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Appendix

Appendix A: Proofs of Lemmas and Propositions

The appendix provides proofs for all lemmas and propositions presented in the paper. Additionally, there is one additional lemma (Lemma A) that is useful in the proofs.

Lemma A. $(p_F(z,x)f_{k_F}(x|z))^{-1}\left(\int_{\underline{x}}^{x} p_F(z,\xi)f_{k_F}(\xi|z)d\xi\right)$ *is non-increasing in* z and $(p_R(z,x)f_{k_R}(x|z))^{-1}\left(\int_{x}^{\overline{x}} p_R(z,\xi)f_{k_R}(\xi|z)d\xi\right)$ *is non-decreasing in* z.

Prova.

From the affiliation inequality, it is possible to see that, for any $\alpha \leq x$ and $z' \leq z$:

$$\frac{f_{k_f}(\alpha|z)}{f_{k_f}(x|z)} \leq \frac{f_{k_f}(\alpha|z')}{f_{k_f}(x|z')}$$

Using **A4**, it is easy to verify that:

$$\frac{p_F(z,\alpha)f_{k_f}(\alpha|z)}{p_F(z,x)f_{k_f}(x|z)} \leq \frac{p_F(z',\alpha)f_{k_f}(\alpha|z')}{p_F(z',x)f_{k_f}(x|z')}$$

Integrating α over $\underline{x}$ and x delivers the desired result. The other result has a similar proof. □

Lemma 1. *Under assumptions **A1**, **A2** and **A4**,bidding* $\beta_2(x) = v(x,x)$ *for all bidders is a Bayes-Nash equilibrium in a uniform buy-or-sell auction with* $s = 0$.

*Prova.*This proof has two main steps. In the first, we argue that the synthetic type is increasing given the assumptions made. In the second step, we prove that is indeed a Bayes-Nash equilibrium strategy to bid the synthetic value on a uniform price auction.

To reach the conclusion that $v(x,y)$ is non decreasing in both arguments, note that the affiliated hypothesis guarantees that $v_F(x,y)$ and $v_R(x,y)$ are increasing in both arguments. Moreover, $v_F(x,y) \leq v_R(x,y)$ also due to the affiliation hypothesis. Combining this facts with **A4**, it is clear that $v(x,y)$ is increasing in both arguments.

Under **A1** and **A2**, the equilibrium strategy of a uniform price auction when other players bid $v(y, y)$ satisfies:

$$\begin{aligned}\beta_2(x) = \underset{b}{argmax} \int_{\underline{x}}^{\beta_2^{-1}(b)} & p_F(x,\zeta)\left[v_F(x,\zeta) - v(\zeta,\zeta)\right] f_{k_F}(\zeta|x)d\zeta + \\ \int_{\underline{x}}^{\beta_2^{-1}(b)} & p_R(x,\zeta)\left[v_R(x,\zeta) - v(\zeta,\zeta)\right] f_{k_R}(\zeta|x)d\zeta \\ = \underset{b}{argmax} \int_{\underline{x}}^{\beta_2^{-1}(b)} & \lambda(x,\zeta)\left[v(x,\zeta) - v(\zeta,\zeta)\right] d\zeta\end{aligned}$$

Given that $v(x, y)$ is non decreasing in both arguments, it is optimal for every bidder (given that all others also follow this strategy) to bid $\beta_2(x) = v(x, x)$.
□

Lemma 2. Existence of the Worst Type: *Under the assumptions **A1** and **A2**, exists x^* that satisfies the equation above and x^* corresponds to the worst type of the buy-or-sell auction.*

*Prova.*Define the function $\Lambda^*(x) \equiv \Lambda(x, x)$, where:

$$\Lambda(x,a) \equiv \int_{\underline{x}}^{a} p_F(x,\xi) f_{k_F}(\xi|x)d\xi - \int_{a}^{\overline{x}} p_R(x,\xi) f_{k_R}(\xi|x)d\xi$$

To begin with, we must prove that $\Lambda^*(x)$ is differentiable. Under **A2**, note that p_F and p_R are continuous, given that they are, respectively, the conditional expected value of the continuous functions P_F and P_R. This property comes from the Dominated Convergence Theorem. To see that, consider the case of p_F for instance. As the continuous function $P_F(s)f_{S_F}(s|x, y)$ is clearly dominated by the integrable function $f_{S_F}(s|x, y)$, it is possible to guarantee that, for any sequence $\{(x_n, y_n)\} \to (x, y)$:

$$\begin{aligned}\lim_{(x_n,y_n)\to(x,y)} p_F(x_n,y_n) &= \lim_{(x_n,y_n)\to(x,y)} \int_{S_F} P_F(s) f_{S_F}(s|x_n,y_n)ds \\ &= \int_{S_F} \lim_{(x_n,y_n)\to(x,y)} P_F(s) f_{S_F}(s|x_n,y_n)ds \\ &= \int_{S_F} P_F(s) f_{S_F}(s|x,y)ds \\ &= p_F(x,y)\end{aligned}$$

