{cases}$$

We have that $\widehat{F}_t$ has a associated porcupine.

We can extend $\widehat{F}_t$ for all $\mathbb{R}^2 \times \mathbb{R}$ and so we have a one parameter family of local diffeomorphism $(\widehat{F}_t)_{t \in [0,1]}$, $\widehat{F}_t \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ such that the maximal invariant set $\widehat{\Lambda}_t$ in $\mathbf{C}$ is a porcupine.

Using the conjugation $\pi \colon \Gamma \to \Sigma_2$ we have a equivalence between such a local diffeomorphism $\widehat{F}_t$ and a skew-product $F_t = \widehat{F}_t \circ \pi \colon \Sigma_2 \times \mathbb{R} \to \Sigma_2 \times \mathbb{R}$.

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