Sebastián Alejandro Pérez Opazo

C^r-stabilisation of non-transverse heterodimensional cycles

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Pontifícia Universidade Católica
do Rio de Janeiro

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Tese de Doutorado

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Advisor: Prof. Lorenzo Justiniano Díaz Casado

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Dedicated to Emilia and Isabel, with all my love.

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Abstract

Pérez Opazo, Sebastián Alejandro; Díaz Casado, Lorenzo Justiniano (Advisor). **C^r-stabilisation of non-transverse heterodimensional cycles**. Rio de Janeiro, 2016. 146p. PhD Thesis – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

A diffeomorphism f has a *heterodimensional cycle* if there are (transitive) hyperbolic sets with different indices (dimension of the unstable bundle) whose invariant sets meet cyclically. The cycle of f is C^r*-robust* if every small C^r-perturbation of f has a cycle associated to the continuations of these hyperbolic sets. When the cycle of f is defined by a pair of hyperbolic saddles we say that this can be C^r*-stabilised* if every C^r -neighbourhood of f contains diffeomorphisms with a robust cycle associated to hyperbolic sets containing the continuations of these saddles. In dimension three we consider non-transverse heterodimensional cycles associated to saddles: the saddles are involved in a heterodimensional cycle and their two dimensional manifolds have some non-transverse intersection. For $r \geq 2$, we study the occurrence of C^r-robust cycles in this setting as well as the C^r-stabilisation of the initial cycles. We prove that for every $r \geq 2$ there exist a class $\mathcal{N}^r$ of three-dimensional diffeomorphisms having non-transverse cycles such that any diffeomorphism in $\mathcal{N}^r$ can be C^r-stabilised. A key ingredient of our method is a renormalisation scheme at the heteroclinic quadratic intersection converging to a Hénon-like family of endomorphism with blender-horseshoes. We also see that this type of bifurcation leads to C^r*-intermingled* homoclinic classes (the homoclinic classes of two saddles with different indices have non-empty intersection) which are non-dominated.

Keywords

Blender-horseshoe; Dominated Splitting; Hénon-like families; Heterodimensional cycle; Homoclinic class; Renormalisation scheme; Robust cycle; Stabilisation of a cycle;

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Resumo

Pérez Opazo, Sebastián Alejandro; Díaz Casado, Lorenzo Justiniano. **C^r-estabilização de ciclos heterodimensionais não transversais**. Rio de Janeiro, 2016. 146p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Um difeomorfismo f tem um *ciclo heterodimensional* se existem conjuntos hiperbólicos (transitivos) com índices diferentes (dimensão do fibrado instável) cujos conjuntos invariantes interseptam-se ciclicamente. O ciclo de f é *C^r-robusto* se toda pequenha C^r-perturbação de f tem um ciclo associado as continuações destes conjuntos hiperbólicos. Quando o ciclo de f é defindo por um par de selas hiperbólicas se diz que este ciclo pode ser *C^r-estabilizado* se toda C^r-vizinhança de f contém difeomorfismos com um ciclo robusto entre conjuntos hiperbólicos que contém as continuações destas selas. No caso tridimensional consideramos ciclos heterodimensionais não transversais associado a selas: as selas definem um ciclo heterodimensional onde as suas variedades dois dimensionais tem alguma intersecão não transversal. Para $r \geq 2$ estudamos a ocorrência de ciclos C^r-robustos neste contexto assim como a C^r-estabilização destes ciclos. Provamos que para cada $r \geq 2$ existe uma classe $\mathcal{N}^r$ de difeomorfismos tridimensionais tendo ciclos não transversais tais que quaisquer difeomorfismo nesta classe pode ser C^r-estabilizado. Um ingrediente chave do nosso método é um esquema de renormalização definido sobre a tangência quadrática do ciclo convergindo para uma família tipo Hénon que tem ferraduras-misturadoras. Também vemos que este tipo de bifurcação leva ao misturamento de classes homoclínicas (as classes homoclínicas de duas selas de índices diferentes tem interseção não-vazia) as quais não suportam decomposições dominadas.

Palavras-chave

Ferradura-misturadora; Decomposição dominada; Famílias tipo Hénon; Ciclos heterodimensionais; Classes homoclínicas; Esquemas de renormalização; Ciclo robusto; Estabilização de um ciclo;

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"Creo en los cafés, en el diálogo, creo en la dignidad de la persona, en la libertad. Siento nostalgia, casi ansiedad de un Infinito, pero humano, a nuestra medida."

Ernesto Sabato, *La Resistencia.*

1
Introduction

Heterodimensional cycles and homoclinic tangencies are two important mechanisms that prevent hyperbolicity. Indeed, Palis density conjecture claims that they are the main mechanisms for the absence of hyperbolicity: every non-hyperbolic system can be approached by a system with either a homoclinic tangency or a heterodimensional cycle[1], see (37).

The simples heterodimensional cycle is obtained by the heteroclinic intersections of the invariant manifolds of two saddles of different indices[2] (this necessarily implies that some intersection is non-transverse). In the case of a homoclinic tangency, the invariant manifolds of a saddle have some non-transverse intersection. Kupka-Smale genericity theorem (47) states that generically (i.e., in a residual set) the invariant manifolds of hyperbolic periodic points meet transversely. Thus, heterodimensional cycles and homoclinic tangencies are associated to saddles are fragile. One can define heterodimensional cycles and homoclinic tangencies associated to hyperbolic sets as above. In this case Kupka-Smale theorem cannot be applied and one may that "robust" heterodimensional cycles and tangencies. In this work we study the robust inherent phenomena to the unfolding of a heterodimensional cycle associated to saddes whose configuration involves tangencies. In particular we are interested in the generation of robust cycles and tangencies. We now discuss these notions in a bit more precise way.

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1.1
Robust cycles and Stabilisation of cycles

Recall that a hyperbolic set Λ of a diffeomorphism f has defined a continuation Λ_g for every g close to f (in particular, Λ and Λ_g are conjugate). Recall that a invariant set is *transitive* if it has a dense orbit. The *index* of a hyperbolic transitive set Λ, $\mathrm{ind}(\Lambda)$, is the dimension of its unstable bundle (transitivity implies that this number is well defined).

[1]This conjecture was proved for surface C^1-diffeomorphisms in (43). Currently, the C^r-case, $r \geq 2$, seems to be beyond reach.

[2]dimension of the unstable bundle

Definition 1.1.1 (Robust cycles and tangencies) Let f be a C^r-diffeomorphism defined on a compact manifold.

– *Robust heterodimensional cycles.* f has a *heterodimensional cycle* associated to the transitive hyperbolic sets Γ and Σ with different indices if
$$W^{\mathrm{u}}(\Gamma) \cap W^{\mathrm{s}}(\Sigma) \neq \emptyset \quad \text{and} \quad W^{\mathrm{s}}(\Gamma) \cap W^{\mathrm{u}}(\Sigma) \neq \emptyset.$$
The cycle above is C^r*-robust* if there is a C^r-neighbourhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a heterodimensional cycle associated to the hyperbolic continuations Γ_g and Σ_g of Γ and Σ.

– *Robust homoclinic tangencies.* f has a *homoclinic tangency* associated to a transitive hyperbolic set Γ if there exists a non-tranverse intersection point between the invariant manifolds $W^{\mathrm{u}}(\Gamma)$ and $W^{\mathrm{s}}(\Gamma)$. We say that the tangency is C^r*-robust* if there exists a C^r-neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the continuation Γ_g of Γ has a homoclinic tangency.

Note that heterodimensional cycles can only occur in manifolds of dimension three or higher.

An important type of cycles are those associated to hyperbolic periodic points. Observe that Kupka-Smale genericity theorem implies that robust cycles and tangencies involve, necessarily, some non-trivial hyperbolic set (containing infinitely many orbits). Thus a crucial point is to determine when these cycles generate robust ones. Moreover, in the positive case, determine the relation of the hyperbolic sets in the robust cycle with the (continuations) of the saddles in the initial cycle. This leads to the following notion introduced in (12):

Definition 1.1.2 (Stabilisation of cycles) Consider a C^r-diffeomorphism $f : M \to M$ having a heterodimensional cycle associated to a pair of saddles P and Q. This cycle can be C^r*-stabilised* if every C^r-neighbourhood $\mathcal{U}$ of f contains a diffeomorphism g having a robust heterodimensional cycle associated to hyperbolic basic sets $\Lambda_g \ni P_g$ and $\Sigma_g \ni Q_g$. Here P_g and Q_g denotes the respective continuations of the initial saddles P and Q. A cycle that cannot be stabilized is called C^r*-fragile.*

Note that, *mutatis mutandi,* the previous definition can be formulated for homoclinic tangencies. Hence, from (33, 39, 45) every homoclinic tangency of a C^2-diffeomorphism is always stabilised.

Let us observe the definition of stabilisation of a cycle was also motivated by Bonnatti's robustness conjecture below and the study of spectral decompo-

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sition of the dynamics into elementary pieces of dynamics in non-hyperbolic settings.

1.2 Generation of robust cycles

In what follows, for simplicity, we restrict our attention to three dimensional diffeomorphisms. In this case the indices of the hyperbolic sets in a heterodimensional cycle are one and two. Let us summarise the main results in this setting:

• Every diffeomorphism with a heterodimensional cycle (associated to a pair of saddles) can be C^1-approximated by diffeomorphisms with robust cycles, (9). If one of the saddles of the cycle has a non-real multiplier (i.e. it is a saddle focus) the the cycle can be C^1-stabilised, (12). A key ingredient in these constructions are the *blenders* and *blender-horseshoes* (very roughly a horseshoe which is immersed in the space with special geometric superposition properties, see the precise discussion in Section 3.4).

• For cycles associated to saddles with real eigenvalues the study is done considering three parameters using the symbols $+$ and $-$. The two first parameters involves the signal of the so-called central eigenvalues of the saddles in the cycle and third parameter involves the central orientation of the transition along the quasi-transverse heteroclinic orbit related to the intersection of the one dimensional manifolds. This leads to eight configurations. In (12) it is shown that seven of these configurations can be stabilised, however (11) provides examples of cycles associated to the remaining configuration that cannot be stabilised.

• About robust tangencies we have the following results: (33) provides examples of C^2-robust homoclinic tangencies of surface diffeomorphisms, (10) provides examples of C^1-robust homoclinic tangency (in dimension three or higher), and (31) proved that surface diffeomorphisms cannot have C^1-robust tangencies. Note that in (33) the underlying mechanism for the generation of robust tangencies are the so-called thick horseshoes while the mechanism in (10, 9) for robust cycles are blenders. These two mechanisms have different nature.

• In (22) was introduced the concept of a *heterodimensional tangency*. In some cases, this type of tangencies can be thought as a "homoclinic tangency" between two dimensional invariant manifolds of a transitive partially hyperbolic set: there are saddles of different indices in the set whose two dimensional invariant manifolds have a non-transverse intersection[3]. As was shown (22), the

[3] The creation of this tangencies appear as a natural transition from partially hyperbolic dynamics to non-dominated dynamics, see the examples in (22).

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C^1-unfolding of a such tangency in the leads to C^1-robustly non-dominated dynamics and in some cases to very intermingled dynamics related to *universal dynamics*[4]. Examples of C^2-robust heterodimensional tangencies are provided in (26) using the thick horseshoes in (33). The authors also see how these robust tangencies can be associated to heterodimensional cycles using blenders. Assuming large codimension of heterodimensional tangencies on manifolds of dimension ≥ 7, in (2) the authors also exhibit C^2-robust heterodimensional tangencies. The above results motivate the problem of generation and stabilisation of cycles in the C^r-topology.

1.3 The setting: Palis and Bonatti's conjectures

Motivated by the Palis density conjecture, Bonatti proposed in (3) to split the space of C^1-diffeomorphisms considering the following three dichotomies (we postpone the precise definitions)

- tame (having robustly finitely many chain recurrence classes whose number is locally constant) versus wild (having robustly infinitely many chain recurrence classes) diffeomorphisms,
- diffeomorphisms far from heterodimensional cycles versus diffeomorphisms with robust heterodimensional cycles,
- diffeomorphisms far from homoclinic tangencies versus diffeomorphisms with robust homoclinic tangencies.

Bonatti's conjecture claims that in each case union of the two opens sets involved in each dichotomy is a dense subset in the space of C^1-diffeomorphisms. Note that when two (transitive) hyperbolic sets of different indices are involved in a robust cycle then they are in the same chain recurrence class. Thus the occurrence of robust cycles has immediate consequences in the description of the *elementary pieces of dynamics* of the diffeomorphisms.

Considering the above dichotomies Bonatti proposed the following types of diffeomorphisms with increasing *complexity*[5] of dynamics:

[4] This phenomenon can be thought as a generalisation of the phenomenon of Newhouse: sets with universal dynamics displays infinitely many pairwise disjoint exhibit non-trivial homoclinic classes; infinitely many non-trivial hyperbolic and non-hyperbolic attractors, and infinitely many non-trivial hyperbolic and non-hyperbolic repellors. In dimension three, the construction of C^1-universal dynamics involves the (robust) presence of saddles with different indices and non-real eigenvalues in the same transitive set, see (7).

[5] In extremely rough terms and considering three dimensional systems, case (ii) corresponds to dominated dynamics with three bundles, case (iii) corresponds to dominated dynamics with two bundles, and (iv) corresponds to non-dominated dynamics, see for instance (49).

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(i) hyperbolic systems,

(ii) tame systems far from homoclinic tangencies (note that these diffeomorphisms may exhibit robust heterodimensional cycles),

(iii) tame systems with robust homoclinic tangencies (in the known examples these systems also have robust heterodimensional cycles), and

(iv) wild diffeomorphisms.

We observe that there are some partial progress in the C^1-topology to describe the dynamics above and that very few is know in the case of wild dynamics (besides some locally generic examples). The understanding of these dynamics with higher regularity is a quite open problem.

1.4 Overview of our main results

The goal of this thesis is to present a semi-local scenario illustrating the occurrence of C^r-robust cycles and other related dynamics. We introduce explicit families of local diffeomorphisms (we are only concerned with some local and semi-local aspects of the dynamics) displaying C^r-robust heterodimensional cycles, $r \geq 2$. More precisely, in the C^r-setting, we consider three dimensional diffeomorphisms having a pair of saddle-focus fixed points, say P and Q, having a heterodimensional cycle. We also assume that the two dimensional invariant manifolds have non-transverse intersection (we call these cycles *non-transverse heterodimensional cycles*). In this way we have a pair of non-transverse heteroclinic orbits, say X and Y, where the α-limit set of X is Q and its ω-limit is P and the ω-limit set of Y is Q and its α-limit is P, see Figure 1.1. In this case the point Y is called a *heterodimensional tangency.*

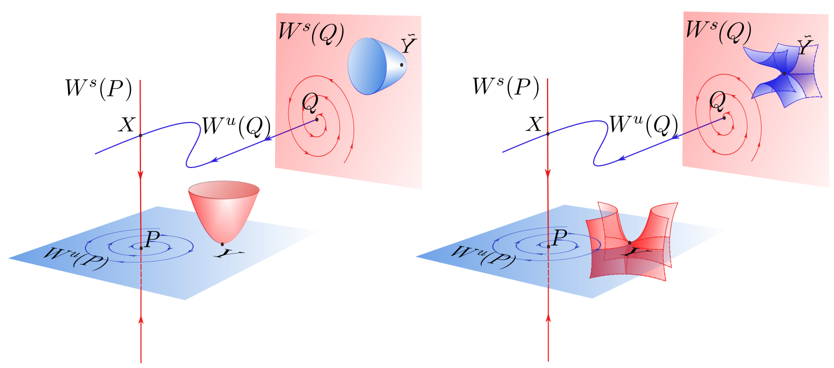

Figure 1.1: Non-transverse heterodimensional cycles

We study the semi-local dynamics of this cycle (i.e, the orbits which remain in the future and in the past in a neighbourhood of the orbits of P, Q, X

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and Y) and see how this configuration may lead to C^r-robust cycles and to the stabilisation of the original cycle. This study has three main ingredients:

- A parametric Hénon-like family exhibiting (agree to the choice of parameters) blender-horseshoes (see Theorem 1) and a pair of fixed saddle-node points related cyclically by the intersection of their invariant manifolds (see Lemma 6.6.3).
- A renormalisation scheme associate to a non-transverse heterodimensional cycle converging to maps of the Hénon-like family (see Theorem 2).
- Applications of the two items above to get robust cycles (Theorem 3) and the stabilisation of the initial cycle (Theorem 4).

The study of the cyclic configuration depicted above is motivated by (21). The configuration in (21) involves saddles P and Q having different indices (say one and two, respectively) and real multipliers of a diffeomorphism f. The authors introduce a renormalisation scheme associated to the heterodimensional tangency converging the *center unstable Hénon-like family*

$$\widetilde{G}_{(\xi,\mu,\kappa,\eta)}(x,y,z) = (\xi\,x + y, \mu + y^2 + \kappa\,x^2 + \eta\,x\,y, y), \quad \xi > 1.$$

For every parameter (ξ,μ,κ,η) close enough to $(1^+,-9,0,0)$ these maps have blenders (see Section 3.4). Indeed we accomplish the construction in (21) proving that these blenders are indeed blender-horseshoes[6]. Following the ideas in (21) we get a sequence of diffeomorphisms f_n converging to f (in the C^r-topology, $r \geq 2$) having blender-horseshoes. These blender-horseshoes have one dimensional stable manifolds. Using these blenders in (21) the authors obtain some additional semi-global information of the diffeomorphisms f_n: small $C^{1+\alpha}$-perturbations of f_n, $\alpha \in (0,1/2)$, yield robust connections between the manifolds one dimensional stable manifold of the blender and the one dimensional unstable manifold of (the continuation of) Q.