Where the first equality comes from the Dominated Convergence Theorem and the second from the fact that the function f is continuous. Therefore, p_F and p_R are continuous. The proof that Λ^* is differentiable is based on the fact that p_F,p_R, f_{k_F} and f_{k_R} are continuous functions. Hence, Λ^* is the sum of integrals of continuous functions. Each integral is differentiable (due to the Fundamental Theorem of Calculus). This proves that Λ^* is differentiable. Given that $\Lambda^*(x)$ is differentiable, $\forall x \in [\underline{x}, \overline{x}]$ we have that:

$$\Lambda^*(\overline{x}) = \int_{\underline{x}}^{\overline{x}} p_F(\overline{x}, \xi) f_{k_F}(\xi|\overline{x}) d\xi > 0$$

and

$$\Lambda^*(\underline{x}) = -\int_{\underline{x}}^{\overline{x}} p_R(\underline{x}, \xi) f_{k_R}(\xi|\underline{x}) d\xi < 0$$

By the Intermediate Value Theorem, it is possible to exhibit a point x^* such that $\Lambda^*(x^*) = 0$. The above inequalities also show that x^* is an intermediate type (is not one of the extremes, at least). To prove that this equation is what defines the worst type, simply apply **A1** to the expected profit equation. □

Lemma 3. Uniqueness of the Worst Type: *Under the assumptions* ***A1****,* ***A2*** *and* ***A5****, there exists only one x^* that solves: $\Lambda(x^*, x^*) = 0$, where:*

$$\Lambda(x, a) \equiv \int_{\underline{x}}^{a} p_F(x, \xi) f_{k_F}(\xi|x) d\xi = \int_{a}^{\overline{x}} p_R(x, \xi) f_{k_R}(\xi|x) d\xi$$

In particular, in an IPV model with constant probabilities $p_F = p_R$ and $k_F = n - k_R = 1$, x^* is the median of the distribution.

*Prova.*As Lemma 2 proves the existence of x^*, the problem now lies in proving that this point is unique. Note that:

$$\frac{d\left[\Lambda^*(x^*)\right]}{dx} = p_F(x^*, x^*) f_{k_F}(x^*|x^*) + p_R(x^*, x^*) f_{k_R}(x^*|x^*) +$$
$$\int_{\underline{x}}^{x^*} \frac{\partial p_F(x^*, \xi)}{\partial x} f_{k_F}(\xi|x^*) d\xi - \int_{x^*}^{\overline{x}} \frac{\partial p_R(x^*, \xi)}{\partial x} f_{k_R}(\xi|x^*) d\xi +$$
$$\int_{\underline{x}}^{x^*} p_F(x^*, \xi) \frac{\partial f_{k_F}(\xi|x^*)}{\partial x} d\xi - \int_{x^*}^{\overline{x}} p_R(x^*, \xi) \frac{\partial f_{k_R}(\xi|x^*)}{\partial x} d\xi$$

The above equation and **A2** yield:

$$\int_{\underline{x}}^{x^*} \frac{\partial p_F(x^*, \xi)}{\partial x} f_{k_F}(\xi|x^*) d\xi - \int_{x^*}^{\overline{x}} \frac{\partial p_R(x^*, \xi)}{\partial x} f_{k_R}(\xi|x^*) d\xi \geq 0$$

Additionally, we have to assess the values of the other terms of the equation. Note that:

$$p_F(x^*,x^*)f_{k_F}(x^*|x^*)+p_R(x^*,x^*)f_{k_R}(x^*|x^*)+ \int_{\underline{x}}^{x^*} p_F(x^*,\xi)\frac{\partial f_{k_F}(\xi|x^*)}{\partial x}d\xi - \int_{x^*}^{\overline{x}} p_R(x^*,\xi)\frac{\partial f_{k_R}(\xi|x^*)}{\partial x}d\xi >$$

$$\int_{\underline{x}}^{x^*}\left[\left(p_F(x^*,\xi)\frac{\frac{\partial f_{k_F}(\xi|x^*)}{\partial x}}{f_{k_F}(\xi|x^*)}\right)+p_F(x^*,x^*)f_{k_F}(x^*|x^*)\right]f_{k_F}(\xi|x^*)d\xi -$$

$$\int_{\underline{x}}^{x^*}\left[\left(p_R(x^*,\xi)\frac{\frac{\partial f_{k_R}(\xi|x^*)}{\partial x}}{f_{k_R}(\xi|x^*)}\right)-p_R(x^*,x^*)f_{k_R}(x^*|x^*)\right]f_{k_R}(\xi|x^*)d\xi > 0$$

