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Buy-or-Sell Auction: Basic Framework

In the multi unit buy-or-sell auction considered in this paper, a risk-neutral auctioneer buys, sells or conducts both operations simultaneously in the auction of multiple units of an indivisible object. In the forward (reverse) auction, there are k_F (k_R) units available to be sold (bought). All $n > k_F + k_R$ risk-neutral participants are not informed *ex-ante* about what operation will be conducted. Each bidder can place a single bid and only buy or sell one unity of each good.

Each participant $i = 1, ..., n$ has a private information with respect to the object, summarized by the realization of the random variable $X_i \in [\underline{x}, \overline{x}]$, denominated as *signal* or *type*. Additionally, consider that exists a vector $S = (S_1, ..., S_m)$ of random variables that affect the value of the object to the bidders.

As in Milgrom and Weber (1982), the value of the object for the bidder i is represented by the non-decreasing continuous function U_i, symmetric in the last $n-1$ arguments:

$$V_i = U_i(S, X) = U(S, X_i, [X_j]_{j \neq i})$$

The last equality means that the value of the object depends on S and signals of others bidders in the same way for any bidder i, where $\mathbb{E}[V_i] < \infty$. Additionally, define $f(s, x)$ as the continuously differentiable joint probability density of (S, X). It is assumed that f is symmetric on the last n final arguments (X_i are drawn from the same distribution) and greater than zero at any point. Finally, it is assumed that (S, X) are affiliated.

The auction proceeds as follows. We consider that the spread, that is, the term that defines the difference between bids and prices for the auctions, is a parameter. Informed of this spread, the participants submit a single bid. If the auction is, *ex-post*, a forward (reverse) auction, the auctioneer sells (buys) the objects (one unit to each bidder) to (from) the k_F highest (k_R lowest) bidders. [1] In terms of payments, if the auction is forward (reverse), the winning bidder pays (receives) his own bid subtracted(added) by s. The expected profit

[1] The expression k_F highest refers to the k_F element if the bids were to be considered in an decreasing order.

function of each participant i that bids b_i is given by:

$$\Pi_i(b_i, x_i) = \mathbb{E}\left[P_F(S_F)(V_i - (b_i - s))\mathbb{1}_{b_i > \bar{b}}|X_i = x_i\right] + \\ \mathbb{E}\left[P_R(S_R)(b_i + s - V_i)\mathbb{1}_{b_i < \underline{b}}|X_i = x_i\right]$$

where P_F and P_R are the probability of each kind of auction (discussed later on this section) and $\bar{b}$ is the notation adopted to represent the element k_F when bids, ignoring b_i, are sorted in a decreasing order. In more simpler terms, in a single unit auction, $\bar{b}$ would be the highest bid not considering the bid made by the individual i (that is: the first bidder that loses the auction). Additionally, $\underline{b}$ is the analog at the reverse auction end. Each participant's objective is to maximize it's own expected profit, given the strategy followed by all others bidders.

In terms of the way we approach the probability of each kind of auction, the function $P_F > 0$ is the probability of occurring a discriminatory price forward auction, in which the auctioneer sells k_F units of the object, one to each of the bidders whose bid belongs to the k_F highest bids, and $P_R > 0$ is the probability of occurring a reverse auction, in which the auctioneer buys k_R units of the object from the k_R lowest bids. Note that might be the case that $P_F + P_R > 1$, given that the auctioneer may conduct both operations. In our model, all bidders have the same function generating these probabilities (the arguments of these functions, however, change from bidder to bidder). Define $Y_1, ..., Y_{n-1}$ the order statistics of $X_1, ...X_{i-1}, X_{i+1}, ..., X_n$. We assume that:

Assumption 1 (A1): $\forall i, k = 1, ..., n$, conditional on $X_i = x, Y_k = y$, S_F and S_R are independent of (S, X).

A1 describes the mathematical way we include the probability functions in our model. We consider that there are two random variables, S_F and S_R, that are affiliated with X, but, conditional on private signals, it does not affect the value of object or any other variable in the vector (S, X). This formulation is natural. The probability of each kind of auction depends on the private signals that bidders observe, since these signals will affect the optimal bidding strategy, but does not directly impact the value attributed to the good. Additionally, we assume that:

Assumption 2 (A2) The continuously differentiable functions $P_F : S_F \to (0, 1)$ and $P_R : S_R \to (0, 1)$ are monotone (P_F increasing and P_R decreasing).

A2 describes how the functions that assign probabilities to each kind of operation in a buy-or-sell auction must behave. This assumption captures the fact that if a bidder knows that his private signal is high and considers that the k_F highest signal is high, then the auctioneer will most likely sell the good, given that bidders assign a high value to it. Note, however, that we are not considering the fact that a bidder may affect this probability function with his own bid directly.