This results are a partial step in the direction of the generation of C^r-robust cycles. In (21) the authors are not able to get intersections between the two dimensional stable manifold of the blender and the two dimensional stable manifold of Q. Thus it is unknown if there is possible to get robust cycles involving the blenders and Q. Finally, it is also unknown if the blender is (homoclinically) related to the saddle P. Two important cons in the

[6] A blender is a hyperbolic set satisfying some superposition geometrical properties, its structure as hyperbolic set is not important and it may fail to be locally maximal. Blender-horseshoes are locally maximal and conjugate to the complete shift of two symbols. This property guarantees a complete description of the local stable manifold of the blender-horseshoes.

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constructions in (21) are that the systems are "slowly recurrent" after the generation of the blender and the "size" of the blenders decrease exponentially fast (this is a difficulty to "relate" the blender to the saddles in the cycle). The naive idea in this paper to bypass these two difficulties is to add extra recurrences to the system considering saddles with non-real multipliers (saddle focus) this allows to get C^r-robust connections between the one unstable manifolds. We also obtain an appropriate control of the size of the blender-horseshoes that allows us we get the intersections between the stable manifold of the blender and the stable manifold of Q.

Organisation of the thesis. This thesis is organised as follows. In Section 2 we give a rough description of the bifurcation setting that we consider. In Section 3 we describe precisely the main ingredients involved in this thesis. In Section 4 we state precisely our main results in this thesis and sketch the key steps of the proofs. Theorem 1 about the existence of blender-horseshoes in Hénon-like families is proved in Section 5. In Section 5.3 we prove Theorem 2 about the convergence of the renormalisation scheme. In Section 6 we construct laminations of the parameter space corresponding to diffeomorphisms with blender-horseshoes and prove the first part of Theorem 4 about stabilisation of cycles. The second part of this theorem is given in Section 7. In Section 6.7 we prove Theorem 3. Finally, in Section 8.3 we apply our results to get non-dominated homoclinic classes.

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2
The bifurcation context: Non-transverse heterodimensional cycles

In this section we discuss our bifurcation setting. In a three dimensional manifold, we study a bifurcation setting of a heterodimensional cycle associated a pair of hyperbolic saddles having non-real eigenvalues such that the one-dimensional manifolds meets quasi-transversely and the two-dimensional invariant manifolds also meet non-transversely. More precisely, let $f : M \to M$ be a diffeomorphism having a pair of fixed saddles P and Q of indices (dimension of unstable bundle) two and one, respectively, such that

- P and Q are *irrational saddle-focus*: $Df(P)$ have a non-real expanding eigenvalue and $Df(Q)$ have a non-real contracting eigenvalue, having both eigenvalues an irrational argument);
- The one-dimensional manifolds meets through a orbit of a *quasi-transverse intersection point* X i.e., $X \in W^{\mathrm{s}}(P,f) \cap W^{\mathrm{u}}(Q,f)$ and

$$T_X W^{\mathrm{s}}(P,f) + T_X W^{\mathrm{u}}(Q,f) = T_X W^{\mathrm{s}}(P,f) \oplus T_X W^{\mathrm{u}}(Q,f);$$

- The two-dimensional manifolds meet through an orbit of a *non-transverse intersection point* Y i.e., the orbit of Y is contained in the set

$$\Big(W^{\mathrm{u}}(P,f) \cap W^{\mathrm{s}}(Q,f)\Big) \setminus \Big(W^{\mathrm{u}}(P,f) \pitchfork W^{\mathrm{s}}(Q,f)\Big).$$

The study of such bifurcations depends on the geometrical constrains (shape and relative positions of the invariant manifold close to the heteroclinic orbits) as well of type of intersections (elliptic and hyperbolic contact of the two-manifolds) in the cycle. The resulting dynamics is determined by four maps:

- the local dynamics in (small) neighbourhoods U_P and U_Q of the saddles P and Q,
- two transition maps $T_{Q,P}$ and $T_{P,Q}$, where $T_{Q,P}$ follows the orbit of the heteroclinic point X and goes from U_Q to a U_P and $T_{P,Q}$ follows the orbit of the heteroclinic point Y and goes from U_P to U_Q.

See Figure 2.1.

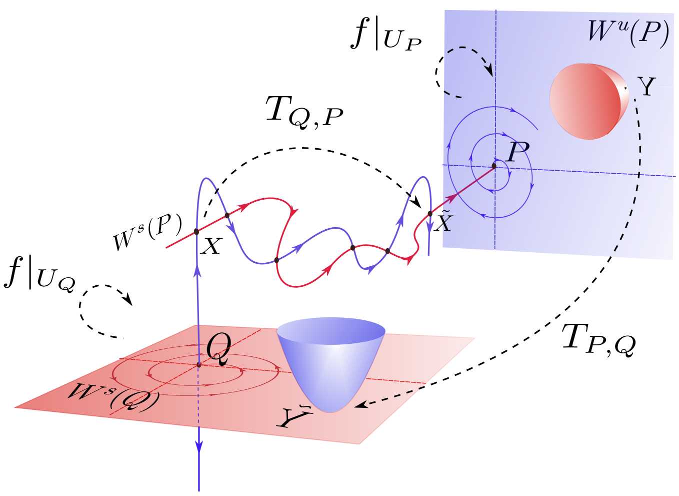

Figure 2.1: Non-transverse heterodimensional cycle satisfying conditions (H), (Q), (T) and (L).

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A key aspect of the dynamics of the bifurcation, using the terminology in (13, Preface) and in very rough terms, heterodimensional cycles correspond to the so-called *non-critical dynamics* and the homoclinic tangencies correspond to the so-called *critical dynamics*. The dynamical configuration in this paper involves the simultaneous occurrence of a heterodimensional cycle and heterodimensional tangency, as a results aspects of critical and non-critical dynamics overlap and there is an interplay between the two types of dynamics. In our setting $T_{Q,P}$ corresponds to a non-critical dynamics and $T_{P,Q}$ to the critical one. Let us also observe that the transitions $T_{P,Q}$ and $T_{Q,P}$ determines the (local) geometry of the invariant manifolds of the saddles in the cycle along the heteroclinic orbits.

We postpone the discussion of these points and emphasise that our results depend on appropriate choices for these configurations, see Section 4.2.

3
Preliminaries

In this section we define and discuss the main objects in our results: non-transverse heterodimensional cycles (Section 3.1), dominated splittings (Section 3.2), homoclinic classes (Section 3.3), blender-horseshoes (Section 3.4), and renormalisation schemes and Hénon-like families (Section 3.5). In what follows, we denote by M a three-dimensional compact boundaryless manifold and for $r \geq 1$, we denote by $\mathrm{Diff}^r(M)$ the metric space of C^r-diffeomorphisms on M endowed with the uniform metric $\|\cdot\|_r$.

3.1
Non-transverse heterodimensional cycles

For the next discussion recall the definition of a heterodimensional cycle, see Definition 1.1.1. Consider a diffeomorphism f with a heterodimensional cycle associated to hyperbolic sets Γ and Σ. Then by definition we can assume that for every $X \in W^{\mathrm{s}}(\Gamma) \cap W^{\mathrm{u}}(\Sigma)$ and every $Y \in W^{\mathrm{u}}(\Gamma) \cap W^{\mathrm{s}}(\Sigma)$ it holds

$$\begin{aligned}
\dim\Big(T_X W^{\mathrm{s}}(\Gamma)\Big) + \dim\Big(T_X W^{\mathrm{u}}(\Sigma)\Big) &< \dim\Big(T_X M\Big),\\
\dim\Big(T_Y W^{\mathrm{u}}(\Gamma)\Big) + \dim\Big(T_Y W^{\mathrm{s}}(\Sigma)\Big) &> \dim\Big(T_Y M\Big).
\end{aligned}$$

Thus the last condition allows two types of intersections between $W^{\mathrm{u}}(\Gamma)$ and $W^{\mathrm{s}}(\Sigma)$: transverse and non-transverse. The set of transverse intersection of such manifolds is denoted by $W^{\mathrm{u}}(\Gamma) \pitchfork W^{\mathrm{s}}(\Sigma)$. This motivates the following definition:

Definition 3.1.1 (Non-transverse heterodimensional cycles) Consider a diffeomorphism $f : M \to M$ having a heterodimensional cycle associated to saddles P and Q such that $\mathrm{ind}(P) > \mathrm{ind}(Q)$. A heteroclinic orbit of the cycle in $W^{\mathrm{u}}(P,f) \cap W^{\mathrm{s}}(Q,f)$ disjoint from $W^{\mathrm{u}}(P,f) \pitchfork W^{\mathrm{s}}(Q,f))$ is called a *heterodimensional tangency*. In such a case we say that cycle is *non-transverse*.

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3.2
Dominated Splittings

There are several notions extending the concept of uniform hyperbolicity. Let us discuss briefly one of this generalisations.

In the context of the stability conjeture, Liao (27), Mañe (28), Pliss (41) were led to the more general notion of hyperbolicity known as *dominated splitting*. This systems support a invariant splitting shaped for two subbundles: one of the bunddle is definitely more contracted (or less expanded) than the other after a fixed time (uniform in whose invariant set) of iterated.

We now give the formal definition of this weaker form of hyperbolicity.

Definition 3.2.1 (Dominated splitting) An f-invariant set Λ has a *dominated splitting* if the tangent bundle $T_\Lambda M$ over Λ splits into two Df-invariant bundles E and F, $T_\Lambda M = E \oplus F$, whose fibers E_x and F_x have constant dimensions, and there exists constants $0 < \lambda < 1$ and $C > 0$, and an integer $l \geq 1$ such that for every point $x \in \Lambda$ it holds that

$$\|Df^l_x|_{E_x}\| < \frac{1}{2}\,\|Df^{-l}_{f^l(x)}|_{F_{f^l(x)}}\|. \tag{3.2.1}$$

In this case, we say that splitting is l-dominated and that F dominates E.

We now list some important properties of a dominated splitting (for details see, for instance, (13)).

Remark 3.2.2

(i) *Continuous dependence of the fibers.* The fibers E_x and F_x of the dominated splitting depend continuously on the point $x \in \Lambda$.

(ii) *Bounded angle.* The angle between the bundles E_x and F_x, $x \in \Lambda$, is uniformly bounded away from below.

(iii) *Extension to the closure.* Suppose that $E \oplus F$ is a dominated splitting defined on an (not necessarily closed) f-invariant set Λ. Then there is a dominated splitting $\widetilde{E} \oplus \widetilde{F}$ defined on the closure $\overline{\Lambda}$ of Λ such that $\widetilde{E}_x = E_x$ and $\widetilde{F}_x = F_x$, for all $x \in \Lambda$.

In this paper we obtain three-dimensional C^r-diffeomophisms f, $r \geq 2$, such that every system sufficiently C^r-close to f has a transitive invariant sets that does not support any splitting dominated. We will use the following simple observation.

Remark 3.2.3 (Obstruction to domination) In dimension three, the simultaneous presence of two saddles-focus of different u-indices in a transitive

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set Λ prevents the existence of dominated splittings defined on the whole Λ. Indeed, let P and Q be two saddle-focus in Λ such that P has a non-real expanding eigenvalue and Q has a non-real contracting eigenvalue. Assume that $T_\Lambda M := E \oplus F$ is a dominated splitting. Assume for instance that E is one-dimensional (the case when F is one dimensional follows analogously interchanging the roles of P and Q). Then $E_Q \cap E^{\mathrm{s}}_Q$ is a one-dimensional Df-invariant sub-bundle contained in E^{s}_Q. This contradicts the fact that the derivate of f at Q has two non-real (contractive) eigenvalues.

3.3
Homoclinic class

Homoclinic classes were introduced by Newhouse in (34) as an abstraction of the basic sets of the Smale theory (see (48)). They are an essential ingredient in the structure of the dynamics of diffeomorphisms and in many relevant cases correspond to the elementary pieces of dynamics (see (13), Chapter 10.4]).

Definition 3.3.1 (Homoclinic class) Let f be a diffeomorphism and P a periodic saddle of f. Denote by $\mathcal{O}(P)$ the orbit of P. The *homoclinic class* of P (or of the orbit of P), denoted by $H(P, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of $\mathcal{O}(P)$. We say that a homoclinic class is *non-trivial* if it contains at least two different orbits.

A homoclinic class can be also defined as the closure of the set of saddles that are *homoclinically related* with P. Recall that a saddle Q is *homoclinically related* with P if the invariant manifolds of the orbits of P and Q meet cyclically and transversely. Note that homoclinically related saddles have the same index.

We observe that the saddles of $H(P, f)$ having the same index as P form a dense subset of the whole class $H(P, f)$. Finally, any homoclinic class $H(P, f)$ is f-invariant and transitive. For these properties of homoclinic classes see, for instance, (36).

We observe that a homoclinic class $H(P, f)$ may fail to be uniformly hyperbolic. Indeed, this, may contain in a robust way hyperbolic saddles having indices different from the one of P (see, for instance, the constructions in (20)).

Denote by $\pitchfork (P, f)$ the dense subset of saddles in $H(P, f)$ that are homoclinically related to P. For each $Q \in \pitchfork (P, f)$ we have the hyperbolic splitting $(E^{\mathrm{s}} \oplus E^{\mathrm{u}})|_{\mathcal{O}(Q)}$ defined over the orbit $\mathcal{O}(Q)$ of Q. The dimensions of the bundles in these splittings do not depend on Q. One aims to extend these splittings to the whole closure of $\pitchfork (P, f)$ (i.e., $H(P, f)$) to get a "nice" splitting defined on $H(P, f)$. Unfortunately, this is not always possible. First, the angles between the bundles can be arbitrarily small (this prevents the extension, recall

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Remark 3.2.2). Second, if such an extension exists it may fail to be dominated. In this work, we get, after unfolding of a non-tranverse heterodimensional cycle, homoclinic classes that are C^r-robustly non-dominated for $r \geq 2$. This is obtained considering a homoclinic class containing a pair of saddle-focus points with different indices. This motivates the following definition.

Definition 3.3.2 (Intermingled homoclinic classes) Let P and Q be saddles of different indices of a diffeomorphism f. The homoclinic classes $H(P, f)$ and $H(Q, f)$ are *intermigled* if they has non-empty intersection. This intermingledness is *C^r-robust* if there exist a C^r-neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the continuations P_g and Q_g of P and Q satisfies $H(P_g, g) \cap H(Q_g, g) \neq \emptyset$.

In the C^1-topology, the persistence of intermingled homoclinic classes of different indices is a phenomenon inherent to the unfolding of heterodimensional cycles. For details and precise statements see (24). On the other hand, by (1, 17) there exist a residual set of C^1-diffeomorphisms such that any two homoclinic classes are either disjoint or coincide. In this work we get two homoclinic classes of sadles with different indices that are intermingled C^r-robustly for $r \geq 2$.

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3.4 Blenders and Blender-horseshoes

We now discuss the definition of a three-dimensional blenders and blender-horseshoes, for further details and generalisations see (5), (10), and (13, Chapter 6). First, we give an axiomatic definition and thereafter sufficient conditions for the existence of a special kind of cu-blender called *blender-horseshoe.*

Definition 3.4.1 (cu-Blender, Definition 3.1 in (10)) Let $f : M \to M$ be a three-dimensional diffeomorphism. A transitive hyperbolic compact set Λ of index two of f is a cu-*blender* if there are a C^1-neighbourhood $\mathcal{U}$ of f and a C^1-open set $\mathcal{D}$ of embeddings of one-discs D into M such that for every $g \in \mathcal{U}$ and every disc $D \in \mathcal{D}$ the local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_g)$ of the continuation Λ_g intersects D. The set $\mathcal{D}$ is called the *region of superposition* of the blender.

3.4.1 Blender-horseshoes

We begin with some preliminary constructions. In what follows we restrict our attention to the three dimensional case. Since the construction is local we assume that the ambient space is $\mathbb{R}^3$.

Consider a cube of the form

$$\Delta = \mathrm{I}_x \times \mathrm{I}_y \times \mathrm{I}_z \subset \mathbb{R}^3, \quad \text{where } \mathrm{I}_x, \mathrm{I}_y, \text{ and } \mathrm{I}_z \text{ are closed intervals.}$$

We divide the boundary $\partial\Delta$ of Δ into three parts as follows:

$$\partial^{\mathrm{ss}}\Delta := \partial\mathrm{I}_x \times \mathrm{I}_y \times \mathrm{I}_z, \quad \partial^{\mathrm{uu}}\Delta := \mathrm{I}_x \times \partial\mathrm{I}_y \times \mathrm{I}_z, \quad \partial^{\mathrm{u}}\Delta := \mathrm{I}_x \times \partial(\mathrm{I}_y \times \mathrm{I}_z).$$

Note that $\partial^{\mathrm{uu}}\Delta \subset \partial^{\mathrm{u}}\Delta$ and $\partial\Delta = \partial^{\mathrm{s}}\Delta \cup \partial^{\mathrm{u}}\Delta$.