Where the last inequality is due to **A5**. Therefore, in any point where the function Λ^* crosses the x-axis, it is increasing. This allows to conclude that exists only one point where $\Lambda^*(x^*)=0$. Finally, in an IPV model with constant probabilities $p_F = p_R$ and $k_F = n - k_R = 1$, x^*, the $\Lambda^*(x^*)=0$ condition reads as:

$$[1-F(x^*)]^{n-1} = [F(x^*)]^{n-1} \Rightarrow F(x^*) = 1/2$$

In this specific model, x^* is the median of the distribution. □

Proposition 1. *Under assumptions **A1-A6** and Lemmas 2 and 3, the symmetric and strictly increasing and Bayes-Nash equilibrium is characterized by :*

(i) $x_i \neq x^*$:

$$\beta(x) = v(x,x) - \int_{x^*}^{x} L(\zeta|x)dt(\zeta)$$

(ii) $x_i = x^*$:

$$\beta(x^*) = v(x^*,x^*)$$

where

$$t(\zeta) = v(\zeta,\zeta)$$

and

$$L(\zeta|x) = \exp\left[-\int_{\zeta}^{x}\frac{\lambda(\psi,\psi)}{\Lambda(\psi,\psi)}d\psi\right]$$

*Prova.*Consider the profit function on the auction:

$$\Pi_i(b_i,x_i) = \mathbb{E}\left[P_F(S_F)(V_i-(b_i-s))\mathbb{1}_{b_i>\overline{b}}|X_i=x_i\right] + \mathbb{E}\left[P_R(S_R)(b_i+s-V_i)\mathbb{1}_{b_i<\underline{b}}|X_i=x_i\right]$$

Applying the Law of Iterated Expectation yields:

$$\Pi_i(b_i,x_i) = \mathbb{E}\left[\mathbb{E}\left(P_F(S_F)(v_i-(b_i-s))\mathbb{1}_{b_i>\bar{b}}|X_i,Y_{k_F}\right)|X_i=x_i\right] + \\ \mathbb{E}\left[\mathbb{E}\left(P_R(S_R)(b_i+s-v_i)\mathbb{1}_{b_i<\underline{b}}|X_i,Y_{k_R}\right)|X_i=x_i\right]$$

Using the definitions made on section 2, the last equation may be written as:

$$\Pi_i(b_i,x_i) = \int_{\underline{x}}^{\beta-1(b_i)} p_F(x_i,\xi)\left[v_F(x_i,\xi)-b_i+s\right] f_{k_F}(\xi|x_i)d\xi + \\ \int_{\beta-1(b_i)}^{\overline{x}} p_R(x_i,\xi)\left[b_i+s-v_R(x_i,\xi)\right] f_{k_R}(\xi|x_i)d\xi$$

The necessary condition for optimality for bidder i is:

$$\Pi_i^{b_i}(\beta(x_i),x_i)=0$$

Using the Leibniz rule and analyzing the FOC in $\beta(x_i)$, we arrive at the following first order ODE:

$$\beta'(x_i)\Lambda(x_i,x_i) = \lambda(x_i,x_i)\left[v(x_i,x_i)-\beta(x_i)\right]$$

From Lemma 3, we know that exists (a unique) x^* in which: $\Lambda(x^*,x^*)=0$. For this specific value, the above equation provides:

$$\beta(x^*) = v(x^*,x^*)$$

For all other values:

$$\begin{aligned}\frac{\lambda(x_i,x_i)}{\Lambda(x_i,x_i)}\left[v(x_i,x_i)-\beta(x_i)\right] &= \frac{\lambda(x_i,x_i)}{\Lambda(x_i,x_i)}\left[\int_{x^*}^{x_i} L(\epsilon|x_i)dt(\epsilon)\right] \\ &= d\left(\left[\int_{x^*}^{x} t(\epsilon)L(\epsilon|x_i)\frac{\lambda(x_i,x_i)}{\Lambda(x_i,x_i)}dt(\epsilon)\right]\right)/dx \\ &= \beta'(x_i)\end{aligned}$$

From the aforementioned, the strategy proposed satisfies the necessary condition for optimality $\forall x_i$. Additionally, we need to show that the proposed strategy satisfies a sufficient condition: it is the optimal strategy globally, not only on the neighborhood of the point (as the FOC guarantees). We want to show that no deviation is profitable for a bidder with private sign z. It is sufficient to show that the function is quasi-concave. From the Intermediate Value Theorem, it is clear that is sufficient to consider bids in the range of β. Hence, we want to show that the following condition holds:

$$\Pi_i^{b_i}(\beta(x),z)(z-x)\geq 0$$

Define the functions A and B as:

$$A(z,x) \equiv v_F(z,x) + s - \beta(x) - \beta'(x)\frac{\int_{\underline{x}}^{x} p_F(z,\xi) f_{k_F}(\xi|z) d\xi}{p_F(z,x) f_{k_F}(x|z)}$$
$$B(z,x) \equiv v_R(z,x) - s - \beta(x) + \beta'(x)\frac{\int_{x}^{\overline{x}} p_R(z,\xi) f_{k_R}(\xi|z) d\xi}{p_R(z,x) f_{k_R}(x|z)}$$

