3
Equilibrium

3.1
Preliminaries

Denote the infimum and supremum of the support of signs X by, respectively, $\underline{x}$ and $\overline{x}$, allowing for $\underline{x} = -\infty$ or/and $\overline{x} = \infty$. We focus our analysis on any bidder i without loss of generality, since we investigate the symmetric equilibrium. First, we define the functions for $\gamma = \{F, R\}$:

$$v_\gamma(x, y) \equiv \mathbb{E}\left[V_i | X_i = x, Y_{k_\gamma} = y\right]$$

Additionally, we must define explicitly the probability functions for $\rho = \{F, R\}$:

$$p_\rho(x, y) \equiv \mathbb{E}\left[P_\rho(S_\rho) | X_i = x, Y_{k_\rho} = y\right]$$

It is important to state that this probability functions are monotone in x and y, as guaranteed by **A2** and Theorem 5 of Milgrom and Weber (1982). In second place, we propose the following ***synthetic value (or type) of the object*** that will appear on the derivation and the final result of equilibrium:

$$v(x, y) \equiv \frac{p_F(x, y) f_{k_F}(y|x)}{\lambda(x, y)} \left[v_F(x, y) + s\right] + \frac{p_R(x, y) f_{k_R}(y|x)}{\lambda(x, y)} \left[v_R(x, y) - s\right]$$

where:

$$\lambda(x, y) \equiv p_F(x, y) f_{k_F}(y|x) + p_R(x, y) f_{k_R}(y|x)$$

The synthetic type is the expected valuation (adjusted by the spreads) on the forward and the reverse auctions, weighted by the probability of each auction occurring and bidder i participating actively. The subscript k_F on density functions represents that this is the conditional density of the Y_{k_F} order statistics. For the reverse auction, we consider that k_R actually corresponds to the conditional density of the Y_{n-k_R} order statistics. The same notation is used on other functions through the paper. For the probability functions, we adopt simply that $p_F(x, y)$ is the probability of the forward auction and $p_R(x, y)$

the probability of the reverse. Using the definitions of $p_F(x, y)$, $p_R(x, y)$ and $\lambda(x, y)$, we make two additional assumptions.

Assumption 3 (A3): $p_F(x, y)$ and $p_R(x, y)$ are log-supermodular.

Assumption 4 (A4): $\frac{p_R(x,y) f_{k_R}(y|x)}{\lambda(x,y)}$ is non-decreasing in x and y.

A3 is an assumption on the general behavior of the probability functions we introduced on our model. We assume that p_F and p_R are log-supermodular in order to guarantee the equilibrium structure, given that the density functions also present this property. This hypothesis is trivially satisfied, for example, if we consider that bidders do not change their probabilities based on their own private signal.

A4 addresses how the relative weights on the synthetic value of the object change when x and y change. It may seem strange at first sight that we consider that the weight of the reverse auction increases when we increase the values of x and y. However, this guarantees that the bidders with highest private signals attach higher values to the object being auctioned. **A4** also considers that, specifically on the worst type, these weights must be the same. As will be proven that the worst type is an intermediate type on a buy-or-sell auction, this hypothesis is reasonable.

To illustrate the concept of synthetic type, we present the Lemma 1. An interesting aspect of this synthetic type is that in a uniform price auction (where bidders do not pay their own bid), it is possible to show that it is a symmetric Bayes-Nash equilibrium to bid the synthetic type when the exogenous spread s is zero.

Lemma 1. *Under assumptions **A1**, **A2** and **A4**,bidding $\beta_2(x) = v(x, x)$ for all bidders is a Bayes-Nash equilibrium in a uniform buy-or-sell auction with $s = 0$.*

3.2 The Worst Type

In auctions where bidders pay their bid, the usual equilibrium derivation strategy consists of finding the value of the bid where the first order derivative of each bidders expected profit is zero (FOC). To determine a boundary condition, most auction models establish that the worst type (the lowest type in a forward auction, for an example) must attend a zero profit condition at the equilibrium and, hence, bid his own valuation. In the model presented on this paper, however, the existence of a type that serves as a boundary condition is not obvious: bidders with types that are close to $\overline{x}$ are more prone to be competitive on the forward auction, while bidders with types approaching $\underline{x}$ are competitive on the reverse auction. This is a difference, for instance, to the models developed in Morgan (2004). When bidders only bid actively to buy and the amount paid is given to the losers of the auction, then the worst type still is the lowest. Mathematically, the worst type x^* of the buy-or-sell auction must satisfy:

$$\int_{\underline{x}}^{x^*} p_F(x^*,\xi) f_{k_F}(\xi|x^*) d\xi = \int_{x^*}^{\overline{x}} p_R(x^*,\xi) f_{k_R}(\xi|x^*) d\xi$$

Lemma 2 shows that the condition above corresponds to the worst type and guarantees the existence of x^*.