Given $\theta > 1$ for each $p \in \Delta \subset \mathbb{R}^3$ we consider the cone fields

$$\begin{aligned} \mathcal{C}^{\mathrm{u}}_\theta(p) &= \Big\{(u,v,w) \in T_p\Delta : \theta|u| < \sqrt{v^2+w^2}\Big\}, \\ \mathcal{C}^{\mathrm{uu}}_\theta(p) &= \Big\{(u,v,w) \in T_p\Delta : \theta\sqrt{u^2+w^2} < |v|\Big\}, \\ \mathcal{C}^{\mathrm{s}}_\theta(p) &= \Big\{(u,v,w) \in T_p\Delta : \theta\sqrt{v^2+w^2} < |u|\Big\}. \end{aligned} \tag{3.4.1}$$

Note that $\mathcal{C}^{\mathrm{uu}}_\theta(p) \subset \mathcal{C}^{\mathrm{u}}_\theta(p)$.

Related to these cone fields we define *vertical* and *horizontal curves* and *vertical strips* as follows:

- A regular curve $L \subset \Delta$ is *vertical* (resp. *horizontal*) if for every point p in L, it holds $T_pL \subset \mathcal{C}^{\mathrm{uu}}_\theta(p)$ (resp. $T_pL \subset \mathcal{C}^{\mathrm{s}}_\theta(p)$) and the end-points of L are contained in different connected components of $\partial^{\mathrm{uu}}\Delta$ (resp. $\partial^{\mathrm{s}}\Delta$). In what follows the vertical curves will be called uu-*disc*.
- A surface $S \subset \Delta$ is called a *vertical strip in* Δ if $T_pS \subset \mathcal{C}^{\mathrm{u}}_\theta(p)$ for every p in S and there exists a C^1-embedding $E : \mathrm{I}_y \times \mathrm{J} \to \Delta$ (where J is a subinterval of I_z) such that $E(\mathrm{I}_y \times \mathrm{J}) = S$ and $L(z) := E(\mathrm{I}_y \times \{z\})$ is a vertical curve for every $z \in \mathrm{J}$. The *width* of S, denoted by $w(S)$, is the infimum of the length of the curves in S which are transverse to $\mathcal{C}^{\mathrm{uu}}_\theta$ and join the two components of $L(\partial J)$.

Note that every horizontal curve W in Δ define two different (free) homotopy classes of vertical segments through Δ and disjoint from W. This allows us to consider uu-discs *to the left* and *to the right* of W (corresponding to two different homotopy classes). We will denote this classes by H^l_W and H^r_W.

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- The *right class* H^r_W (resp. *left class* H^ℓ_W) consist of every vertical curve L in Δ such that $L \cap W = \emptyset$ and L is (freely) homotopic to $\{x_0\} \times \mathrm{I}_y \times \{z^+\}$ (resp. $\{x_0\} \times \mathrm{I}_y \times \{z^-\}$) for some $x_0 \in \mathrm{I}_x$, where $\mathrm{I}_z = [z^-, z^+]$. If $L \in \mathrm{H}^r_W$ (resp. $L \in \mathrm{H}^\ell_W$) we say that L is *at the right* (resp. *at the left*) of W. Observe that if W_1 and W_2 are different horizontal curves in Δ, then $\mathrm{H}^r_{W_1} \cap \mathrm{H}^\ell_{W_2} \neq \emptyset$ or $\mathrm{H}^r_{W_2} \cap \mathrm{H}^\ell_{W_1} \neq \emptyset$.

  Similarly, a vertical strip S through Δ is *at the right* (resp. *at the left*) of W if it is foliated by vertical curves at the right (resp. *at the left*) of W.

We now borrow the following definition from (10).

Definition 3.4.2 (Blender-Horseshoes) The maximal invariant $\Lambda_F := \cap_{i\in\mathbb{Z}} F^i(\Delta)$ of a local diffeomorphism $F : \Delta \to \mathbb{R}^3$ is a *blender-horseshoes* if F satisfies conditions (BH1)-(BH6) below:

(BH1) *Vertical legs of the blender*: The intersection $F(\Delta) \cap (\mathbb{R} \times \mathrm{I}_y \times \mathbb{R})$ consists of two connected components, denoted $F(\mathcal{A})$ and $F(\mathcal{B})$ satifiying

$$F(\mathcal{A}) \cup F(\mathcal{B}) \subset \mathrm{int}(\mathrm{I}_x) \times \mathrm{I}_y \times \mathbb{R} \quad \text{and} \quad (\mathcal{A} \cup \mathcal{B}) \cap \partial^{\mathrm{uu}}\Delta = \emptyset.$$

Here $\mathrm{int}(X)$ denote the interior of X.

(BH2) (i) *Strict invariance of cone fields*: There exist $\theta > 1$ such that for every $p \in F(\mathcal{A}) \cap F(\mathcal{B})$ and $q \in \mathcal{A} \cap \mathcal{B}$ then

$$DF^{-1}_p(\mathcal{C}^{\mathrm{ss}}_\theta(p)) \subset \mathcal{C}^{\mathrm{ss}}_\theta(F^{-1}(p)), \;\; DF_q(\mathcal{C}^*_\theta(q)) \subset \mathcal{C}^*_\theta(F(q)), \quad * = \mathrm{u}, \mathrm{uu},$$

(ii) *Expansion/Contraction the cone fields*: The derivatives $DF|_{\mathcal{C}^{\mathrm{u}}}$ and $DF^{-1}|_{\mathcal{C}^{\mathrm{s}}}$ are uniformly expanding and contracting, respectively.

(BH3) *Markov partition*: Consider the connected components of $F^{-1}(\Delta) \cap \Delta$:

$$\mathbb{A} := F^{-1}(F(\mathcal{A}) \cap \Delta), \quad \mathbb{B} := F^{-1}(F(\mathcal{B}) \cap \Delta).$$

Then,

$$F(\mathbb{A}) \cup F(\mathbb{B}) \subset \mathrm{int}(\mathrm{I}_x) \times \mathrm{I}_y \times \mathrm{I}_z, \quad \mathbb{A} \cup \mathbb{B} \subset \mathrm{I}_x \times \mathrm{int}(\mathrm{I}_y \times \mathrm{I}_z).$$

Conditions (BH2) and (BH3) imply the existence of two saddles $P \in \mathbb{A}$ and $Q \in \mathbb{B}$. We define the local stable manifolds of P and Q by

$$W^{\mathrm{s}}_{\mathrm{loc}}(P) := \text{connected componet of } W^{\mathrm{s}}(P) \cap \Delta \text{ containig } P,$$

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$$W^{\mathrm{s}}_{\mathrm{loc}}(Q) := \text{connected componet of } W^{\mathrm{s}}(Q) \cap \Delta \text{ containig } Q.$$

These local manifolds are horizontal curves in Δ. Note that either $\mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)} \cap \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)} \neq \emptyset$ or $\mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(P)} \cap \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)} \neq \emptyset$ and assume that the first case holds. We say that a vertical curve is *in between* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ *and* $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ if it belongs to $\mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)} \cap \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$. We use the notation

$$\mathrm{H}^b := \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)} \cap \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}.$$

We say that he saddles P and Q are the *reference saddles* of Λ_F, where P is the *left* saddle and Q is the *right* saddle. The family of discs in H^b is called *superposition region* of the blender-horseshoe.

(BH4) uu-*discs through the local stable manifolds of* P *and* Q: Let D and D' be uu-discs such that $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(P) \neq \emptyset$ and $D' \cap W^{\mathrm{s}}_{\mathrm{loc}}(Q) \neq \emptyset$. Then

$$D \cap \left(\partial^{\mathrm{u}}\Delta \setminus \partial^{\mathrm{uu}}\Delta\right) = \emptyset, \quad D' \cap \left(\partial^{\mathrm{u}}\Delta \setminus \partial^{\mathrm{uu}}\Delta\right) = \emptyset$$

(BH5) *Positions of images of* uu-*discs*: Let D be a uu-disc in Δ and write

$$D_{\mathcal{A}} := D \cap \mathcal{A} \quad \text{and} \quad D_{\mathcal{B}} := D \cap \mathcal{B}.$$

There are the following six cases:

(1) if $D \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$ then $F(D_{\mathcal{A}}) \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$,
(2) if $D \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$ then $F(D_{\mathcal{A}}) \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$,
(3) if $D \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$ then $F(D_{\mathcal{B}}) \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$,
(4) if $D \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$ then $F(D_{\mathcal{B}}) \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$,
(5) if $D \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$ or $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(P) \neq \emptyset$ then $F(D_{\mathcal{B}}) \in \mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}$, and
(6) if $D \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$ or $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(Q) \neq \emptyset$ then $F(D_{\mathcal{A}}) \in \mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}$.

(BH6) *Positions of images of* uu-*discs* H^b: Let $D \in \mathrm{H}^b$. Then either $F(D_{\mathcal{A}})$ or $F(D_{\mathcal{B}})$ belongs to H^b.

Figure 3.1 illustrates a prototypical blender-horseshoes dynamics.

Remark 3.4.3 (Consequences of (BH1)-(BH6), Section 3.2.4 in (10))

- Condition (BH3) is equivalent to

$$\left(F(\mathbb{A}) \cup F(\mathbb{B})\right) \cap \partial^{\mathrm{s}}\Delta = \emptyset, \quad \text{and} \quad \left(\mathbb{A} \cup \mathbb{B}\right) \cap \partial^{\mathrm{u}}\Delta = \emptyset.$$

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- From (BH3), one gets that $\{\mathbb{A}, \mathbb{B}\}$ is a Markov partition generating Λ_F. In particular, the set Λ_F contains exactly two fixed points of F, $P \in \mathbb{A}$ and $Q \in \mathbb{B}$. Therefore, (BH2) and (BH3) imply that the dynamics of F in Λ_F is hyperbolic and conjugate to the full shift of two symbols.
- The conditions (BH1)-(BH6) are C^1-open ones. Hence if Λ_F is a blender-horseshoe of F for every G close enough to F the continuation Λ_G of Λ_F is a blender-horseshoe.

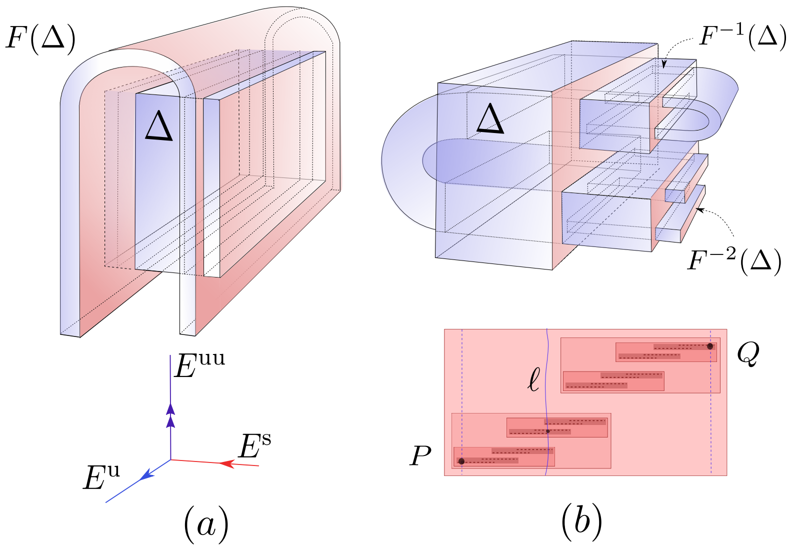

Figure 3.1: (a) The diffeomophism F satisfies conditions (BH1)-(BH6) in Δ. (b) Projection of the region of superposition H^b between the reference saddles P and Q of the blender-horseshoe Λ_F. The curve ℓ is a uu-disc in H^b.

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We consider the following local stable manifold of Λ_F,

$$W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_F) := \bigcap_{n\in\mathbb{N}} F^n(\Delta) \subset W^{\mathrm{s}}(\Lambda_F).$$

We borrow the following lemma (and its proof) from (4) stating the *distinctive property* of a blender.

Lemma 3.4.4 (Lemma 3.13 in (4)) *For every $D \in H^b$ it holds $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_F) \neq \emptyset$ for every $D \in H^b$.*

Proof. Consider $D = D_0 \in H^b$. By condition (BH6), $F(D)$ contains a disc $D_1 \in H^b$. Write $F^{-1}(D_1) = D_1' \subset D_0$. We now proceed inductively, assuming defined $D_n \in H^b$ with $D_n \subset F(D_{n-1})$ and $F^{-n}(D_n) = D_n' \subset D_0$, we define $D_{n+1} \in H^b$ contained in $F(D_n)$ and let $F^{-n-1}(D_n) = D_n' \subset D_0$. The sequence D_n' is nested and hence $\emptyset \neq \cap_n D_n' \subset D_0$. Note that by construction $\cap_n D_n' \subset W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_F)$. ■

We also the following refinement of the above lemma. First we say that a vertical strip is *in between* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ *and* $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ if it is foliated by curves in H^b.

Lemma 3.4.5 *Every vertical strip in between* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ *and* $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ *intersects transversely* $W^{\mathrm{s}}(P)$.

Proof. Since Λ_F is the maximal invariant set in Δ, then $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_F) = \cap_{i\in\mathbb{N}} F^{-i}(\Delta)$. Thus, it is sufficient to see that the stable manifold $W^{\mathrm{s}}(P^*, F)$ of $P^* \in \Lambda_F$ intersects transversally every vertical strip S through Δ to the right of W^{s}_0 (note that any vertical segment D through Δ can be seen as an intersection of a nested sequence of vertical strip S throughout Δ). To see that, note that the conditions (H1)–(H6) above imply that the width of vertical strips in Δ grows exponentially after iterations by F (i.e., the image $F(S)$ contains a strip S' such that $w(S') > c'w(S)$ for some $c' > 1$). This implies that the stable manifold $W^{\mathrm{s}}(P^*, F)$ of P^* intersects transversally every vertical strip S through Δ to the right of W^{s}_0. In particular, we have that $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_F) \cap S \neq \emptyset$, ending the proof of the lemma. ■

3.4.2 Blender-horseshoes for endomorphisms

Now we reformulate the definition of a blender-horseshoe for endomorphisms:

Definition 3.4.6 (Blender-horseshoes for endomorphisms) The generalised maximal invariant $\widehat{\Lambda}_G := \cap_{i\in\mathbb{Z}}\big(\Delta \cap G^i(\Delta)\big)$ of a local endomorphism $G : \Delta \to \mathbb{R}^3$ is a *blender-horseshoes* if G satisfies the conditions below:

(BH1') *Vertical legs of the blender*: The intersection $G(\Delta) \cap (\mathbb{R} \times \mathrm{I}_y \times \mathbb{R})$ consists of two connected components, denoted $G(\mathcal{A})$ and $G(\mathcal{B})$ satifiying

$$G(\mathcal{A}) \cup G(\mathcal{B}) \subset \mathrm{int}(\mathrm{I}_x) \times \mathrm{I}_y \times \mathbb{R} \quad \text{and} \quad (\mathcal{A} \cup \mathcal{B}) \cap \partial^{\mathrm{uu}}\Delta = \emptyset,$$

where $\mathcal{A}$ and $\mathcal{B}$ are connected subsets of Δ.

(BH2') (i) *Strict invariance of cone fields*: There exist $\theta > 1$ such that if $p \in \mathcal{A} \cap \mathcal{B}$ then

$$\mathcal{C}^{\mathrm{ss}}_\theta(G(p)) \subset DG_p\Big(\mathcal{C}^{\mathrm{ss}}_\theta(p)\Big), \quad DG_q\Big(\mathcal{C}^{*}_\theta(p)\Big) \subset \mathcal{C}^{*}_\theta(G(p)), \quad * = \mathrm{u}, \mathrm{uu},$$

(ii) *Expansion/Contraction the cone fields*: The derivatives $DG|_{\mathcal{C}^{\mathrm{u}}}$ and $DG|_{\mathcal{C}^{\mathrm{s}}}$ are uniformly expanding and contracting, respectively.

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(BH3') *Quasi Markov partition*: Consider connected components of $G^{-1}(\Delta)\cap\Delta$:

$$\mathbb{A}\subset G^{-1}(G(\mathcal{A})\cap\Delta)\cap\Delta,\quad \mathbb{B}\subset G^{-1}(G(\mathcal{B})\cap\Delta)\cap\Delta.$$

such that

$$G(\mathbb{A})=G(\mathcal{A})\cap\Delta,\quad G(\mathbb{B})=G(\mathcal{B})\cap\Delta.$$

Then,

$$G(\mathbb{A})\cup G(\mathbb{B})\subset \mathrm{int}(\mathrm{I}_x)\times\mathrm{I}_y\times\mathrm{I}_z,\quad \mathbb{A}\cup\mathbb{B}\subset\mathrm{I}_x\times\mathrm{int}(\mathrm{I}_y\times\mathrm{I}_z).$$

Conditions (BH2') and (BH3') imply the existence of two (reference) saddles $P\in\mathbb{A}$ and $Q\in\mathbb{B}$. Thus, the local stable manifolds of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ are horizontal segments in Δ. We assume that $\mathrm{H}^r_{W^{\mathrm{s}}_{\mathrm{loc}}(P)}\cap\mathrm{H}^\ell_{W^{\mathrm{s}}_{\mathrm{loc}}(Q)}\neq\emptyset$.