Rewriting the FOC:

$$\begin{aligned}\Pi_i^{b_i}(\beta(x),z) &= \frac{\lambda(z,x)}{\beta'(x)}\left[v(z,x) - \beta(x) - \beta'(x)\frac{\Lambda(z,x)}{\lambda(z,x)}\right] \\ &= \frac{p_F(z,x) f_{k_F}(x|z)}{\beta'(x)} A(z,x) + \frac{p_R(z,x) f_{k_R}(x|z)}{\beta'(x)} B(z,x)\end{aligned}$$

We know that $\Pi_i^{b_i}(\beta(z),z) = 0$. In order to prove the quasi-concavity condition holds, our focus is to prove that $\Pi_i^{b_i}(\beta(x),z)$ is nonnegative for x $<$ z and non-positive for x $>$ z. Note that $s \leq \overline{s}$ guarantees that:

$$A(x^*,x) \leq B(x^*,x)$$

As this condition is valid for $z = x^*$ is valid for any x, since:

$$x^* = \underset{x}{argmin}\left[v_R(x,x) - v_F(x,x) + \int_{\underline{x}}^{x} L_F(\zeta|x) dt_F(\zeta) + \int_{x}^{\overline{x}} L_R(\zeta|x) dt_R(\zeta)\right]$$

Additionally, $A(z,x)$ and $B(z,x)$ are increasing in z (from Lemma A). Finally, we must address the question of the relative weights between the terms. In first place, note that at:

$$\Pi_i^{b_i}(\beta(x),z)(z-x) \geq 0 \Leftrightarrow \Pi_i^{b_i}(\beta(x),z)\left[\lambda(z,x)\right]^{-1}(z-x) \geq 0$$

This equivalence guarantees that we can analyze how terms change relatively to $\lambda(z,x)$. Note that **A4** guarantees that $\frac{p_R(x,y) f_{k_R}(y|x)}{\lambda(x,y)}$ is non-decreasing. Therefore, it is possible to write the expected profit as the product of a function $\Psi(x,z) \geq 0$ and $z - x$:

$$\Pi_i^{b_i}(\beta(x),z) = \Psi(x,z)(z-x) \Rightarrow \Pi_i^{b_i}(\beta(x),z)(z-x) = \Psi(x,z)(z-x)^2 \geq 0$$

This condition guarantees that we indeed derived an global optimum and that the strategy presented in Proposition 1 is indeed an equilibrium. □

Lemma 4. *Under assumptions **A1** and **A2**In the buy-or-sell auction, the symmetric equilibria when one of the probability functions is zero is given by:*

(i) $p_R(x,y) = 0, \forall x, y \in [\underline{x} < \overline{x}]$: $\beta_F(x,s) = s + \int_{\underline{x}}^{x} v_F(\xi,\xi) dL_F(\xi|x)$, *where*:

$$L_F(\xi|x) = exp\left[-\int_{\xi}^{x} \frac{p_F(\zeta,\zeta) f_{k_F}(\zeta|\zeta)}{\int_{\underline{x}}^{\zeta} p_F(\zeta,\psi) f_{k_F}(\psi|\zeta) d\psi} d\zeta\right]$$

(ii) $p_F(x,y) = 0, \forall x, y \in [\underline{x} < \overline{x}]$: $\beta_R(x,s) = \int_{x}^{\overline{x}} v_R(\xi,\xi) dL_R(\xi|x) - s$, *where*:

$$L_R(\xi|x) = exp\left[\int_{\xi}^{x} \frac{p_R(\zeta,\zeta) f_{k_R}(\zeta|\zeta)}{\int_{\zeta}^{\overline{x}} p_R(\zeta,\psi) f_{k_R}(\psi|\zeta) d\psi} d\zeta\right]$$

*Prova.*We will provide the proof for the forward auction, since the reverse auction is similar. In first place, we will derive the FOC and arrive at a necessary condition for the optimal bidding strategy. After that, we will argue that this condition is sufficient and that the FOC provides a global maximum. Consider the profit function on the auction:

$$\Pi_i(b_i, x_i) = \mathbb{E}\left[P_F(S_F)(V_i - (b_i - s))\mathbb{1}_{b_i > \bar{b}} | X_i = x_i\right]$$

Applying the Law of Iterated Expectation yields:

$$\Pi_i(b_i, x_i) = \mathbb{E}\left[\mathbb{E}\left(P_F(S_F)(v_i - (b_i - s))\mathbb{1}_{b_i > \bar{b}} | X_i, Y_{k_F}\right) | X_i = x_i\right]$$

Using the definitions made on section 2, the last equation may be written as:

$$\Pi_i(b_i, x_i) = \int_{\underline{x}}^{\beta_F^-1(b_i)} p_F(x,\xi) \left[v_F(x_i,\xi) - b_i + s\right] f_{y_{k_F}}(\xi|x_i) d\xi$$

