4 Equilibrium Properties

4.1 Auction Type Revelation

This section presents some properties on the equilibrium described in Propositions 1. The conclusions on this section shed light on one of the main objectives of the paper, which was to conclude if is profitable for the auctioneer to reveal the type of auction being conducted. As we compare different types of auction, it is necessary to assume that the support of private valuations is compact (to guarantee the existence of equilibrium in the forward and reverse auctions). Hence, we assume that $-\infty < \underline{x} < \overline{x} < \infty$.

In first place, we need to derive the optimal bidding strategies of a symmetric Bayes-Nash equilibrium of the buy-or-sell auction in case one of the probability functions is identically zero, which is done in Lemma 4.

Lemma 4. *Under assumptions **A1** and **A2**, the symmetric equilibria when one of the probability functions is zero is given by:*

(i) $p_R(x, y) = 0, \forall x, y \in [\underline{x} < \overline{x}]$: $\beta_F(x, s) = s + \int_{\underline{x}}^{x} v_F(\xi, \xi) dL_F(\xi|x)$, *where*:

$$L_F(\xi|x) = exp\left[-\int_{\xi}^{x} \frac{p_F(\zeta,\zeta) f_{k_F}(\zeta|\zeta)}{\int_{\underline{x}}^{\zeta} p_F(\zeta,\psi) f_{k_F}(\psi|\zeta) d\psi} d\zeta\right]$$

(ii) $p_F(x, y) = 0, \forall x, y \in [\underline{x} < \overline{x}]$: $\beta_R(x, s) = \int_{x}^{\overline{x}} v_R(\xi, \xi) dL_R(\xi|x) - s$, *where*:

$$L_R(\xi|x) = exp\left[\int_{\xi}^{x} \frac{p_R(\zeta,\zeta) f_{k_R}(\zeta|\zeta)}{\int_{\zeta}^{\overline{x}} p_R(\zeta,\psi) f_{k_R}(\psi|\zeta) d\psi} d\zeta\right]$$

Lemma 5A proves that there exists an envelopment of bidding strategies, that is: buyers who buy in a buy-or-sell auction pay more for sufficiently small values of s. Therefore, not only the expected profit of a buy-or-sell auction is higher, but also the actual profit is always higher on a buy-or-sell auction when spreads are sufficiently small.

Lemma 5A. Envelopment of Optimal Bidding Strategies *Under* *the Proposition 1 and Lemma 4,* $s \leq \overline{s}$ *implies that:*

$$\beta_F(x) - s < \beta(x) - s < \beta(x) + s < \beta_R(x) + s$$

The result in the above lemma is rather intuitive: in a buy-or-sell auction, bidders face conflicting incentives. Therefore, they bid something that takes the forward and reverse auctions in account. Additionally, from the auctioneer point of view, the buy-or-sell auction presents a set of information that is intrinsically different from an auction where the type is known. This difference lies on the fact that on a forward auction, for instance, all participants know what operation will be conducted (the auctioneer sells the good). On a buy-or-sell auction, on the other hand, only the auctioneer knows what auction will be conducted. Hence, its expected profit is higher, while the expected profit for bidders are reduced.

A reasonable criticism to the property of Lemma 5A is: if the auctioneer conducts an only forward or only reverse auctions, why bidders would consider that there is a probability of the occurring? Lemma 5B addresses this topic. The result is that even if bidders believe that the auction is surely forward or reverse, the result in Lemma 5A is valid. For that, we define the particular cases of the functions of Lemma 4, which account for the fact that bidders are sure of which type of auction might be occurring: $\beta_{F,1}(x,s) = s + \int_{\underline{x}}^{x} v_F(\xi,\xi) dL_{F,1}(\xi|x)$, *where*:

$$L_{F,1}(\xi|x) = exp\left[-\int_{\xi}^{x} \frac{f_{k_F}(\zeta|\zeta)}{F_{k_F}(\zeta|\zeta)} d\zeta\right]$$

and $\beta_{R,1}(x,s) = \int_{x}^{\overline{x}} v_R(\xi,\xi) dL_{R,1}(\xi|x) - s$, where:

$$L_{R,1}(\xi|x) = exp\left[\int_{\xi}^{x} \frac{p_R(\zeta,\zeta) f_{k_R}(\zeta|\zeta)}{1 - F_{k_R}(\zeta|\zeta)} d\zeta\right]$$

Lemma 5B. Envelopment of Optimal Bidding Strategies *Under the Proposition 1 and Lemma 4,* $s \leq \overline{s}$ *implies that:*

$$\beta_{F,1}(x) - s < \beta(x) - s < \beta(x) + s < \beta_{R,1}(x) + s$$

4.2 Expected Profit Analysis

This subsection analyses, given the strategy described in Proposition 1, the participation conditions and the expected profit for bidders. On a buy-or-sell auction, bidders suffer from the double sided winner's curse phenomenon. Obviously, the most extreme types have also a considerable amount of informational rent. On the IPV model, only the second effect exists. In this context, it is possible to show that when the spread is zero, the worst type bidder has zero expected profit in a buy-or-sell auction in the IPV model, while a negative expected profit at the affiliated model from the double sided winner's curse effect. Lemma 6 states this difference. The superscript IPV states that this is the profit in the IPV model and AFF the model with strictly affiliated preferences (that is, the affiliation inequality is strict).