(BH4') The uu-discs D and D' in Δ such that

$$D\cap W^{\mathrm{s}}_{\mathrm{loc}}(P)\neq\emptyset\quad\text{and}\quad D'\cap W^{\mathrm{s}}_{\mathrm{loc}}(Q)\neq\emptyset.$$

satisfies the condition (BH4) in Definition 3.4.2.

(BH5') Let D be a uu-disc in Δ and denote $D_{\mathcal{A}}:=D\cap\mathcal{A}$ and $D_{\mathcal{B}}:=D\cap\mathcal{B}$. Then the discs $G(D_{\mathcal{A}})$ and $G(D_{\mathcal{B}})$ satifies the posibilities of (BH5) in Definition 3.4.2.

(BH6') Let D be a uu-disc in H^b. Then either $G(D_{\mathcal{A}})$ or $G(D_{\mathcal{B}})$ is in H^b.

We now reformulate the item about continuations of blender-horseshoes for endomorphisms.

Remark 3.4.7 (Continuations of blender-horseshoes endomorphisms) Suposse that the endomorphism G has a blender-horseshoe in Δ. Then every diffeomorphism F such that $F|_\Delta$ is sufficiently close to $G|_\Delta$ has a blender-horseshoe in Δ.

3.5 Renormalisation schemes and Hénon-like families

First, let us observe that renormalisation methods play an important role in the study of homoclinic bifurcations (dynamics at homoclinic tangencies). This method leads to the approximation of dynamics by quadratic families (Hénon-like families) and allows to translate some properties of such families (as existence of strange attractors and sinks, or thick hyperbolic sets) to the renormalised diffeomorphisms, see for instance (38, Chapter 6.4).

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Analogously to the case of homoclinic bifurcations, in (21) is introduced a renormalisation scheme for a three-dimensional C^r-diffeomorphism f, $r \geq 2$, having a non-transverse heterodimensional cycles between saddles P and Q (in (21) both saddles in the cycle have real multipliers) of u-index two and one, respectively, whose dynamic limit it the *center unstable Hénon-like family*

$$\widetilde{G}_{(\xi,\mu,\kappa,\eta)}(x,y,z) = (\xi\, x + y, \mu + y^2 + \kappa\, x^2 + \eta\, x\, y, y), \quad \xi > 1. \tag{3.5.1}$$

This Hénon-like family exhibits, for an appropriate open set of parameters, blender-horseshoes, see Theorem 1.

Recall our bifurcation setting in Section 2 of a diffeomorphism $f \in \mathrm{Diff}^r(M)$ having a non-transverse heterodimensional tangency point Y and a quasi-transverse heteroclinic point X. A *renormalisation scheme* to f at Y is a 4-tuple

$$\mathcal{R}_k(f) :\, = \Big(\{\Psi_k\}_k, \{f_k\}_k, \{\ell(k)\}_k, R_\infty(f)\Big) \tag{3.5.2}$$

where

- $\Psi_k : \mathbb{R}^3 \to M$ is a sequence of local coordinates such that $\Psi_k(K) \to \{Y\}$ for every compact set K in $\mathbb{R}^3$;
- $f_k : M \to M$ is a sequence of diffeomorphisms (obtained by an unfolding of the cycle - both heteroclinic connections X and Y) converging to f in the C^r-topology;
- $\ell(k) \in \mathbb{N}$ is a sequence of *return times* of f to the heterodimensional tangency; and
- $R_\infty(f) : \mathbb{R}^3 \to \mathbb{R}^3$ is an endomorphism,

such that the *renormalised sequence*

$$R_k(f) :\, = \Psi_k^{-1} \circ f_k^{\ell(k)} \circ \Psi_k,$$

converges on compact sets to $R_\infty(f)$ in the C^r-topology. The endomorphism $R_\infty(f)$ is called *dynamic limit*.

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4
Precise statements of the main results

4.1
Hénon-like families with blender-horseshoes

Next result is a version of (21, Theorem 1.1) where blenders are replaced by blender-horseshoes.

Theorem 1 *Consider the center unstable Hénon-like family of endomorphisms*

$$G_{(\xi,\mu,\kappa,\eta)}(x,y,z) := (y, \mu + y^2 + \kappa\, y\, z + \eta\, z^2, \xi\, z + y), \quad \xi > 1,$$

and the cube

$$\Delta := [-4,4]^2 \times [-40,22].$$

Then there is $\varepsilon > 0$ such that for every

$$\bar{\nu} = (\xi,\mu,\kappa,\eta) \in \mathcal{O} := (1.18, 1, 19) \times (-10,-9) \times (-\varepsilon,\varepsilon)^2$$

the endomorphism $G_{\bar{\nu}}$ has a blender-horseshoe in Δ.

As a consequence, every diffeomorphism sufficiently C^1-close to $G_{\bar{\nu}}$ has a blender-horseshoe in Δ.

Remark 4.1.1 The endomorphism $\widetilde{G}_{(\xi,\mu,\kappa,\eta)}$ in (3.5.1) and $G_{(\xi,\mu,\kappa,\eta)}$ are conjugated by the map $\widetilde{\Theta}(x,y,z) = (z,y,x)$.

4.2
A bifurcation setting for C^r-robust cycles

Recall that M denotes a three-dimensional compact Riemannian manifold. Consider a diffeomorphism $f \in \mathrm{Diff}^r(M)$ having a pair of periodic saddles P and Q of u-*indice* two and one, respectively, related by a heterodimensional cycle satisfying the following conditions (L), (H), (Q) and (T):

(**L**) *Linearising and Spectral conditions.* Suppose that there exist C^r-linearising local chart U_P and U_Q at the the saddles P and Q. We assume also, that the saddles P and Q has non-real eigenvalues with spectrum are given by

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$$\mathrm{Spec}\Big(Df(P)\Big) = \Big\{\lambda_P, \sigma_P\, e^{\pm 2\pi i \varphi_P}\Big\}, \quad \mathrm{Spec}\Big(Df(Q)\Big) = \Big\{\sigma_Q, \lambda_Q\, e^{\pm 2\pi i \varphi_Q}\Big\},$$

where $0 < |\lambda_P|, |\lambda_Q| < 1 < |\sigma_P|, |\sigma_Q|$, and $\varphi_P, \varphi_Q \in \mathbb{Q}^c$. We called to this type of saddles *irrational saddle-focus*. We assume some technical conditions (including non-resonance like ones) on the parameters

$$\mathrm{LocDyn}(f) := (\lambda_P, \sigma_P, \lambda_Q, \sigma_Q) \in \mathbb{R}^4,$$

given by a open and non-empty set $\mathcal{P}$ (see Lemma 6.5.3). For simplicity, we relegate the explicit formulation of these conditions to Section 5.3.1 (see (5.3.2)).

(**H**) *Heterodimensional tangency.* The two dimensional manifolds $W^{\mathrm{u}}(P, f)$ and $W^{\mathrm{s}}(Q, f)$ have a *heterodimensional tangency* $Y \in W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$, see Defintion 3.1.1.

Taking backward iterates of Y, if necessary, we can assume that $Y \in W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$. Identify a linearising neighbourhood of P with a neighbourhood of $\mathbf{0} \in \mathbb{R}^3$, we know, by the classification of quadratic surfaces in $\mathbb{R}^3$ that Y has the form as illustrated in following Fig. 2.1.

(**Q**) *Quasi-transverse intersections.* The one-dimensional manifolds $W^{\mathrm{s}}(P, f)$ and $W^{\mathrm{u}}(Q, f)$ meet *quasi-transversely* along the orbit of a heteroclinic point $X \in W^{\mathrm{s}}(P, f) \cap W^{\mathrm{u}}(Q, f)$ i.e., $T_X W^{\mathrm{s}}(P, f) + T_X W^{\mathrm{u}}(Q, f) = T_X W^{\mathrm{s}}(P, f) \oplus T_X W^{\mathrm{u}}(Q, f)$.

Replacing the heteroclinic points $X \in W^{\mathrm{s}}(P, f) \cap W^{\mathrm{u}}(Q, f)$ and $Y \in W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$ by some backward iterates we can assume that $X \in U_Q$ and $Y \in U_P$. Associated to these heteroclinic points we define a pair of *transition maps* corresponding to suitable iterations of the diffeomorphism f in small neighbourhoods of X and Y.

Definition 4.2.1 (The set $\mathcal{T}_{\mathrm{quad}}$ of allowed quadratic transitions) Denote by $\mathcal{T}_{\mathrm{quad}}$ the space of polynomials of $\mathbb{R}^3$ fixing the origin of the form

$$q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 x + a_2 y + a_3 z \\ b_1 x + b_2 y^2 + b_3 z^2 + b_4 yz \\ c_1 x + c_2 y + c_3 z \end{pmatrix},$$

such that

$$b_1\, c_2\, (a_3 - a_2) \neq 0, \quad b_2 + b_3 + b_4 \neq 0, \quad c_2 = c_3.$$

Note that q is a local diffeomorphism at the origin. We identify the map $q \in \mathcal{T}_{\mathrm{quad}}$ with the vector $v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathbb{R}^9$ and write

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$q = q_v$. We denote $\mathrm{Q} \subset \mathbb{R}^9$ the set of vectors v such that $q_v \in \mathcal{T}_{\mathrm{quad}}$ and

$$\Pi : \mathcal{T}_{\mathrm{quad}} \to \mathrm{Q}, \quad \Pi(q_v) = v. \tag{4.2.1}$$

(**T**) *Transition maps.* There are natural numbers N_1 and N_2 (called *transition times*) such that $f^{N_1}(X) \in U_P$ and $f^{N_2}(Y) \in U_Q$ such that

- in a small neighbourhood of X, the transition map f^{N_1} is C^r-close to a translation: there are small neighbourhoods $U_X \subset U_Q$ of X and $U_{f^{N_1}(X)} \subset U_P$ and local coordinates in these neighbourhoods (where X and $f^{N_1}(X)$ are identified with the origin) such that

$$f^{N_1} : U_X \to U_{f^{N_1}(X)}, \quad f^{N_1}(Z) = Z + \widetilde{H}(Z),$$

where the map $\widetilde{H}$ is a higher order term (of order at least two) satisfying the flat conditions in (5.3.8) at $Z = \mathbf{0}$.

- in a small neighbourhood of Y, the transition map f^{N_2} is a C^r-map of quadratic type: there are small neighbourhoods $U_Y \subset U_P$ of Y and $U_{f^{N_2}(Y)} \subset U_Q$ and local coordinates in these neighbourhoods (where Y and $f^{N_2}(Y)$ are identified with the origin) such that

$$f^{N_2} : U_Y \to U_{f^{N_2}(Y)}, \quad f^{N_2}(Z) = \mathrm{Quad}(f)_{Y,N_2}(Z) + H(Z),$$

where $\mathrm{Quad}(f)_{Y,N_2} \in \mathcal{T}_{\mathrm{quad}}$ and H denotes the high order terms (of order at least two) satisfying flat conditions in (5.3.11) and (**??**) at $Z = \mathbf{0}$[1]

The map $\mathrm{Quad}(f)_{Y,N_2}$, called the *quadratic transition* of f, describes, up to a C^r-error of order two, the tangential contact between $W^{\mathrm{u}}(P, f)$ and $W^{\mathrm{s}}(Q, f)$ at Y.

Definition 4.2.2 For $r \geq 1$, we define the subset $\mathcal{N}^r_{P,Q}(\mathcal{T}_{\mathrm{quad}})$ of $\mathrm{Diff}^r(M)$ consisting of diffeomorphisms f having a non-transverse heterodimensional cycle associated to irrational saddle-focus P_f and Q_f satisfying conditions (L), (H), (Q), and (T). Given a subset $\mathcal{T} \subset \mathcal{T}_{\mathrm{quad}}$ we denote by $\mathcal{N}^r_{P,Q}(\mathcal{T})$ the subset of $\mathcal{N}^r_{P,Q}(\mathcal{T}_{\mathrm{quad}})$ of diffeomorphisms f such that $\mathrm{Quad}(f) \in \mathcal{T}$.

[1]The flat conditions of H imply that there is no "interference" between $\mathrm{Quad}(f)_{Y,N_2}$ and H: the first and the third components of H have no linear terms, the second component of H has no x, y^2, z^2, yz terms.

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4.3 The renormalisation scheme

Following the constructions in the proof of (21, Theorem 1.2), we get a renormalisation scheme of diffeomorphisms having a non-transverse heterodimensional cycle between irrational saddles-focus, converging to the family of endomorphisms

$$E_{(\xi,\mu,\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5)}(x,y,z) = (\xi\, x + \varsigma_1\, y,\ \mu + \varsigma_2\, y^2 + \varsigma_3\, x^2 + \varsigma_4\, x\, y,\ \varsigma_5\, y). \quad (4.3.1)$$

For an appropriate choice of parameters this family is C^∞-conjugate to the family $G_{\xi,\mu,\kappa,\eta}$ in Theorem 1 (see Lemma 6.4.1).

Theorem 2 (Renormalisation scheme) *Consider a diffeomorphism f in the set $\mathcal{N}^r_{P,Q}(\mathcal{T}_{\mathrm{quad}})$, $r \geq 2$ with quadratic transition $\mathrm{Quad}(f) = q_v$. Then there is a unfolding family $\mathfrak{F} = \{f_{\bar{v}}\}_{\bar{v}\in\mathbb{R}^8}$ in $\mathrm{Diff}^r(M)$ bifurcating the non-transverse heterodimensional cycle of $f = f_{\bar{0}} \in \mathfrak{F}$ satisfying the following property:*

For every $\xi > 0$, there exist a renormalisation scheme $\mathcal{R}(\xi, \mathfrak{F}, f)$ consisting of

- *$m, n \geq 1$ sequences of natural numbers;*
- *$\Psi_{m,n} : \mathbb{R}^3 \to M$ sequence of parameterisations of the manifold;*
- *$\bar{v}_{m,n} : \mathbb{R}^3 \to \mathbb{R}^8$ sequence of functions parameterizing the bifurcating family $f_{\bar{v}}$;*
- *$\mathcal{R}_{m,n}\Big(f_{\bar{v}_{m,n}}\Big) : M \to M$, sequence of rescaled diffeomorphisms defined by*

$$\mathcal{R}_{m,n}\Big(f_{\bar{v}_{m,n}}\Big) := \Big(f_{\bar{v}_{m,n}}\Big)^{N_2+m+N_1+n}$$

- *rational maps $\varsigma_i : \mathrm{Dom}(\varsigma_i) \subset \mathbb{R} \times \mathbb{R}^9 \to \mathbb{R}$, $i = 1, 2, 3, 4, 5$;*

satisfying the following conditions:

- *for compact sets L, Δ in $\mathbb{R}^3$ we have the convergence:*

$$\bar{v}_{m,n}(L) \to \{\mathbf{0}\}, \quad \Psi_{m,n}(\Delta) \to \{Y\}, \quad \textit{when} \quad m, n \to +\infty,$$

  here Y is the point of heterodimensional tangency;
- *for each $(\mu, \tilde{\alpha}, \alpha) \in \mathbb{R}^3$ the corresponding renormalised sequence*

$$\Psi^{-1}_{m,n} \circ \mathcal{R}_{m,n}\Big(f_{\bar{v}_{m,n}}\Big) \circ \Psi_{m,n}, \quad \textit{where} \quad \bar{v}_{m,n} = \bar{v}_{m,n}(\mu, \tilde{\alpha}, \alpha), \quad (4.3.2)$$

  converges in the C^r-topology and on compact sets of $\mathbb{R}^3$ to the endomorphism

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$$E_{(\xi,\mu,\bar{\varsigma}(\xi,v))}(x,y,z) = (\xi\, x + \varsigma_1\, y,\ \mu + \varsigma_2\, y^2 + \varsigma_3\, x^2 + \varsigma_4\, x\, y,\ \varsigma_5\, y), \quad (4.3.3)$$

where $\bar{\varsigma} := (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5)$ *and* $\varsigma_i = \varsigma_i(\xi, v)$, $i = 1, 2, 3, 4, 5$.

4.4 C^r-robust cycles and C^r-stabilisation

We give a first result asserting that for every $r \geq 2$ there is a class of diffeomorphisms in $\mathcal{N}^r_{P,Q}(\mathcal{T}_{\mathrm{quad}})$ such that every diffeomorphism in this class can be C^r-approximated by diffeomorphisms having C^r-robust (heterodimensional) cycles.

Theorem 3 (C^r-robust cycles) *Let $r \geq 2$. There exist a seven-dimensional sub-manifold $\mathcal{T}^{RC}$ in $\mathcal{T}_{\mathrm{quad}}$ such that every diffeomorphism in $\mathcal{N}^r_{P,Q}(\mathcal{T}^{RC})$ can be C^r-approximated by diffeomorphisms having C^r-robust heterodimensional cycles.*

The previous theorem does not provide any relation between the hyperbolic sets involved in the robust cycle and the continuations of the saddles in the initial cycle. This motivates the next result. Recall that a *lamination* is a locally trivial partition of a set (contained in a manifold) into sub-manifolds.