The necessary condition for optimality for bidder i is:

$$\Pi_i^{b_i}(\beta_F(x_i), x_i) = 0$$

Using the Leibniz rule and analyzing the FOC in $\beta_F(x_i)$, we arrive at the following first order ODE:

$$\beta_F'(x_i)\left[\int_{\underline{x_i}}^{x_i} p_F(x_i,\xi) f_{k_F}(\xi|x_i)\xi\right] = p_F(x_i,x_i) f_{k_F}(x_i|x_i)\left[v_F(x_i,x_i) + s - \beta_F(x_i)\right]$$

which is satisfied at our equilibrium strategy. Additionally, the boundary condition on this model is that: $\beta_F(\underline{x}) = v_F(\underline{x}, \underline{x}) + s$. This condition comes from the zero-expected profit of the worst type on a forward auction. Additionally, we need to show that the proposed strategy satisfies a sufficient condition: it is the optimal strategy globally, not only on the neighborhood of the point (as the FOC guarantees). We want to show that no deviation is profitable for a bidder with private sign z. It is sufficient to show that the function is quasi-concave. From the Intermediate Value Theorem, it is clear that is sufficient to consider bids in the range of β. Hence, we want to show that the following condition holds:

$$\Pi_i^{b_i}(\beta_F(x), z)(z - x) \geq 0$$

We know that $\Pi_i^{b_i}(\beta_F(z), z) = 0$. In order to prove the quasi-concavity condition holds, our focus is to prove that $\Pi_i^{b_i}(\beta_F(x), z)$ is nonnegative for $x < z$ and non-positive for x > z. Rewriting the FOC:

$$\Pi_i^{b_i}(\beta(x), z) = \frac{p_F(z,x) f_{k_F}(x|z)}{\beta_F'(x)} \Bigg[v_F(z,x) + s - \beta_F(x) - \beta_F'(x) \frac{\int_{\underline{x}}^{x} p_F(z,\xi) f_{k_F}(\xi|z) d\xi}{p_F(z,x) f_{k_F}(x|z)} \Bigg]$$

Which, given Lemma A and the monotonicity of β_F and v_F, is clearly nonnegative when $z > x$ and non-positive for x > z. □

Lemma 5A. Envelopment of Optimal Bidding Strategies *Under the Proposition 1 and Lemma 4, $s \leq \overline{s}$ implies that:*

$$\beta_F(x) - s < \beta(x) - s < \beta(x) + s < \beta_R(x) + s$$

*Prova.*There are two different ways to prove this lemma. The first way is a direct way: given the equations for the equilibrium strategies, we prove that the inequalities in the lemma statement are valid. However, this proof is only valid for small values of s. To approach the problem when s is higher, we introduce also a second proof. The reason we do not focus entirely on the second proof is that the first proof has some clarifying concepts and illustrates well the relationship of the variables on the model.

First, consider , without loss of generality, that X = [0,1]. This demonstration will have two steps: the first one, we show that the proposed inequality for $x = 0, 1$. The second step consists in exploring the crossing conditions of this continuous functions. For $x = 1$:

$$\beta_F(1) = v_F(1,1) + s - \int_0^1 L_F(\zeta|1) dt_F(\zeta) < v(1,1) - \int_{x^*}^1 L(\zeta|1) dt(\zeta)$$
$$< v_R(1,1) - s$$

The second inequality is trivial. The first, however, needs some detail. In first place, notice that: $\forall \zeta > x^*, L_F(\zeta|1) > L(\zeta|1)$ given that:

$$\forall \psi \in [\zeta, 1] : \frac{p_F(\psi,\psi) f_{k_F}(\psi|\psi)}{\int_{\underline{x}}^{\psi} p_F(\alpha,\psi) f_{k_F}(\alpha|\psi) d\alpha} < \frac{\lambda(\psi,\psi)}{\Lambda(\psi,\psi)}$$
$$\Rightarrow exp\left(-\int_{\zeta}^{1} \frac{\lambda(\psi,\psi)}{\Lambda(\psi,\psi)} d\psi\right) < exp\left(-\int_{\zeta}^{1} \frac{p_F(\psi,\psi) f_{k_F}(\psi|\psi)}{\int_{\underline{x}}^{\psi} p_F(\alpha,\psi) f_{k_F}(\alpha|\psi) d\alpha} d\psi\right)$$
$$\Rightarrow L_F(\zeta|1) > L(\zeta|1)$$

However, this is not enough to guarantee the inequality, since we have both $t(\zeta)$ and $t_F(\zeta)$ at the integral, which are different. Define the function:

$$\tau(x) \equiv p_R(x,x) \frac{f_{y_R}(x|x)}{\lambda(x,x)} \left[v_R(x,x) - v_F(x,x) - 2s\right]$$