Lemma 2. Existence of the Worst Type: *Under the assumptions **A1** and **A2**, exists x^* that satisfies the equation above and x^* corresponds to the worst type of the buy-or-sell auction.*

To also prove the uniqueness of x^*, we assume that:

Assumption 5 (A5): $\frac{|f_k^2(\hat{x}\hat{x})|}{f_k(\hat{x}|\hat{x})} < f_k(x^*|x^*)$ $\forall k = 1, ..., n$ and $\hat{x} = \{\underline{x}, \overline{x}\}$, where $f_k^2(y|x) = \frac{\partial f_k(y|x)}{\partial x}$ and x^* is the worst type on the auction (defined in Lemma 2).

The intuition behind this assumption is that bidders cannot be too affiliated, that is: the effect of the other bidders information is limited by the conditional density of the worst type.Using the assumptions made on the paper, we are able to prove Lemma 3.

Lemma 3. Uniqueness of the Worst Type: *Under the assumptions **A1**, **A2** and **A5**, there exists only one x^* that solves: $\Lambda(x^*, x^*) = 0$, where:*

$$\Lambda(x,a) = \int_{\underline{x}}^{a} p_F(x,\xi) f_{k_F}(\xi|x) d\xi = \int_{a}^{\overline{x}} p_R(x,\xi) f_{k_R}(\xi|x) d\xi$$

In particular, in an IPV model with constant probabilities $p_F = p_R$ and $k_F = n - k_R = 1$, x^* is the median of the distribution.

It is possible to make a comparison with a result derived in Myerson and Satterwhite (1983). In their paper, they analyze a particular mechanism for a partnership dissolution between two agents. One of the agents is assigned at random to play after the other. The agent that plays first declares a price and the other agent decides if it is going to buy or sell at that price. They concluded that in a partnership dissolution between two agents that has the structure presented, all types will bid a price that is an weighted average of their own valuation and the median of the distribution of types. Hence, the only truth telling type is the median of the distribution. In our model, a similar effect appears: the type that provides the boundary condition and bid its own synthetic value is an intermediate type.

3.3 Derivation

In this subsection we derive the symmetric equilibrium of the buy-or-sell auction. We assume that the spread not too high in order to guarantee that the equilibrium structured is maintained (**A6**).

Assumption 6 (A6) $s \leq \overline{s}$, *where*:

$$\overline{s} \equiv \frac{1}{2}\left[v_R(x^*,x^*) - v_F(x^*,x^*) + \int_{\underline{x}}^{x^*} L_F(\zeta|x^*)dt_F(\zeta) + \int_{x^*}^{\overline{x}} L_R(\zeta|x^*)dt_R(\zeta)\right]$$

where $t_F(x) = v_F(x,x)$ *and* $t_R(x) = v_R(x,x)$.

This is assumption is natural, since it guarantees that the spread is not higher than the distance between the bids on an only forward and an only reverse auctions. Under all assumptions and the definition of the worst type from the previous sections, we are able to fully characterize the Bayes-Nash symmetric equilibrium of the buy-or-sell auction.

Proposition 1. *Under assumptions **A1-A6** and Lemmas 2 and 3, the symmetric and strictly increasing and Bayes-Nash equilibrium is characterized by :*

(i) $x_i \neq x^*$:

$$\beta(x) = v(x,x) - \int_{x^*}^{x} L(\zeta|x)dt(\zeta)$$

(ii) $x_i = x^*$:

$$\beta(x^*) = v(x^*,x^*)$$

where

$$t(\zeta) = v(\zeta,\zeta)$$

and

$$L(\zeta|x) = \exp\left[-\int_{\zeta}^{x} \frac{\lambda(\psi,\psi)}{\Lambda(\psi,\psi)} d\psi\right]$$

An important detail that is hidden on the above equations is the fact that $L(\xi|x)$ is a distribution over $[\min(x, x^*), \max(x, x^*)]$. The interpretation of the optimal bidding is that bidders calculate an average using these hidden distributions and this is their optimal bid. To see this interpretation clearly, it is possible to rewrite the optimal bidding strategy as:

$$\beta(x) = \int_{x^*}^{x} v(\zeta,\zeta) dL(\zeta|x)$$

Another hidden feature is that the spread s is considered in the synthetic value. Therefore, although is does not explicitly appear in the equations at Proposition 1, it does play a central role in our model. Furthermore, as expected, the worst type (x^*) bids his synthetic value of the object as it happens in the auctions where the auctioneer reveals for agents what auction is being conducted.