Theorem 4 (C^r-stabilisation) *Let $2 \leq r < +\infty$. There are an open subset $\mathcal{B} \subset \mathbb{R}^3$ and a local lamination $\mathcal{T}_{\mathcal{B}} := (\mathcal{T}_{\bar{b}})_{\bar{b}\in\mathcal{B}}$ whose leaves are sub-manifolds of $\mathcal{T}_{\mathrm{quad}}$ of dimension seven satisfying the following properties:*

- *(I) For every $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}_{\bar{b}})$ there is a sequence of diffeomorphisms f_k converging to f in the C^r-topology such that every f_k has a blender-horseshoe Λ_{f_k} (of index two) accumulating to the heterodimensional tangency of f.*
- *(II) There is a subset $\mathcal{B}' \subset \mathcal{B}$ such that for every $\bar{b} \in \mathcal{B}'$ there exist a open subset $\mathcal{T}_0$ of $\mathcal{T}_{\bar{b}}$ such that if $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}_0)$ then for every k large enough it holds*
  - *(i) Λ_{f_k} and the saddle Q_{f_k} form a robust cycle,*
  - *(ii) Λ_{f_k} is a homoclinically related to the saddle P_{f_k}, and*
  - *(iii) the homoclinic classes of P_{f_k} and Q_{f_k} are C^r-robustly intermingled.*

In particular, the initial cycle of every diffeomorphism in $\mathcal{N}^r(\mathcal{T}_0)$ can be C^r- stabilised by diffeomorphisms having homoclinic classes C^r-robustly non-dominated.

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4.5
Strategy of the proofs of Theorems 3 and 4

We now sketch the main steps in the proofs.

• *Renormalisation scheme.* Bearing in mind Theorems 1 (specially equation (4.3.2)) and 2 and recalling the conjugation $\Theta_{(\varsigma_1,\varsigma_2,\varsigma_5)}$ in Lemma 6.4.1 between the Hénon-like family in (3.5.1) and the quadratic family of endomorphisms in (4.3.1),

$$\Theta^{-1}_{(\varsigma_1,\varsigma_2,\varsigma_5)} \circ E_{(\xi,\mu,\bar{\varsigma})} \circ \Theta_{(\varsigma_1,\varsigma_2,\varsigma_5)} = G_{(\xi,\mu,\kappa,\eta)},$$

we get the following convergence result. Recall the maps $\varsigma_1 = \varsigma_1(\xi, v)$, $\varsigma_2 = \varsigma_2(\xi, v)$, and $\varsigma_5 = \varsigma_5(\xi, v)$ in Theorem 2 and consider

$$\kappa(\xi, v) = \varsigma_1(\xi, v)^2\, \varsigma_2(\xi, v)^{-1}\, \varsigma_3(\xi, v), \quad \eta(\xi, v) = \varsigma_1(\xi, v)\, \varsigma_2(\xi, v)^{-1}\, \varsigma_4(\xi, v). \tag{4.5.1}$$

Proposition 1 *For appropriate choice of parameters ξ, μ, and v the following C^r-convergence holds*

$$\Theta^{-1}_{(\varsigma_1,\varsigma_2,\varsigma_5)} \circ \Psi^{-1}_{m,n} \circ \mathcal{R}_{m,n}\Big(f_{\bar{v}_{m,n}}\Big) \circ \Psi_{m,n} \circ \Theta_{(\varsigma_1,\varsigma_2,\varsigma_5)} \to G_{(\xi,\mu,\kappa,\eta)}.$$

Next step is know the dynamics of $G_{(\xi,\mu,\kappa,\eta)}$.

• *Blenders and saddle-node points in the renormalisation scheme: proof of Theorem 3.* To explain the proof of this theorem we begin by recalling some ingredients in (9). To get robust cycles in (9) the authors first consider a series of genuine C^1-perturbations leading to a configuration called *strong homoclinic intersections of a saddle-node* (roughly, the strong unstable and stable manifolds of the saddle node meets quasi-transversely). This configuration yields robust cycles after small C^∞-perturbations. In our setting the existence of strong homoclinic intersections of a saddle-node occurs naturally in Hénon-like families, we now discuss this point.

For the family $G_{(\xi,\mu,\kappa,\eta)}$ there are two important set of parameters:

– an open set where the maps have blender-horseshoes (Theorem 1) and

– parameters of the form $(1, \mu, 0, 0)$, with $\mu \sim -9$, where there are a pair of saddle-nodes which are "homoclinically related" (the strong invariant one-dimensional manifolds meet quasi-transversally and cyclically). After small perturbations this configuration leads to strong homoclinic intersections associated to saddle-nodes.

Here we use the Hénon-like families in the place of C^1-perturbations to get the strong homoclinic intersections.

Summarising, the main step of the proof of Theorem 3 is the following:

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Proposition 2 *Let $\{F_k\}_k$ be a sequence in* $\mathrm{Diff}^r(\mathbb{R}^3)$ *such that for every compact set $K \subset \mathbb{R}^3$ it holds*

$$\lim_{k\to+\infty} \|(F_k - G_{(1,\mu,0,0)})|_K\|_r = 0.$$

Then there exist $\epsilon_k \to 0$ and an ϵ_k-C^r-perturbation G_k of F_k such that G_k has strong homoclinic intersection associated to a saddle-node for every large k.

• *Laminations and admissible leaves in Theorems 3 and 4.* The proof of the stabilisation theorem involves a careful choice of an open subset $\mathcal{B} \subset \mathbb{R}^3$ and a local lamination $\mathcal{T}_{\mathcal{B}} := (\mathcal{T}_{\bar b})_{\bar b\in\mathcal{B}}$ whose leaves are sub-manifolds of $\mathcal{T}_{\mathrm{quad}}$. For that recall the maps $\Pi : \mathcal{T}_{\mathrm{quad}} \to \mathbb{R}^9$ in (4.2.1), and κ and η in (4.5.1). Consider the family of maps

$$\gamma_\xi : \Pi(\mathcal{T}_{\mathrm{quad}}) \to \mathbb{R}^2, \quad \gamma_\xi(v) = (\kappa(\xi, v), \eta(\xi, v)) \in \mathbb{R}^2 \quad (\xi \in \mathbb{R}).$$

For regular values (κ_0, η_0) of γ_ξ (we denote such set by $\mathrm{RV}(\gamma_\xi)$) we consider the sub-manifold $\gamma_\xi^{-1}(\kappa_0, \eta_0)$ of dimension seven in $\mathbb{R}^9$.

Recall the Hénon-like family $G_{(\xi,\mu,\kappa,\eta)}$ and take the open subset $\mathcal{O} = I \times J \times \mathcal{V} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ of parameters providing blender-horseshoes for this family in Theorem 1. Let $\mathcal{B} = I \times \mathcal{V}$, then we will see that for every ξ we have that $\mathrm{RV}(\gamma_\xi) \subset \mathcal{V}$ and hence the leaves $\mathcal{T}_{\bar b}$ of the lamination $\mathcal{T}_{\mathcal{B}}$ are given by

$$\mathcal{T}_{\bar b} : = \Pi^{-1}\Big(\gamma_\xi^{-1}(w)\Big), \quad \bar b : = (\xi, w) \in \mathcal{B}.$$

We also note that the sub-manifold $\mathcal{T}^{RC}$ of $\mathcal{T}_{\mathrm{quad}}$ in Theorem 3 is $\Pi^{-1}(\gamma_1^{-1}(0,0))$.

• *Homoclinic and heteroclinic relations leading to Theorem 4.* The stabilisation of the initial cycle is related to the existence of additional heteroclinic intersections between the saddles in the cycle. The existence of these heteroclinic intersections depends on the initial configuration the cycle, this leads to split $\mathcal{T}_{\mathrm{quad}}$ according to geometrical constrains such as the type of tangency (elliptic or hyperbolic contact) and the relative position of the invariant manifolds of the saddles. The choice of the sub-regions of $\mathcal{T}_{\mathrm{quad}}$ guarantees the following properties:

⋆ *New quasi-transverse orbits.* The new quasi-transverse orbits have a suitable unfolding independent of the renormalisation scheme.

⋆ *Robust cycle between the blender-horseshoe and the saddle Q in the cycle.* Consider the sequence of diffeomorphisms $f_{\bar v_{m,n}}$ and their corresponding horseshoe-blender $\Lambda_{m,n}$ in Theorem 4. Bifurcating a new quasi-transverse orbit

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(by a small C^r-perturbation) we obtain a uu-disc simultaneously contained in the unstable manifold of Q and in superposition region of blender $\Lambda_{m,n}$. This provides a robust intersection between the one-dimensional invariant manifolds of the saddle and the blender.

To generate the robust cycle we need to get a transverse intersection between the two-dimensional invariant manifolds of the saddle and the blender. The key property guaranteeing this intersection is obtained analysing the blender of the Hénon-like family. Let us give some additional details.

Consider $G_{(\xi,\mu,\kappa,\eta)}|_{\Delta}$. The properties of this family implies that if D is a uu-disc then $G_{(\xi,\mu,\kappa,\eta)}|_{\Delta}(D)$ contains a "uu-disc" of size close to $|D|^2$ (this claim is depicted in Figure 8.2). This assertion, in particular, holds for unstable manifolds of the references saddles of the blender. With this in mind we can take increasing domains $\Delta_{m,n} \supset \Delta$ such that, restricted to the set $\Delta_{m,n}$ and in the C^r-topology we have the following: let

$$\Phi_{m,n} = \Psi_{m,n} \circ \Theta_{(\varsigma_1,\varsigma_2,\varsigma_5)}$$

then

$$\Phi_{m,n}^{-1} \circ \mathcal{R}_{m,n}\Big(f_{\bar{\upsilon}_{m,n}}\Big) \circ \Phi_{m,n} \to G_{(\xi,\mu,\kappa,\eta)}.$$

and

$$\Phi_{m,n}^{-1}\Big(W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{\upsilon}_{m,n}})\Big) \pitchfork W^{\mathrm{u}}(P^*, G_{(\xi,\mu,\kappa,\eta)}|_{\Delta_{m,n}}) \neq \emptyset.$$

This implies that

$$\Phi_{m,n}^{-1}\Big(W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{\upsilon}_{m,n}})\Big) \pitchfork W^{\mathrm{u}}(P^*_{\bar{\upsilon}_{m,n}}, \Phi_{m,n}^{-1} \circ \mathcal{R}_{m,n}\Big(f_{\bar{\upsilon}_{m,n}}\Big) \circ \Phi_{m,n}|_{\Delta_{m,n}}) \neq \emptyset,$$

where $P^*_{\bar{\upsilon}_{m,n}}$ is the continuation of P^*. This completes our sketch of the generation of the robust cycle. To get the stabilisation of the cycle it remains to connect the saddle P and the blender homoclinically.

$\star$ *The blender-horseshoe and the saddle P are homoclinically related.* The cycle configuration implies that $W^{\mathrm{u}}(P, f_{\bar{\upsilon}_{m,n}})$ meets transversely $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{\upsilon}_{m,n}})$. The irrational argument of the saddle Q and the one-dimensional intersection of the robust cycle imply that $W^{\mathrm{s}}(P^*_{\bar{\upsilon}_{m,n}}, f_{\bar{\upsilon}_{m,n}})$ is dense in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{\upsilon}_{m,n}})$. From this we obtain that

$$W^{\mathrm{s}}(P^*_{\bar{\upsilon}_{m,n}}, f_{\bar{\upsilon}_{m,n}}) \pitchfork W^{\mathrm{u}}(P, f_{\bar{\upsilon}_{m,n}}) \neq \emptyset.$$

To obtain the remaining transverse intersection, we note that after of generation of the robust cycle there is "surviving" quasi-transverse heteroclinic orbit. Using this orbit and the irrational argument of saddle Q, we have that the

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stable manifold $W^{\mathrm{s}}(P, f_{\bar{v}_{m,n}})$ is dense in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{v}_{m,n}})$. On the other hand, using the transverse intersection of the robust cycle, we have $W^{\mathrm{u}}(P^{*}_{\bar{v}_{m,n}}, f_{\bar{v}_{m,n}})$ meets transversely $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{v}_{m,n}})$. Thus, we get that

$$W^{\mathrm{u}}(P^{*}_{\bar{v}_{m,n}}, f_{\bar{v}_{m,n}}) \pitchfork W^{\mathrm{s}}(P, f_{\bar{v}_{m,n}}) \neq \emptyset.$$

This completes the sketch of the construction.

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5
Blender-horseshoes in Hénon-like families: Proof of Theorem 1

In this section we prove Theorem 1. By the C^1-robustness of properties (BH1')-(BH6') in Definition 3.4.6 (recall Remark 3.4.3 and 3.4.7), it is sufficient to show the following theorem.

Theorem 5.0.1 *For every* $(\xi, \mu) \in (1.18, 1, 19) \times (-10, -9)$, *the Hénon like endomorphism*

$$G_{(\xi,\mu,0,0)}(x, y, z) = (y, \mu + y^2, \xi\, z + y)$$

has a blender-horseshoes in $\Delta := [-4, 4]^2 \times [-40, 22]$.

To prove this theorem we first investigate some properties of the endomorphisms $G_{\xi,\mu} := G_{(\xi,\mu,0,0)}$ on Δ to the parameters $(\xi, \mu) \in (1.18, 1.19) \times (-10, -9)$.

5.1
Properties of the Hénon like family

Let

$$\mathcal{P} := (1.18, 1.19) \times (-10, -9). \tag{5.1.1}$$

Lemma 5.1.1 (Hyperbolic fixed points) *For every* $(\xi, \mu) \in \mathcal{P}$, *the endomorphism* $G_{\xi,\mu}$ *has two hyperbolic fixed saddles* $P^{\pm}_{\xi,\mu} = (x^{\pm}_{\xi,\mu}, y^{\pm}_{\xi,\mu}, z^{\mp}_{\xi,\mu}) \in \Delta$ *where*

$$\begin{aligned} x^{\pm}_{\xi,\mu} = y^{\pm}_{\xi,\mu} &= \mu + (y^{\pm}_{\xi,\mu})^2 = (1-\xi)\, z^{\mp}_{\xi,\mu}, \\ y^{\pm}_{\xi,\mu} = y^{\pm}_{\mu} &:= \frac{1 \pm (1 - 4\,\mu)^{1/2}}{2}. \end{aligned} \tag{5.1.2}$$

Proof. The condition $\mu \in (-10, -9)$ implies that $-2.7 < y^{-}_{\mu} < -2.5$ and $3.5 < y^{+}_{\mu} < 3.71$. Thus, for every $(\xi, \mu) \in \mathcal{P}$ we get following estimates for $z^{\mp}_{\xi,\mu}$:

$$-20.6 < z^{-}_{\xi,\mu} = \frac{y^{+}_{\mu}}{(1-\xi)} < -18.4, \quad 13 < z^{+}_{\xi,\mu} = \frac{y^{-}_{\mu}}{(1-\xi)} < 15.$$

Therefore, $P^{\pm}_{\xi,\mu} \in \Delta$.