Therefore:

$$\begin{aligned}
\beta_F(1) &= v_F(1,1) + s - \int_0^1 L_F(\zeta|1) dt_F(\zeta) \\
&= t(1) - \tau(1) - \int_0^1 L_F(\zeta|1) dt_F(\zeta) \\
&= t(1) - \tau(1) + \int_0^1 L_F(\zeta|1) d\left(t(\zeta) - t_F(\zeta)\right) - \int_0^1 L_F(\zeta|1) dt(\zeta) \\
&< t(1) + \int_0^1 L_F(\zeta|1) d\left(t(\zeta) - t_F(\zeta) - \tau(\zeta)\right) - \int_0^1 L_F(\zeta|1) dt(\zeta) \\
&\leq t(1) + \int_0^1 L_F(1|1) d\left(t(\zeta) - t_F(\zeta) - \tau(\zeta)\right) - \int_0^1 L_F(\zeta|1) dt(\zeta) \\
&= t(1) - \int_0^1 L_F(\zeta|1) dt(\zeta) \\
&< v(1,1) - \int_{x^*}^1 L(\zeta|1) dt(\zeta) = \beta(1) < v_R(1,1) - s = \beta_R(1)
\end{aligned}$$

Analogously to the last reasoning, it is possible to demonstrate that: $\beta_F(0) < \beta(0) < \beta_R(0)$. The second step consists of demonstrating a condition that must be valid when the functions cross each other. Assume that $\exists x : \beta(x) = \beta_F(x)$. In this case, combining the FOC's of the two problems:

$$\begin{aligned}
&\Lambda(x,x)\beta'(x) + \lambda(x,x)\left(\beta(x) - v(x,x)\right) = \\
&\qquad = \beta_F'(x)\left[\int_{\underline{x}}^{x} p_F(x,\xi) f(\xi|x) d\xi\right] + \left(\beta(x) - v_F(x,x) - s\right) p_F(x,x) f_{k_F}(x|x)
\end{aligned}$$

Hence, the above equation yields:

$$p_R(x,x)\left[v_R(x,x)-s-\beta(x)\right]f_{k_R}(x|x)+\left[\int_x^{\overline{x}} p_R(x,\xi)f_{k_R})(\xi|x)d\xi\right]\beta^{'}(x)=$$
$$=\left(\beta^{'}(x)-\beta_F'(x)\right)\left[\int_{\underline{x}}^{x} p_F(x,\xi)f(\xi|x)d\xi\right]$$

Note that:

$$\begin{aligned} v_R(x,x)-s-\beta(x) &= v_R(x,x)-s-\beta_F(x)\\ &> v_F(x,x)+s-\beta_F(x)\\ &> 0 \end{aligned}$$

This allows us to conclude that $\beta(x)=\beta_F(x)\Rightarrow\beta'(x)>\beta_F'(x)$. It is also possible to show that $\beta(x)=\beta_R(x)\Rightarrow\beta'(x)>\beta_R'(x)$. Considering that all bid functions are continuous, we have proved our proposition.

Second, on the proof of Proposition 1, we argued that $\forall s\leq\overline{s}$:

$$A(z,x)\leq B(z,x)$$

where A and B were defined in the proof of Proposition 1. In an equilibrium strategy, it must be true that:

$$A(z,z)\leq 0\leq B(z,z)$$

since the FOC must be satisfied. This guarantees that the inequalities of the lemma statement are valid.

Lemma 5B. Envelopment of Optimal Bidding Strategies *Under the Proposition 1 and Lemma 4, $s\leq\overline{s}$ implies that:*

$$\beta_{F,1}(x)-s<\beta(x)-s<\beta(x)+s<\beta_{R,1}(x)+s$$

*Prova.*Define the optimal bidding strategy of an auction where bidders are certain of what operation will be conducted as $\beta_{F,1}$ for the forward and $\beta_{R,1}$ for the reverse. This yields:

$$\beta_{F,1}\leq\beta_F\leq\beta_R\leq\beta_{R,1}$$

The equality in the middle is due to Lemma 5A. We now detail the others.

This proof follows a similar logic to the direct proof of Lemma 5A:

$$
\begin{aligned}
\beta_{F,1}(x) &= v_F(x,x) - \int_{\underline{x}}^{x} exp\left[-\int_{\xi}^{x} \frac{f_{k_F}(\zeta|\zeta)}{F_{k_F}(\zeta|\zeta)} d\zeta\right] dt_F(\xi) \\
&= v_F(x,x) - \int_{\underline{x}}^{x} exp\left[-\int_{\xi}^{x} \frac{f_{k_F}(\zeta|\zeta)}{\int_{\underline{x}}^{\zeta} f_{k_F}(\psi|\zeta) d\psi} d\zeta\right] dt_F(\xi) \\
&< v_F(x,x) - \int_{\underline{x}}^{x} exp\left[-\int_{\xi}^{x} \frac{f_{k_F}(\zeta|\zeta)}{\int_{\underline{x}}^{\zeta} \frac{p_F(\zeta,\psi)}{p_F(\zeta,\zeta)} f_{k_F}(\psi|\zeta) d\psi} d\zeta\right] dt_F(\xi) \\
&= v_F(x,x) - \int_{\underline{x}}^{x} exp\left[-\int_{\xi}^{x} \frac{p_F(\zeta,\zeta) f_{k_F}(\zeta|\zeta)}{\int_{\underline{x}}^{\zeta} p_F(\zeta,\psi) f_{k_F}(\psi|\zeta) d\psi} d\zeta\right] dt_F(\xi) \\
&= \beta_F(x)
\end{aligned}
$$