Lemma 6. Expected Profits: *If* $s = 0$*, then* $\Pi^{IPV}(x^*, \beta(x^*)) = 0 > \Pi^{AFF}(x^*, \beta(x^*))$

A result that is possible to derive from the above lemma is the consequence of the following situation: if an agent that observes the private signal corresponding to the worst type has negative profit, it would not voluntarily participate of the auction and all other participants believe that there is no agent with that particular private signal. If no participating agent observes x^* as a private signal, then there might be less informational rent for other types. This might create a snow ball process that leads the affiliated model to collapse in situations where s is zero.

A cautionary note is necessary before proceeding to the proposition. What ee are referring here is a situation where for agents with specific private valuations participation might not be profitable. Note that we are not defining or considering that from n individuals of a determined buy-or-sell auction, some of the will exit. In other words, we are not considering that this number changes in any situation, but that, given a spread s, all participants know exactly what possible types the other $n-1$ might have their private valuation and that this set might be different from the original set of possible private valuations. The study of this model where agents endogenously choose to participate or not in the auction is not on the scope of this paper.

Proposition 2. The Spread Effect on Profits: *In the affiliated model with* $s = 0$*, even if bidders of specific private signals do not participate, there is no symmetric Bayes-Nash pure strategy equilibrium where all types have positive expected profit.*

Another important aspect of the model is the expected profit of bidders. Define the spread value s_1 as the value that guarantees that the *ex-ante* (i.e., before knowing the private signal) expected profit is non-negative, that is:

$$\mathbb{E}_{x_i}\left(\Pi(\beta(x_i), x_i, s_1)\right) = 0$$

Or, in explicit terms:

$$s_1 = \left[\int_{\underline{x}}^{\overline{x}}\int_{\underline{x}}^{\zeta} p_F(\zeta,\xi) f_{k_F}(\xi|\zeta) d\xi \ f_x(\zeta) d\zeta + \int_{\underline{x}}^{\overline{x}}\int_{\zeta}^{\overline{x}} p_R(\zeta,\xi) f_{k_R}(\xi|\zeta) d\xi \ f_x(\zeta) d\zeta\right]^{-1} \left[\int_{\underline{x}}^{\overline{x}}\int_{\underline{x}}^{\zeta} p_F(\zeta,\xi)\left[\beta(\zeta) - v_F(\zeta,\xi)\right] f_{k_F}(\xi|\zeta) d\xi \ f_x(\zeta) d\zeta + \int_{\underline{x}}^{\overline{x}}\int_{\zeta}^{\overline{x}} p_R(\zeta,\xi)\left[v_R(\zeta,\xi) - \beta(\zeta)\right] f_{k_R}(\xi|\zeta) d\xi \ f_x(\zeta) d\zeta\right]$$

Additionally, define the spread value s_2 as the value that guarantees that the *interim* (i.e., after observing the private signal but before the auction is conducted) expected profit is non-negative, that is:

$$\Pi(\beta(x_i), x_i, s_2) = 0$$

Or, in explicit terms:

$$s_2 = \frac{\int_{x^*}^{\overline{x}} p_R(x^*,\xi) v_R(x^*,\xi) f_{y_{K_R}}(\xi|x^*) d\xi - \int_{\underline{x}}^{x^*} p_F(x^*,\xi) v_F(x^*,\xi) f_{y_{K_F}}(\xi|x^*) d\xi}{\int_{\underline{x}}^{x^*} p_F(x^*,\xi) f_{y_{K_F}}(\xi|x^*) d\xi + \int_{x^*}^{\overline{x}} p_R(x^*,\xi) f_{y_{K_R}}(\xi|x^*) d\xi}$$

The next lemma of this section guarantees that the equilibrium of Proposition 1 is feasible, in the sense that we argue that there is an interval of possible values of the spread that induce full participation (with an *ex-ante* or interim participation condition) and the equilibrium is the described in Proposition 1. The importance of this lemma lies on the fact that the condition that $s \leq \overline{s}$ is sufficient to guarantee that the equilibrium is the described in Proposition 1. On the other hand, the values s_1 and s_2 illustrate what is the minimum value of the spread to guarantee participation in different settings.

Lemma 7. *The equilibrium of Proposition 1 is feasible if we consider an ex-ante participation condition, that is:*

$$s_1 \leq \overline{s}$$

Additionally, if $p_R(x^*, x^*) f_{k_R}(x^*|x^*) = p_F(x^*, x^*) f_{k_F}(x^*|x^*)$, then a interim participation condition is feasible, that is::

$$s_1 \leq s_2 \leq \overline{s}$$

If we believe we are in a context were participation restrictions are substantial, the above lemma tells that there are values of the spread that are high enough to guarantee that all agents will want to participate, but small enough to guarantee that the equilibrium derived in proposition 1 is maintained. Note that if s is smaller than the value relevant to guarantee full participation, the problem of endogenous entry is not trivial. This question is out of the scope of this paper. Future studies might focus on this problem and how this specificity of a buy-or-sell auction creates a trade-off between participation and extraction of informational rent by the auctioneer.

Finally, there is an effect on the bidding strategy and on payments when the spread changes. Lemma 8 addresses this question.

Lemma 8. *Under Proposition 1, for increasing values of s, the optimal bidding strategy considering the payments decreases for the forward (i.e., $\beta - s$ are decreasing in s) and increases in the reverse end of the auction.*

The response of the optimal bidding strategy is not obvious in the affiliated model, but the impact on payments is the one expected: an increasing spread reduces the payments on the forward auction end and increase in the reverse auction end. Note that, although expected, this result should be verified, since the optimal bidding strategy is affected by the magnitude of the spread on a non-trivial way.