It remains to check the hyperbolicity of these points. Note that $\lambda^{\mathrm{s}} = 0$ and $\lambda^{\mathrm{c}} = \xi > 1$ are the eigenvalues of $DG_{\xi,\mu}(P^{\pm}_{\xi,\mu})$ associated to eigenspaces

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spanned by the vectors $(1,0,0)$ and $(0,0,1)$, respectively. The (strong) expanding eigenvalue of $DG_{\xi,\mu}(P^{\pm}_{\xi,\mu})$ is given by $\lambda^{\mathrm{uu}}(P^{\pm}_{\xi,\mu}) = 2\,y^{\pm}_{\mu}$ that is associated to the eigenvector

$$\Big(2\,y^{\pm}_{\mu} - \xi, 2\,y^{\pm}_{\mu}\,(y^{\pm}_{\mu} - \xi), 2\,y^{\pm}_{\mu}\Big).$$

Note that $|\lambda^{\mathrm{uu}}(P^{+}_{\xi,\mu})| = 2|\,y^{+}_{\mu}| > 7$ and $|\lambda^{\mathrm{uu}}(P^{-}_{\xi,\mu})| = 2|\,y^{-}_{\mu}| > 5$. This ends the proof of lemma. ■

Remark 5.1.2 (Invariant directions) Consider the straight lines

$$\Big\{(y^{\pm}_{\mu}, y^{\pm}_{\mu}, z^{\pm}_{\xi,\mu} + t) : t \in \mathbb{R}\Big\}, \quad \Big\{(y^{\pm}_{\mu} + t, y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu}) : t \in \mathbb{R}\Big\}.$$

These lines are are $G_{\xi,\mu}$-invariant, contains to $P^{\pm}_{\xi,\mu}$ in its interior, are tangent to $E^{\mathrm{c}}(P^{\pm}_{\xi,\mu}) := \{0\} \times \{0\} \times \mathbb{R}$ and $E^{\mathrm{s}}(P^{\pm}_{\xi,\mu}) := \mathbb{R} \times \{0\} \times \{0\}$ (respectively), and

$$\begin{aligned} G_{\xi,\mu}(y^{\pm}_{\mu}, y^{\pm}_{\mu}, z^{\pm}_{\xi,\mu} + t) &= (y^{\pm}_{\mu}, y^{\pm}_{\mu}, z^{\pm}_{\xi,\mu} + \xi\, t), \quad t \in \mathbb{R}, \\ G_{\xi,\mu}(y^{\pm}_{\mu} + t, y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu}) &= (y^{\pm}_{\mu}, y^{\pm}_{\mu}, z^{\mp}_{\xi,,\mu}), \quad t \in \mathbb{R}. \end{aligned} \tag{5.1.3}$$

We put

$$\begin{aligned} W^{\mathrm{c},\pm}_{\xi,\mu}(\Delta) &= \Big\{(y^{\pm}_{\mu}, y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu} + t) : t \in \mathbb{R}\Big\} \cap \Delta, \\ W^{\mathrm{s},\pm}_{\xi,\mu}(\Delta) &= \Big\{(y^{\pm}_{\mu} + t, y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu}) : t \in \mathbb{R}\Big\} \cap \Delta. \end{aligned} \tag{5.1.4}$$

Remark 5.1.3 (Invariant foliation) Consider $g_{\mu,\xi} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g_{\xi,\mu}(y,z) := \Pi_1 \circ G_{\xi,\mu}(x,y,z) = (\mu + y^2, \xi\, z + y). \tag{5.1.5}$$

Note that, for every $(\xi,\mu) \in \mathcal{P}$, the map $\Pi_1 \circ G_{\xi,\mu}(x,y,z)$ does not depend on x. Note that also $g_{\xi,\mu}$ preserves the vertical foliation $\Big\{\{y\} \times \mathbb{R} : y \in \mathbb{R}\Big\}$ of $\mathbb{R}^2$. In particular, the lines

$$W^{\mathrm{c},\pm}_{\xi,\mu} := \{(y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu} + t)\} : t \in \mathbb{R},$$

are invariant and $g_{\xi,\mu}(y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu} + t) = (y^{\pm}_{\mu}, z^{\mp}_{\xi,\mu} + \xi\, t)$.

5.2 Proof of Theorem 5.0.1

We now see that for every $(\xi,\mu) \in \mathcal{P}$ the endomorphism $G_{\xi,\mu}|_{\Delta}$ satisfies the Definition 3.4.6. The proof of these properties is organised as follows. Conditions (BH1') and (BH3') follow from Lemmas 5.2.1 and 5.2.5. Condition (BH2') is given in Lemma 5.2.7. Condition (BH5') is given in Lemma 5.2.11

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and condition (BH6') in Lemma 5.2.14. Finally, condition (BH4') follows from Remark 5.2.15.

Lemma 5.2.1 (Condition (BH1')) *For every $(\xi, \mu) \in \mathcal{P}$ there is two connected components $\mathcal{A}_{\xi,\mu}$ and $\mathcal{B}_{\xi,\mu}$ in Δ such that $G_{\xi,\mu}(\mathcal{A}_{\xi,\mu})$ and $G_{\xi,\mu}(\mathcal{B}_{\xi,\mu})$ are the connected component of the intersection $G_{\xi,\mu}(\Delta) \cap \Delta$. Moreover, it holds that*

$$G_{\xi,\mu}(\mathcal{A}_{\xi,\mu}) \cup G_{\xi,\mu}(\mathcal{B}_{\xi,\mu}) \subset (-4,4) \times [-4,4] \times \mathbb{R}, \quad (\mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}) \cap \partial^{\mathrm{uu}}\Delta = \emptyset.$$

Proof. Let $\Pi_3, \Pi_1 : \mathbb{R}^3 \to \mathbb{R}^2$ be the projections

$$\Pi_3(x, y, z) = (x, y), \quad \Pi_1(x, y, z) = (y, z).$$

Note that

$$\Pi_3\Big(G_{\xi,\mu}(\Delta)\Big) := \Big\{(y, \mu + y^2) : |y| < 4\Big\},$$

and that $\Pi_3\Big(G_{\xi,\mu}(\Delta)\Big) \cap [-4,4]^2$ consists of the two curves

$$\begin{aligned} \ell_{+,\mu} &:= \Big\{(y, \mu + y^2) : y \in [y^-_{+,\mu}, y^+_{+,\mu}]\Big\}, \\ \ell_{-,\mu} &:= \Big\{(y, \mu + y^2) : y \in [y^-_{-,\mu}, y^+_{-,\mu}]\Big\}, \end{aligned} \tag{5.2.1}$$

where

$$\begin{aligned} y^-_{+,\mu} &:= \sqrt{-4-\mu}, \quad y^+_{+,\mu} := \sqrt{4-\mu}, \\ y^-_{-,\mu} &:= -\sqrt{4-\mu}, \quad y^+_{-,\mu} := -\sqrt{-4-\mu}. \end{aligned} \tag{5.2.2}$$

Thus, we get that

$$G_{\xi,\mu}(\Delta) \cap [-4,4]^2 = (\ell_{+,\mu} \cup \ell_{-,\mu}) \times [-40, 22].$$

Condition $\mu \in (-10, -9)$ gives the following estimates to $y^{\pm}_{\mp,\mu}$:

$$\begin{aligned} -\sqrt{14} < y^-_{-,\mu} < -\sqrt{13}, &\quad -\sqrt{6} < y^+_{-,\mu} < -\sqrt{5}, \\ \sqrt{5} < y^-_{+,\mu} < \sqrt{6}, &\quad \sqrt{13} < y^+_{+,\mu} < \sqrt{14}. \end{aligned} \tag{5.2.3}$$

This imply that $\ell_{+,\mu} \cup \ell_{-,\mu} \subset (-4,4) \times [-4,4]$.

We now consider the following subsets in Δ:

$$\begin{aligned} \mathcal{A}_{\xi,\mu} = \mathcal{A}_\mu &:= [-4,4] \times [y^-_{+,\mu}, y^+_{+,\mu}] \times [-40, 22], \\ \mathcal{B}_{\xi,\mu} = \mathcal{B}_\mu &:= [-4,4] \times [y^-_{-,\mu}, y^+_{-,\mu}] \times [-40, 22]. \end{aligned}$$

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Then, for every $(\xi, \mu) \in \mathcal{P}$ we have that

$$G_{\xi,\mu}(\mathcal{A}_\mu) \cup G_{\xi,\mu}(\mathcal{B}_\mu) \subset (-4,4) \times [-4,4] \times \mathbb{R}, \quad (\mathcal{A}_\mu \cup \mathcal{B}_\mu) \cap \partial^{\mathrm{uu}}\Delta = \emptyset.$$

This completes the lemma. ■

Remark 5.2.2 Observe that for every $\mu \in (-10,-9)$, we have that $y_\mu^+ \in (y_{+,\mu}^-, y_{+,\mu}^+)$ and $y_\mu^- \in (y_{-,\mu}^-, y_{-,\mu}^+)$.

Scholium 5.2.3 We investigate a little bit thoroughly intersection $G_{\xi,\mu}(\Delta) \cap \Delta$. Roughly speaking, the next claim assert that for every $(\xi, \mu) \in \mathcal{P}$ the intersection $\Delta \backslash \Big(G_{\xi,\mu}(\mathcal{A}_{\xi,\mu}) \cup G_{\xi,\mu}(\mathcal{B}_{\xi,\mu})\Big)$ consist of three connected components. This will be used to obtain the condition (BH3') to $G_{\xi,\mu}$.

Claim 5.2.4 (Covering property) *Consider the sub sets*

$$\Delta^+ := \Delta \cap \{y > 0\} \quad \textit{and} \quad \Delta^- := \Delta \cap \{y < 0\}.$$

Then, for every $(\xi, \mu) \in \mathcal{P}$ it holds that

$$\Pi_1\Big(\Delta\Big) \subset \Pi_1\Big(G_{\xi,\mu}(\Delta^+)\Big) \cap \Pi_1\Big(G_{\xi,\mu}(\Delta^-)\Big).$$

Proof.

Recall the definition of $g_{\xi,\mu}$ in (5.1.5). Note that $\Pi_1\Big(\Delta\Big) = [-4,4] \times [-40,22]$. We calculate

$$g_{\xi,\mu}\Big(\partial([-4,4] \times [-40,22])\Big).$$

Observe that $g_{\xi,\mu}$ maps the lines $\{0\} \times [-40,22]$ and $\{4\} \times [-40,22]$ respectively in

$$\{\mu\} \times [-40\,\xi, 22\,\xi] \quad \text{and} \quad \{\mu+16\} \times [-40\,\xi+4, 22\,\xi+4].$$

Conditions $(\xi, \mu) \in \mathcal{P}$ imply that

$$6 < \mu{+}16 < 7, \quad -40\,\xi < -40, \quad 22 < 22\,\xi, \quad -40\,\xi{+}4 < -40, \quad 22 < 22\,\xi{+}4.$$

On the other hand, $g_{\xi,\mu}([0,4] \times \{22\})$ and $g_{\xi,\mu}([0,4] \times \{-40\})$ are contained (respectively) in

$$[\mu, \mu+16] \times [22\,\xi, 22\,\xi+4], \quad \text{and} \quad [\mu, \mu+16] \times [-40\,\xi, -40\,\xi+4].$$

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This imply that $\Pi_1\big(\Delta\big) \subset \Pi_1\Big(G_{\xi,\mu}(\Delta^+)\Big)$. Let us now see that $\Pi_1\big(\Delta\big) \subset \Pi_1\Big(G_{\xi,\mu}(\Delta^-)\Big)$. Analogously to the previous case, we have that

$$g_{\xi,\mu}\Big(\{-4\} \times [-40, 22]\Big) = \{\mu + 16\} \times [-40\,\xi - 4, 22\,\xi - 4]$$

and

$$\begin{aligned} g_{\xi,\mu}\Big([-4, 0] \times \{22\}\Big) &\subset [\mu, \mu + 16] \times [22\,\xi - 4, 22\,\xi], \\ g_{\xi,\mu}\Big([-4, 0] \times \{-40\}\Big) &\subset [\mu, \mu + 16] \times [-40\,\xi - 4, -40\,\xi]. \end{aligned}$$

Since $\xi \in (1.18, 1.19)$ we have that $22 < 22\,\xi - 4$. This implies that $\Pi_1\big(\Delta\big) \subset \Pi_1\Big(G_{\xi,\mu}(\Delta^-)\Big)$, ending the proof of the claim. ■ Recall the definitions of $\ell_{\pm,\mu}$ in (5.2.1) and consider the sets

$$A'_\mu = \ell_{+,\mu} \times [-40, 22] \quad \text{and} \quad B'_\mu = \ell_{-,\mu} \times [-40, 22]. \tag{5.2.4}$$

Then for every $(\xi, \mu) \in \mathcal{P}$ it holds

$$G_{\xi,\mu}(\Delta) \cap \Delta = G_{\xi,\mu}(\mathcal{A}_{\xi,\mu}) \cup G_{\xi,\mu}(\mathcal{B}_{\xi,\mu}) = A'_\mu \cup B'_\mu,$$

ending this completes the scholium.

Lemma 5.2.5 (Condition (BH3')) *Let $\mathcal{A}_{\xi,\mu}$ and $\mathcal{B}_{\xi,\mu}$ be as Lemma 5.2.5. Then, for every $(\xi, \mu) \in \mathcal{P}$ there exist parallelepipeds*

$$\mathbb{A}_{\xi,\mu} \subset G^{-1}_{\xi,\mu}\Big(G_{\xi,\mu}(\mathcal{A}_{\xi,\mu}) \cap \Delta\Big) \quad \textit{and} \quad \mathbb{B}_{\xi,\mu} \subset G^{-1}_{\xi,\mu}\Big(G_{\xi,\mu}(\mathcal{B}_{\xi,\mu}) \cap \Delta\Big),$$

such that

$$\begin{aligned} \mathbb{A}_{\xi,\mu} \subset \mathcal{A}_{\xi,\mu}, \quad G_{\xi,\mu}(\mathbb{A}_{\xi,\mu}) &= G_{\xi,\mu}(\mathcal{A}_{\xi,\mu}) \cap \Delta, \\ \mathbb{B}_{\xi,\mu} \subset \mathcal{B}_{\xi,\mu}, \quad G_{\xi,\mu}(\mathbb{B}_{\xi,\mu}) &= G_{\xi,\mu}(\mathcal{B}_{\xi,\mu}) \cap \Delta \end{aligned}$$

and

$$\Big(\mathbb{A}_{\xi,\mu} \cup \mathbb{B}_{\xi,\mu}\Big) \cap \partial^{\mathrm{uu}}\Delta = \emptyset.$$

Proof. Recall the terms $y^{\pm}_{\mp,\mu}$ in (5.2.2). Consider the parallelepiped $\mathbb{A}'_{\xi,\mu} \subset \mathbb{R}^2$ whose boundary consists in the following four curves:

$$\begin{aligned} \ell^1_{\xi,\mu} &:= \Big\{\Big(y, \xi^{-1}\,(22 - y)\Big) : y \in [y^-_{+,\mu}, y^+_{+,\mu}]\Big\}, \\ \ell^2_{\xi,\mu} &:= \Big\{\Big(y, \xi^{-1}\,(-40 - y)\Big) : y \in [y^-_{+,\mu}, y^+_{+,\mu}]\Big\}, \\ \ell^3_{\xi,\mu} &:= \Big\{y^-_{+,\mu}\Big\} \times [\xi^{-1}\,(-40 - y^-_{+,\mu}), \xi^{-1}\,(22 - y^-_{+,\mu})], \\ \ell^4_{\xi,\mu} &:= \Big\{y^+_{+,\mu}\Big\} \times [\xi^{-1}\,(-40 - y^+_{+,\mu}), \xi^{-1}\,(22 - y^+_{+,\mu})]. \end{aligned}$$

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Thus, for every $(\xi,\mu)\in\mathcal{P}$ we have that $\mathbb{A}'_{\xi,\mu}\subset(-4,4)\times(-40,22)$ and

$$g_{\xi,\mu}\Big(\ell^1_{\xi,\mu}\cup\ell^2_{\xi,\mu}\cup\ell^3_{\xi,\mu}\cup\ell^4_{\xi,\mu}\Big)=\partial\Big([-4,4]\times[-40,22]\Big).$$

Then, $g_{\xi,\mu}(\mathbb{A}'_{\xi,\mu})=[-4,4]\times[-40,22]$. Thus, recalling the set A'_μ in (5.2.4) we get that for every $(\mu,\xi)\in\mathcal{P}$ the set $\mathbb{A}_{\xi,\mu}:=[-4,4]\times\mathbb{A}'_{\xi,\mu}$ defines a parallelepiped in Δ such that $G_{\xi,\mu}(\mathbb{A}_{\xi,\mu})=A'_\mu$.

Analogously, we consider the parallelepiped $\mathbb{B}'_{\xi,\mu}$ bounded by the curves

$$\begin{aligned}
\tilde\ell^1_{\xi,\mu}&:=\Big\{\Big(y,\xi^{-1}(22-y)\Big):y\in[y^-_{-,\mu},y^+_{-,\mu}]\Big\},\\
\tilde\ell^2_{\xi,\mu}&:=\Big\{\Big(y,\xi^{-1}(-40-y)\Big):y\in[y^-_{-,\mu},y^+_{-,\mu}]\Big\},\\
\tilde\ell^3_{\xi,\mu}&:=\{y^-_{-,\mu}\}\times[\xi^{-1}(-40-y^+_{-,\mu}),\xi^{-1}(22-y^+_{-,\mu})],\\
\tilde\ell^4_{\xi,\mu}&:=\{y^+_{-,\mu}\}\times[\xi^{-1}(-40-y^-_{-,\mu}),\xi^{-1}(22-y^-_{-,\mu})],
\end{aligned}$$

and recall the set B'_μ in (5.2.4). Then for every $(\xi,\mu)\in\mathcal{P}$ it holds

$$g_{\xi,\mu}(\mathbb{B}'_{\xi,\mu})=[-4,4]\times[-40,22],\quad \mathbb{B}'_{\mu,\xi}\subset(-4,4)\times(-40,22)$$

and

$$G_{\xi,\mu}(\mathbb{B}_{\xi,\mu})=B'_\mu,\quad \mathbb{B}_{\xi,\mu}=[-4,4]\times\mathbb{B}'_{\xi,\mu}.$$

This completes the proof of lemma. ■

Remark 5.2.6 Note that $\xi^{-1}(-40-y^-_{-,\mu})<z^-_{\xi,\mu}$ and $z^+_{\xi,\mu}<\xi^{-1}(22-y^+_{+,\mu})$. These conditions imply that the projection into the z-coordinate of the union $\ell^4_{\xi,\mu}\cup\tilde\ell^3_{\xi,\mu}$ covers the interval $[z^-_{\xi,\mu},z^+_{\xi,\mu}]$. See Figure 5.1.