The reverse case is similar. □

Lemma 6. Expected Profits: *If* $s = 0$*, then* $\Pi^{IPV}(x^*, \beta(x^*)) = 0 > \Pi^{AFF}(x^*, \beta(x^*))$

*Prova.*n first place, we will approach the IPV model and show that $\Pi^{IPV}(x^*, \beta(x^*)) = 0$. When valuations are independent and $s = 0$, the expected profit might be written as:

$$
\begin{aligned}
\Pi_i(\beta(x_i), x_i) = &\int_{\underline{x}}^{x^*} p_F(x^*)\left[x^* - \beta(x^*)\right] f_{k_F}(\xi) d\xi + \\
&\int_{x^*}^{\overline{x}} p_R(x^*)\left[\beta(x^*) - x^*\right] f_{k_R}(\xi) d\xi \\
&= (\beta(x^*) - x^*)\left(\int_{\underline{x}}^{x^*} p_F(x^*) f_{k_F}(\xi) d\xi - \int_{x^*}^{\overline{x}} p_R(x^*) f_{k_R}(\xi) d\xi\right) \\
&= 0
\end{aligned}
$$

where the last equality comes from the definition of x^*. In second place, we will consider the expected profit of a strictly affiliated model (that is, we assume that the functions v_k are strictly increasing in all arguments $\forall k = 1, ..., n$). This excludes the independent case, for example. evaluated at x^*. The expected profit is given by:

$$\begin{aligned}\Pi_i(\beta(x^*), x^*) = &\int_{\underline{x}}^{x^*} p_F(x^*,\xi)\left[v_F(x^*,\xi) - v(x^*,x^*)\right] f_{y_{k_F}}(\xi|x^*)d\xi + \\ &\int_{x^*}^{\overline{x}} p_R(x^*,\xi)\left[v(x^*,x^*) - v_R(x^*,\xi)\right] f_{y_{k_R}}(\xi|x^*)d\xi \\ = &\int_{\underline{x}}^{x^*} p_F(x^*,\xi)v_F(x^*,\xi)f_{y_{k_F}}(\xi|x^*)d\xi \\ &- \int_{x^*}^{\overline{x}} p_R(x^*,\xi)v_R(x^*,\xi)f_{y_{k_R}}(\xi|x^*)d\xi \\ < 0 &\end{aligned}$$

The equalities all come from the mathematical condition that the worst type must satisfy. The inequality comes from the fact that $\zeta > \xi \Rightarrow v_F(x^*,\xi) < v_R(x^*,\zeta)$. □

Proposition 2. The Spread Effect on Profits: *In the affiliated model with $s = 0$, even if bidders of specific private signals do not participate, there is no symmetric Bayes-Nash pure strategy equilibrium where all types have positive expected profit.*

*Prova.*The idea of this proof is to show that there will always exist at least one private signal that satisfies a condition close to the one presented in Lemma 3. However, this proof focuses only on the existence (and not the uniqueness) of this point. Define $\Phi_L(x)$ ($\Phi_H(x)$) as the set of private signals that do participate in the auction and are lower (higher) than x, $\hat{f}$ as the density functions conditional on participation and $\hat{\Lambda}(x,a)$ as:

$$\hat{\Lambda}(x,a) = \int_{\Phi_L(a)} p_F(x,\xi)\hat{f}_{k_F}(\xi|x)d\xi - \int_{\Phi_H(a)} p_R(x,\xi)\hat{f}_{k_R}(\xi|x)d\xi$$

Using a similar approach to the presented on Lemma 3, it is possible to show that the function $\hat{\Lambda}^*(x) = \hat{\Lambda}(x,x)$ is continuous and, again, just as in Lemma 3:

$$\hat{\Lambda}^*(\overline{x}) = \int_{\Phi_L(\overline{x})} p_F(\overline{x},\xi)\hat{f}_{k_F}(\xi|\overline{x})d\xi > 0$$

And

$$\hat{\Lambda}^*(\overline{x}) = -\int_{\Phi_R(\underline{x})} p_F(\underline{x},\xi)\hat{f}_{k_F}(\xi|\underline{x})d\xi < 0$$

Therefore, $\exists\ \hat{x}$ such that $\hat{\Lambda}^*(\hat{x}) = 0$. Just as in the case with full participation, this condition yield a negative expected profit to a bidder with the private signal $\hat{x}$. □