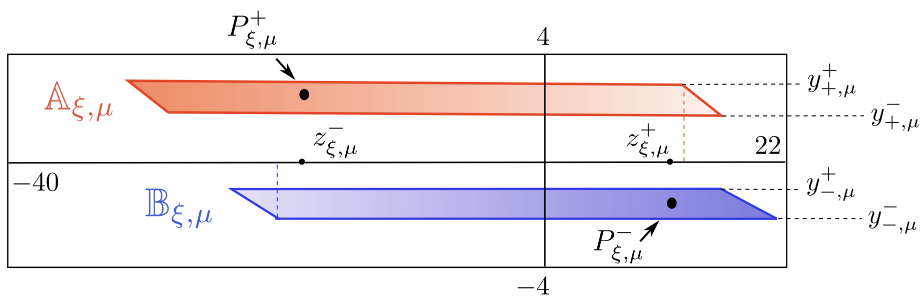

Figure 5.1: Projection of Markov partition of $G_{\xi,\mu}|_\Delta$

Lemma 5.2.7 (Condition (BH2)) *For every $(\xi,\mu)\in\mathcal{P}$ the endomorphism $G_{\xi,\mu}$ satisfies the following properties:*

(i) If $p\in\mathcal{A}_{\xi,\mu}\cup\mathcal{B}_{\xi,\mu}$ then $\mathcal{C}^s_2\Big(G_{\xi,\mu}(p)\Big)\subset D(G_{\xi,\mu})_p\Big(\mathcal{C}^s_2(p)\Big)$.

(ii) If $p \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ *then* $D(G_{\xi,\mu})_p\Big(\mathcal{C}^{\mathrm{u}}_2(p)\Big) \subset \mathcal{C}^{\mathrm{u}}_2\Big(G_{\xi,\mu}(p)\Big)$.

(iii) If $p \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ *then* $D(G_{\xi,\mu})_p\Big(\mathcal{C}^{\mathrm{uu}}_2(p)\Big) \subset \mathcal{C}^{\mathrm{uu}}_2\Big(G_{\xi,\mu}(p)\Big)$.

(iv) $DF|_{\mathcal{C}^{\mathrm{s}}_2}$ *is uniformly contracting and* $DF|_{\mathcal{C}^{\mathrm{u}}_2}$ *is uniformly expanding.*

Proof. For $p = (x, y, z) \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ we have that $G_{\xi,\mu}(p) \in A'_\mu \cup B'_\mu$. We put $\mathbf{v} = (u, v, w) \in T_p\Delta$ and

$$(u_1, v_1, w_1) := D(G_{\xi,\mu})_p\mathbf{v} = (v, 2yv, v + \xi\, w).$$

Next claim asserts that the derivative $D(G_{\xi,\mu})_p$ "opens" the cones C^{s} and "closes" the cones C^{u}.

Claim 5.2.8 (Items (i)-(ii)) *Let* $p = (x, y, z) \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ *and* $\boldsymbol{v} = (u, v, w) \in T_p\Delta \setminus \mathrm{spanned}\{(1, 0, 0)\}$ *then* $D(G_{\xi,\mu})_p\boldsymbol{v} \in C^{\mathrm{u}}\Big(G_{\xi,\mu}(p)\Big)$.

Proof. Note that if $p = (x, y, z) \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ then $|y| > \sqrt{5}$, see (5.2.3). Thus

$$\sqrt{v_1^2 + w_1^2} \geq |v_1^2| = 2|y||v| > 2\sqrt{5}|v| > 2|u_1|,$$

proving the claim. ∎

Claim 5.2.9 (Item (iii)) *Let* $p \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ *and* $\boldsymbol{v} \in \mathcal{C}^{\mathrm{uu}}(p)$. *Then* $D(G_{\xi,\mu})_p(p)\boldsymbol{v} \in \mathcal{C}^{\mathrm{uu}}\Big(G_{\xi,\mu}(p)\Big)$.

Proof. We need to check that if $\mathbf{v} = (u, v, w)$ and $D(G_{\xi,\mu})_p(p)\mathbf{v} = (u_1, v_1, w_1)$ we have that

$$\sqrt{u^2 + w^2} < \frac{1}{2}|v| \quad \Rightarrow \quad \sqrt{u_1^2 + w_1^2} < \frac{1}{2}|v_1|$$

Note that $\sqrt{u^2 + w^2} < \frac{1}{2}|v|$ implies that $|w| < \frac{1}{2}|v|$ and hence

$$u_1^2 + w_1^2 = v^2 + (v + \xi\, w)^2 \leq 2\, v^2 + 2\,\xi\, |v||w| + \xi^2\, |w|^2 \leq \left(2 + \xi + \left(\frac{\xi}{2}\right)^2\right) v^2.$$

Note that the condition $\xi \in (1.18, 1.19)$ implies that

$$\left(2 + \frac{\xi}{2} + \left(\frac{\xi}{2}\right)^2\right) < 4$$

and that $p = (x, y, z) \in \mathcal{A}_{\xi,\mu} \cup \mathcal{B}_{\xi,\mu}$ implies that $|y| > \sqrt{5}$. Thus

$$2\sqrt{u_1^2 + w_1^2} < 4|v| < 2|y||v| = |v_1|,$$

proving the claim. ∎

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Claim 5.2.10 (Item (iv)) *$DG_{\xi,\mu}|_{\mathcal{C}^s}$ is uniformly contracting and $DG_{\xi,\mu}|_{\mathcal{C}^s}$ is uniformly expanding.*

Proof.

The uniform contraction of the cone field $\mathcal{C}^s$ follows from the fact that $D(G_{\xi,\mu})_p$ is an endomorphism whose eigenspace associated the eigenvalue 0 is spanned by $(1,0,0)$.

To study uniform expansion of $D(G_{\xi,\mu})\mathbf{v}$ we consider the norm

$$|(u,v,w)|_* := \max\left\{|u|, \sqrt{v^2+w^2}\right\}.$$

Take $\mathbf{v} = (u,v,w) \in \mathcal{C}_2^{\mathrm{u}}(p)$ and write $D(G_{\xi,\mu})_p\mathbf{v} = (u_1,v_1,w_1) = (v, 2yv, v + \xi w)$. We will check that if $\mathbf{v} \in \mathcal{C}_2^{\mathrm{u}}(p)$ then $|(D(G_{\xi,\mu})_p\mathbf{v}|_* > |\mathbf{v}|_*$. By compactness this implies that $|(D(G_{\xi,\mu})_p\mathbf{v}|_* > c_0|\mathbf{v}|_*$, for some uniform $c_0 > 1$. We divide the proof into two cases: $6.5|v| \geq |w|$ and $6.5|v| \leq |w|$.

Note that for $\mathbf{v} = (u,v,w) \in \mathcal{C}_2^{\mathrm{u}}(p)$ we have $|\mathbf{v}|_* = \sqrt{v^2+w^2}$ and

$$v_1^2 + w_1^2 = 4v^2y^2 + (v + \xi\, w)^2 \geq 4v^2y^2 + v^2 - 2\xi|v||w| + \xi^2 w^2.$$

Then $6.5|v| \geq |w|$ and using the conditions $\xi \in (1.18, 1.19)$ and $|y| > \sqrt{5}$ we get that

$$4v^2y^2 - 2\xi|v||w| > (20 - 13\xi)\, v^2 > 4v^2 \geq 0.$$

Therefore, $|(D(G_{\xi,\mu})_p\mathbf{v}|_* > |\mathbf{v}|_*$, completing the first case.

Similarly, in the case $6.5|v| \leq |w|$ we have

$$v_1^2 + w_1^2 \geq 4y^2v^2 + \xi^2\, w^2 - 2\,\xi\,|v||w| + v^2 > 4y^2v^2 + \xi^2\, w^2 - 2\,\xi\,(6.5)^{-1}\, w^2 + v^2.$$

The condition $\xi \in (1.18, 1.19)$ implies that

$$\xi^2 - 2\,\xi\,(6.5)^{-1} > 1.$$

Hence, $|(D(G_{\xi,\mu})_p\mathbf{v}|_* > |\mathbf{v}|_*$ ending the proof of the claim. ■ This completes the proof of the lemma. ■

Lemma 5.2.11 (Condition (BH5')) *Let ℓ be a* uu-*disc in Δ. For every $(\xi,\mu) \in \mathcal{P}$ consider the discs*

$$\ell_{\mathcal{A}_{\xi,\mu}} := \ell \cap \mathcal{A}_{\xi,\mu} \quad \textit{and} \quad \ell_{\mathcal{B}_{\xi,\mu}} := \ell \cap \mathcal{B}_{\xi,\mu}.$$

Then, for every $(\xi,\mu) \in \mathcal{P}$, the curves $G_{\xi,\mu}\Big(\ell_{\mathcal{A}_{\xi,\mu}}\Big)$ and $G_{\xi,\mu}\Big(\ell_{\mathcal{B}_{\xi,\mu}}\Big)$ are uu-*discs satisfying condition (BH5').*

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Proof.

We extend the cone field $\mathcal{C}^{\mathrm{uu}}$ to the set $\Delta_\infty := [-4,4]^2 \times \mathbb{R}$ by

$$\mathcal{C}^{\mathrm{uu}}(p) := \left\{(u,v,w) \in \mathbb{R}^3 : \sqrt{u^2+w^2} < \frac{1}{2}|v|\right\}, \quad p \in \Delta_\infty.$$

We begin with the following claim.

Claim 5.2.12 *Let ℓ be a* uu-*disc in Δ_∞. Then $G_{\xi,\mu}(\ell)$ contains two* uu-*discs in Δ_∞.*

Proof. Define the subsets $\mathbb{A}_{\infty,\mu}$ and $\mathbb{B}_{\infty,\mu}$ of Δ_∞ by

$$\mathbb{A}_{\infty,\mu} := [-4,4] \times [y^-_{+,\mu}, y^+_{+,\mu}] \times \mathbb{R}, \quad \mathbb{B}_{\infty,\mu} := [-4,4] \times [y^-_{-,\mu}, y^+_{-,\mu}] \times \mathbb{R}.$$

Consider a parameterised C^1-curve $\gamma : [-4,4] \to \Delta_\infty$, $\gamma(t) = \big(\gamma_1(t), t, \gamma_3(t)\big)$, contained in the cone field $\mathcal{C}^{\mathrm{uu}}|_{\Delta_\infty}$, thus

$$\sqrt{\gamma_1'(t)^2 + \gamma_3'(t)^2} < 1/2.$$

Recalling that $|y^{\pm}_{\mp,\mu}| > \sqrt{5}$ in (5.2.3) and noting that for every $\xi \in (1.18, 1.19)$ it holds that

$$\sqrt{1 + \left(1 + \frac{\xi}{2}\right)^2} < \sqrt{5}.$$

Thus, for every t in $[y^-_{+,\mu}, y^+_{+,\mu}] \cup [y^-_{-,\mu}, y^+_{-,\mu}]$, we have that

$$\sqrt{1 + \left(1 + \frac{\xi}{2}\right)^2} < |t|.$$

The claim follows from the following subclaim.

Sub-claim 5.2.13 *The curves $G_{\xi,\mu}\Big(\mathbb{A}_{\infty,\mu} \cap \gamma([-4,4])\Big)$ and $G_{\xi,\mu}\Big(\mathbb{B}_{\infty,\mu} \cap \gamma([-4,4])\Big)$ are* uu-*disk in Δ_∞.*

Proof. Note that the curves $G_{\xi,\mu}\Big(\mathbb{A}_{\infty,\mu} \cap \gamma([-4,4])\Big)$ and $G_{\xi,\mu}\Big(\mathbb{B}_{\infty,\mu} \cap \gamma([-4,4])\Big)$ are respectively parameterised by

$$\gamma_{\mathcal{A}_{\xi,\mu}} : [y^-_{+,\mu}, y^+_{+,\mu}] \to \Delta_\infty, \quad \gamma_A(t) := G_{\xi,\mu} \circ \gamma(t) = (t, \mu + t^2, \xi\gamma_3(t) + t),$$
$$\gamma_{\mathcal{B}_{\xi,\mu}} : [y^-_{+,\mu}, y^+_{+,\mu}] \to \Delta_\infty, \quad \gamma_B(t) := G_{\xi,\mu} \circ \gamma(t) = (t, \mu + t^2, \xi\gamma_3(t) + t).$$

We now check that $\gamma_{\mathcal{A}_{\xi,\mu}}([y^-_{+,\mu}, y^+_{+,\mu}])$ and $\gamma_{\mathcal{B}_{\xi,\mu}}([y^-_{-,\mu}, y^+_{-,\mu}])$ are tangent to $\mathcal{C}^{\mathrm{uu}}|_{\Delta_\infty}$. Note that

$$(G_{\xi,\mu} \circ \gamma)'(t) = (1, 2\,t, \xi\gamma_3'(t) + 1).$$

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Thus, for every t in $[y^{-}_{+,\mu}, y^{+}_{+,\mu}]$, we have that

$$\gamma'_{\mathcal{A}_{\xi,\mu}}(t) \in \mathcal{C}^{\mathrm{uu}}(\gamma(t)) \quad \text{if, and only if,} \quad \sqrt{1 + (\xi\gamma'_3(t) + 1)^2} < |t|.$$

Since $\gamma'(t) \in \mathcal{C}^{\mathrm{uu}}(\gamma(t))$ for every $t \in [-4, 4]$, we have that

$$|\gamma'_3(t)| \leq \sqrt{(\gamma'_1(t))^2 + (\gamma'_3(t))^2} < \frac{1}{2}.$$

Therefore, for every $\xi \in (1.18, 1, 19)$ and every $t \in [y^{-}_{+,\mu}, y^{+}_{+,\mu}]$ follows that

$$\sqrt{1 + (\xi\gamma'_3(t) + 1)^2} < \sqrt{1 + \left(\frac{\xi}{2} + 1\right)^2} < \sqrt{5} < |t|.$$

Therefore, $\gamma_{\mathcal{A}_{\xi,\mu}}$ is a uu-disk. Note that the above also applies to the curve $\gamma_{\mathcal{B}_{\xi,\mu}}$. This ends the proof of the sub-claim. ■ The proof of the claim is now complete. ■

To complete the proof of Lemma 5.2.11, it is enough to study the dynamics of $G_{\xi,\mu}$ along the central directions of the saddles $P^{\pm}_{\xi,\mu}$ in Lemma 5.1.1. We now focus on the sets contained in Δ.

From Remark 5.1.2 it follows that $G_{\xi,\mu}$ restrict to these central directions is given by the the multiplication by $\xi > 1$. Recall the definition of the sets $W^{\mathrm{c},\pm}_{\xi,\mu}$ in (5.1.4). We denote by $\phi^{\pm}_{\xi,\mu}$ the restrictions $G_{\xi,\mu}|_{W^{\mathrm{c},\pm}_{\xi,\mu}}$. The corresponding iterated function system $\{\phi^{+}_{\xi,\mu}, \phi^{-}_{\xi,\mu}\}$ in $\mathrm{I} = [-40, 22]$ is given by

$$\phi^{+}_{\xi,\mu}(z) = \xi\, z + (1 - \xi)\, z^{+}_{\xi,\mu}, \quad \phi^{-}_{\xi,\mu}(z) = \xi\, z + (1 - \xi)\, z^{-}_{\xi,\mu}.$$

Consider the points

$$\begin{aligned} a^{+}_{\xi,\mu} &= \xi^{-1}(-40 - (1-\xi)z^{+}_{\mu,\xi}), \quad b^{+}_{\mu,\xi} = \xi^{-1}(22 - (1-\xi)z^{+}_{\mu,\xi}), \\ a^{-}_{\xi,\mu} &= \xi^{-1}(-40 - (1-\xi)z^{-}_{\mu,\xi}), \quad b^{-}_{\mu,\xi} = \xi^{-1}(22 - (1-\xi)z^{-}_{\mu,\xi}). \end{aligned}$$

Note that the intervals

$$\mathrm{I}^{+}_{\xi,\mu} := [a^{+}_{\mu,\xi}, b^{+}_{\mu,\xi}] \quad \text{and} \quad \mathrm{I}^{-}_{\xi,\mu} := [a^{-}_{\mu,\xi}, b^{-}_{\mu,\xi}]$$

satisfy $\phi^{+}_{\xi,\mu}(\mathrm{I}^{\pm}_{\xi,\mu}) = \mathrm{I}$, see Figure 5.2.