Lemma 7. *The equilibrium of Proposition 1 is feasible if we consider an ex-ante participation condition, that is:*

$$s_1 \leq \overline{s}$$

Additionally, if $p_R(x^*, x^*) f_{k_R}(x^*|x^*) = p_F(x^*, x^*) f_{k_F}(x^*|x^*)$, then a interim participation condition is feasible, that is::

$$s_1 \leq s_2 \leq \overline{s}$$

*Prova.*In first place, we prove that $s_1 \leq \overline{s}$. Define the function $\Pi^*(x, s) = \Pi(\beta(x), x)$ when the spread on the buy-or-sell auction is s. Then, trivially:

$$\Pi^*(x^*, \overline{s}) > 0 > \Pi^*(x^*, s_1)$$

This comes from the fact that is always possible for all agents to have a non-negative expected profit if spread is $\overline{s}$, given that they could bid their private valuations and have an expected profit of zero. Additionally, as the worst type, x^* obviously has a negative profit under s_1, since this is the value of the spread that makes the expected value (on private signals) of the expected profit zero.

As $\Pi^*(x, s)$ is increasing in s, we know that:

$$\Pi^*(x^*, \overline{s}) \geq \Pi^*(x^*, \underline{s}) \Rightarrow \overline{s} \geq s_1$$

In second place, we prove that $_2 \leq \overline{s}$. . Let s_1^* and s_2^* be the highest values to which the following inequalities are still valid:

$$\beta_F(x^*, x^*) - s_1^* \leq \beta(x^*, x^*) - s_1^* < \beta(x^*, x^*) + s_2^* \leq \beta_R(x^*, x^*) + s_2^*$$

By our definition of $\overline{s}$, we have that:

$$2\overline{s} = s_1^* + s_2^*$$

In the profit function:

$$\begin{aligned}
\Pi^*(x^*, \overline{s}) = & \int_{\underline{x}}^{x^*} p_F(x^*, \xi)\left[v_F(x^*, \xi) - \beta(x^*) + \overline{s}\right] f_{k_F}(\xi|x^*)d\xi + \\
& \int_{x^*}^{\overline{x}} p_R(x^*, \xi)\left[\beta(x^*) + \overline{s} - v_R(x^*, \xi)\right] f_{k_R}(\xi|x^*)d\xi \\
= & \int_{\underline{x}}^{x^*} p_F(x^*, \xi)\left[v_F(x^*, \xi) - \beta(x^*) + s_1^*\right] f_{k_F}(\xi|x^*)d\xi + \\
& \int_{x^*}^{\overline{x}} p_R(x^*, \xi)\left[\beta(x^*) + s_2^* - v_R(x^*, \xi)\right] f_{k_R}(\xi|x^*)d\xi \\
= & \int_{\underline{x}}^{x^*} p_F(x^*, \xi)\left[v_F(x^*, \xi) - \beta_F(x^*) + s_1^*\right] f_{k_F}(\xi|x^*)d\xi + \\
& \int_{x^*}^{\overline{x}} p_R(x^*, \xi)\left[\beta_R(x^*) + s_2^* - v_R(x^*, \xi)\right] f_{k_R}(\xi|x^*)d\xi \\
> & \ 0 = \Pi^*(x^*, s_2) > \Pi^*(x^*, s_1)
\end{aligned}$$

The second equality is due to the property that $2\overline{s} = s_1 + s_2$. The inequality is due to the fact that x^* is not the worst type on the forward and reverse auctions only. The last equality is due to the definition $\underline{s}$. Again, as $\Pi^*(x, s)$ is increasing in s, we know that:

$$\Pi^*(x^*, \overline{s}) \geq \Pi^*(x^*, \underline{s}) \Rightarrow \overline{s} \geq \underline{s} \qquad \square$$

Lemma 8. *Under Proposition 1, for increasing values of s, the optimal bidding strategy considering the payments decreases for the forward (i.e., $\beta - s$ are decreasing in s) and increases in the reverse end of the auction.*

*Prova.*We provide the proof to the fact that for increasing values of s, the optimal bidding strategy considering the payments decreases for the forward (i.e., $\beta - s$ are decreasing in s) and increase in the reverse end of the auction. For values of the spread $s \in [\underline{s}, \overline{s}]$, we know that:

$$\begin{aligned}
\left|\frac{\partial \beta(x, s)}{\partial s}\right| = & \left|\int_{x^*}^{x} \frac{p_F(\xi, \xi) f_{k_F}(\xi|\xi) - p_R(\xi, \xi) f_{k_R}(\xi|\xi)}{\lambda(\xi, \xi)} dL(\xi|x)\right| \\
< & \ 1
\end{aligned}$$

The last inequality is due to the fact that $L(\xi|x)$ is a distribution and the integral kernel is lower than one in absolute terms. $\square$