Moreover, since

$$z^{\mp}_{\xi,\mu}(1 - \xi) = y^{\pm}_{\mu}, \quad y^{+}_{\mu} \in (y^{-}_{+,\mu}, y^{+}_{+,\mu}), \quad y^{-}_{\mu} \in (y^{-}_{-,\mu}, y^{+}_{-,\mu})$$

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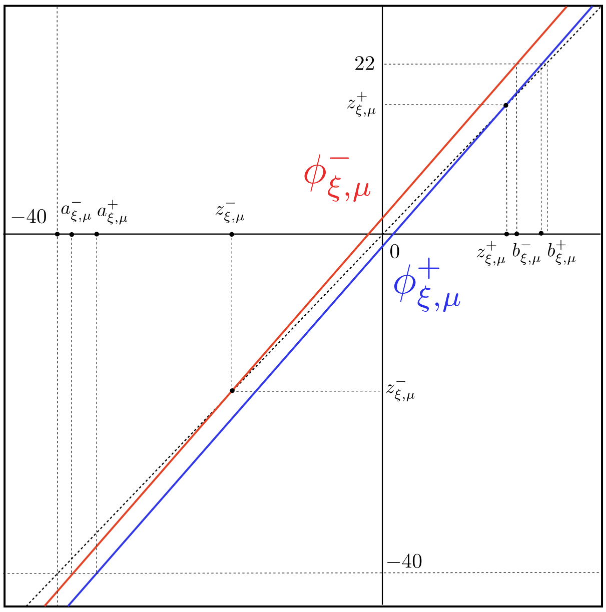

Figure 5.2: Iterated function system

it follows from Remark 5.2.6 that

$$a^+_{\xi,\mu} = \xi^{-1}(-40 - (1-\xi)z^+_{\xi,\mu}) = \xi^{-1}(-40 - y^+_\mu) < \xi^{-1}(-40 - y^-_{-,\mu}) < z^-_{\xi,\mu},$$
$$z^+_{\xi,\mu} < \xi^{-1}(22 - y^+_{+,\mu}) < \xi^{-1}(22 - y^+_\mu) = \xi^{-1}(22 - (1-\xi)z^-_{\xi,\mu}) = b^-_{\xi,\mu}.$$

Therefore, $[z^-_{\xi,\mu}, z^+_{\xi,\mu}] \subset \mathrm{I}^+_{\xi,\mu} \cap \mathrm{I}^-_{\xi,\mu}$.

Using the iterated function system $\phi^\pm_{\xi,\mu}$ it is easy to see that if ℓ is a uu-disc in Δ then the discs $G_{\xi,\mu}(\ell_{\mathcal{A}_{\xi,\mu}})$ and $G_{\xi,\mu}(\ell_{\mathcal{B}_{\xi,\mu}})$ satisfy the possibilities in (BH5'). This completes the proof of the lemma. ■

Recalling the definitions of the sets $W^{s,\pm}_{\xi,\mu}(\Delta)$ in (5.1.4) the following lemma implies condition (BH6').

Lemma 5.2.14 (Region of superposition, (BH6')) *Consider ℓ a* uu-*disk in between $W^{s,+}_{\xi,\mu}(\Delta)$ and $W^{s,-}_{\xi,\mu}(\Delta)$. Then either $G_{\xi,\mu}(\ell_{\mathcal{A}_{\xi,\mu}})$ or $G_{\xi,\mu}(\ell_{\mathcal{B}_{\xi,\mu}})$ is a* uu-*disc in between $W^{s,+}_{\xi,\mu}(\Delta)$ and $W^{s,-}_{\xi,\mu}(\Delta)$.*

Proof. This follows immediately from $[z^-_{\xi,\mu}, z^+_{\xi,\mu}] \subset \mathrm{I}^+_{\xi,\mu} \cap \mathrm{I}^-_{\xi,\mu}$. ■

Remark 5.2.15 (Condition (BH4')) Let $\mathrm{H}^b_{\xi,\mu}$ the region of superposition associated to $G_{\xi,\mu}|_\Delta$ in Lemma 5.2.14. Note that one has that

$$\overline{\bigcup_{D\in \mathrm{H}^b_{\xi,\mu}} D} \cap \left(\overline{\partial^{\mathrm{u}}\Delta \setminus \partial^{\mathrm{uu}}\Delta}\right) = \emptyset.$$

This immediately implies condition (BH4').

This completes the proof of Theorem 5.0.1.

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5.3 Renormalisation scheme: Proof of Theorem 2

In this section we describe the renormalisation scheme associated to our bifurcation setting and prove Theorem 2. Sections 5.3.1 and 5.3.2 are dedicated to the local (dynamics at the saddle points) and semi-local (transition maps) aspects of the dynamics of the cycle. In Section 5.3.3 we describe the types of unfolding of the cycle that we consider. Finally, in Section 5.3.4 we prove the convergence of the renormalisation scheme ending the proof or the theorem.

5.3.1 Local dynamics at the saddle points

Without loss of generality, let us assume that P and Q are fixed points.

(L) We assume the existence of local C^r-linearised charts at the saddles P and Q, $U_P \simeq [-3,3]^3$ and $U_Q \simeq [-3,3]^3$, (the saddles are identified with the respective origins) such that the expression of f in these neighbourhoods is of the form

$$
\begin{aligned}
f|_{U_P} &= \begin{pmatrix} \lambda_P & 0 & 0 \\ 0 & \sigma_P\cos(2\pi\varphi_P) & -\sigma_P\sin(2\pi\varphi_P) \\ 0 & \sigma_P\sin(2\pi\varphi_P) & \sigma_P\cos(2\pi\varphi_P) \end{pmatrix}, \quad \text{and} \\
f|_{U_Q} &= \begin{pmatrix} \lambda_Q\cos(2\pi\varphi_Q) & 0 & -\lambda_Q\sin(2\pi\varphi_Q) \\ 0 & \sigma_Q & 0 \\ \lambda_Q\sin(2\pi\varphi_Q) & 0 & \lambda_Q\cos(2\pi\varphi_Q) \end{pmatrix},
\end{aligned}
\tag{5.3.1}
$$

where $\lambda_P, \lambda_Q, \sigma_P, \sigma_Q \in \mathbb{R}$ and $\varphi_P, \varphi_Q \in (0,1)$ are such that

$$
0 < |\lambda_P|, |\lambda_Q| < 1 < |\sigma_P|, |\sigma_Q| \quad \text{and} \quad \varphi_P \neq \varphi_Q.
$$

In this case, we say that *local dynamic* of f at U_P and U_Q (or shortly *local dynamic* of f) is of type $(\lambda_P, \sigma_P, \varphi_P, \lambda_Q, \sigma_Q, \varphi_Q)$ and we denote this last 6-ple as $\mathrm{LocDyn}(f)$.

We assume the following condition on these multiplies:

$$
0 < \left| \left(\left(\lambda_P\right)^{\frac{1}{2}} \sigma_P \right)^{\eta} \sigma_Q \right| < 1, \quad \text{where} \quad \eta = \frac{\log|\lambda_Q^{-1}|}{\log|\sigma_P|}. \tag{5.3.2}
$$

We called to the equation (5.3.2) *spectral condition* of f.

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We define the local stable and unstable manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(P,f)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$ of P as the connected components of $W^{\mathrm{s}}(P,f)\cap U_P$ and $W^{\mathrm{u}}(P,f)\cap U_P$ containing P. We similarly define $W^{\mathrm{s}}_{\mathrm{loc}}(Q,f)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(Q,f)$.

5.3.1.1
Range of the eigenvalues of $Df(P)$ and $Df(Q)$

We now discuss the spectral condition (5.3.2) on the eigenvalues of the saddles P and Q in the cycle. The main role of this range of values is to ensure the convergence of the renormalisation scheme in Theorem 2. Due to the irrelevance of the arguments φ_P and φ_Q in the vector $\mathrm{LocDyn}(f)$ (see Section 5.3.1) in the description of the spectral condition (5.3.4) also they will be omitted and we write

$$\mathrm{LocDyn}^*(f) = (\lambda_P, \sigma_P, \lambda_Q, \sigma_Q). \tag{5.3.3}$$

Taking the square of f, if necessary, we can assume that the coordinates of $\mathrm{LocDyn}^*(f)$ are all positives. Let $\mathcal{P}$ be the set of points $(\tilde{\lambda}, \tilde{\sigma}, \lambda, \sigma)$ in $\mathbb{R}^4$ such that $\lambda, \tilde{\lambda} < 1 < \tilde{\sigma}, \sigma$ and

$$0 < (\tilde{\lambda}^{\frac{1}{2}}\,\tilde{\sigma})^{\eta}\sigma < 1, \quad \text{where} \quad \eta = \frac{\log \lambda^{-1}}{\log \tilde{\sigma}}. \tag{5.3.4}$$

In the following proposition we prove that the condition 5.3.4 are non degenerated. As consequence the set of diffeomorphisms f described in condition (L), i.e., diffeomorphisms f such that $\mathrm{LocDyn}^*(f)$ satisfy the spectral condition (5.3.4) is a non-empty set.

Proposition 5.3.1 *The set $\mathcal{P}$ is non-empty and open in $\mathbb{R}^4$.*

Proof. The condition in (5.3.4) is clearly open one. Thus it remains to show the existence of numbers satisfying these inequalities. For this, consider the set

$$\widetilde{\mathcal{Z}} := \left\{(\tilde{\lambda}, \tilde{\sigma}) \in (0,1) \times (1, +\infty) : 0 < \tilde{\lambda}^{\frac{1}{2}}\,\tilde{\sigma} < 1\right\} \subset \mathbb{R}^2 \tag{5.3.5}$$

the proof of proposition follows following lemma.

Lemma 5.3.2 *For every point $(\tilde{\lambda}, \tilde{\sigma}, \lambda)$ in $\widetilde{\mathcal{Z}} \times (0,1)$ there is a interval $I_{(\lambda,\tilde{\lambda},\tilde{\sigma})} := (1, \sigma^*_{(\lambda,\tilde{\lambda},\tilde{\sigma})})$, such that every point $(\tilde{\lambda}, \tilde{\sigma}, \lambda, \sigma)$ in $\widetilde{\mathcal{Z}} \times (0,1) \times I_{(\lambda,\tilde{\lambda},\tilde{\sigma})}$ satisfying conditions* (5.3.4).

Proof. Note that the inequality in (5.3.4) is equivalent to

$$\frac{\log \lambda^{-1}}{\log \tilde{\sigma}}\, \log(\tilde{\lambda}^{\frac{1}{2}}\,\tilde{\sigma}) + \log \sigma < 0.$$

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With this in mind, we observe that every every point $(\tilde{\lambda}, \tilde{\sigma}, \lambda)$ in $\widetilde{\mathcal{Z}} \times (0,1)$ satisfies
$$\frac{\log \lambda^{-1}}{\log \tilde{\sigma}} \log(\tilde{\lambda}^{\frac{1}{2}}\, \tilde{\sigma}) < 0.$$
Thus, for every $\sigma > 1$ such that
$$\frac{\log \lambda^{-1}}{\log \tilde{\sigma}} \log(\tilde{\lambda}^{\frac{1}{2}}\, \tilde{\sigma}) < -\log \sigma. \tag{5.3.6}$$
The interval $I_{(\lambda,\tilde{\lambda},\tilde{\sigma})}$ is given by $(1, \sigma^*_{(\lambda,\tilde{\lambda},\tilde{\sigma})})$, where $\sigma^*_{(\lambda,\tilde{\lambda},\tilde{\sigma})}$ it is the supreme of the $\sigma > 1$ satisfiying (5.3.6). This completes the lemma. ■ This completes the proof of the proposition. ■

5.3.2
Transitions along heteroclinic orbits

Consider $X \in W^{\mathrm{s}}(P,f) \cap W^{\mathrm{u}}(Q,f)$ a quasi-transverse intersection point and consider $Y \in W^{\mathrm{u}}(P,f) \cap W^{\mathrm{s}}(Q,f)$ a tangency point. Let ud now to describe the transitions from U_Q to U_P and from U_P to U_Q along the heteroclinic orbits of X and Y, respectively. We consider first the transition from U_Q to U_P along the orbit of X.

5.3.2.1
Transition along the quasi-transverse orbit

Recall that $W^{\mathrm{u}}(Q)$ intersects $W^{\mathrm{s}}(P)$ quasi-transversally at a point X. Replacing X by some backward iterate, if necessary, we can assume that $X \in W^{\mathrm{u}}_{\mathrm{loc}}(Q,f)$. We can also assume that in the local coordinates $X = (0,1,0) \in U_Q$. Then, there exists a positive integer N_1 (called *transition time from Q to P*) such that $\widetilde{X} = f^{N_1}(X) \in W^{\mathrm{s}}_{\mathrm{loc}}(P,f) \subset U_P$. We also can assume $\widetilde{X} = (1,0,0) \in U_P$.

We consider the transition from a small neighbourhood U_X of $X \in U_Q$ to a small neighbourhood $U_{\widetilde{X}}$ of $\widetilde{X} \in U_P$ by the map f^{N_1},
$$f^{N_1} : \begin{pmatrix} x \\ y+1 \\ z \end{pmatrix} \to \begin{pmatrix} 1 + \alpha_1 x + \alpha_2 y + \alpha_3 z + \widetilde{H}_1(x,y,z) \\ \beta_1 x + \beta_2 y + \beta_3 z + \widetilde{H}_2(x,y,z) \\ \gamma_1 x + \gamma_2 y + \gamma_3 z + \widetilde{H}_3(x,y,z) \end{pmatrix}, \tag{5.3.7}$$
where $\alpha_i, \beta_i, \gamma_i$, $i = 1,2,3$ are constants. In our construction, we assume that
$$\alpha_1 = \beta_2 = \gamma_3 = 1;$$
$$\alpha_2 = \alpha_3 = \beta_1 = \beta_3 = \gamma_1 = \gamma_2 = 0.$$

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For each $i = 1, 2, 3$, $\widetilde{H}_i$ is a term of order at last two satisfying the following conditions:

$$\widetilde{H}_i(\mathbf{0}) = \frac{\partial}{\partial x}\widetilde{H}_i(\mathbf{0}) = \frac{\partial}{\partial y}\widetilde{H}_i(\mathbf{0}) = \frac{\partial}{\partial z}\widetilde{H}_i(\mathbf{0}) = 0. \tag{5.3.8}$$

5.3.2.2
Transition along the heterodimensional tangency orbit

We now consider the transition from U_P to U_Q along the orbit of heterodimensional tangency point Y. By replacing Y by some backward iterate we can assume that $Y \in W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$. There is a positive integer N_2 (called *transition time from P to Q*) such that $\widetilde{Y} = f^{N_2}(Y)$ is contained in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f)$. By some linear coordinate change in U_P and in U_Q, one may set $Y = (0, 1, 1) \in U_P$ and $f^{N_2}(Y) = (1, 0, 1) \in U_Q$, respectively. Note that this coordinate change can be done independently of the previous one involving X and $\widetilde{X}$.

We consider the transition from a small neighbourhood U_Y of $Y \in U_P$ to a small neighbourhood of $U_{\widetilde{Y}}$ of $\widetilde{Y} \in U_Q$ by the map f^{N_2}. We assume that

$$f^{N_2} : \begin{pmatrix} x \\ 1+y \\ 1+z \end{pmatrix} \to \begin{pmatrix} 1 + a_1 x + a_2 y + a_3 z + H_1(x, y, z) \\ b_1 x + b_2 y^2 + b_3 z^2 + b_4 yz + H_2(x, y, z) \\ 1 + c_1 x + c_2 y + c_3 z + H_3(x, y, z) \end{pmatrix}, \tag{5.3.9}$$

where a_i, b_i, c_i, $i = 1, 2, 3$ are constants satisfying the conditions

$$b_2 + b_3 + b_4 \neq 0, \quad c_2 = c_3, \quad b_1\, c_2\, (a_3 - a_2) \neq 0. \tag{5.3.10}$$

Remark 5.3.3 The first two conditions in the equation (5.3.10) are merely techniques while the last follows from fact that f is a diffeomorphism. This condition, imply in particular, that $b_1 \neq 0$, $c_2 \neq 0$ and $a_2 \neq a_3$.

For each $i = 1, 2, 3$, H_i is a term of order at last two satisfying the following conditions:

$$\begin{aligned} &H_i(\mathbf{0}) = \frac{\partial}{\partial x}H_i(\mathbf{0}) = \frac{\partial}{\partial y}H_i(\mathbf{0}) = \frac{\partial}{\partial z}H_i(\mathbf{0}) = 0, \\ &\frac{\partial^2}{\partial y^2}H_2(\mathbf{0}) = \frac{\partial^2}{\partial z^2}H_2(\mathbf{0}) = \frac{\partial^2}{\partial y\,\partial z}H_2(\mathbf{0}) = 0, \end{aligned} \tag{5.3.11}$$

5.3.3

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The unfolding family

Here we describe the family $\mathfrak{F}$ bifurcating the cycle of f. The cycle of f has two parts with "independent" unfolding: the heterodimensional tangency and the quasi-transverse heteroclinic point. This unfolding involves an eight-parameter family $\{f_{\bar{v}}\}_{\bar{v}}$ in $\mathrm{Diff}^r(M)$, $r \geq 2$, such that $f_{\bar{0}} = f$. The parameter $\bar{v}$ it is the form $(\bar{\mu}, \bar{\nu}, \tilde{\alpha}, \alpha) \in [-\epsilon, \epsilon]^8$, $\epsilon > 0$, where

- $\bar{\mu} \in \mathbb{R}^3$ unfolds the heterodimensional tangency,
- $\bar{\nu} \in \mathbb{R}^3$ the quasi-transverse intersection, and
- the pair $(\tilde{\alpha}, \alpha) \in \mathbb{R} \times \mathbb{R}$ controls the arguments of the non-real eigenvalues of P and Q, respectively.

The family $\{f_{\bar{v}}\}_{\bar{v}}$ is obtained by local perturbations near the quasi-transverse intersection $\widetilde{X}$ and the heterodimensional tangency $\widetilde{Y}$ and by rescaling the arguments φ_P and φ_Q of the eigenvalues of the saddles P and Q. We now go to the details of this construction.

Let $U_{\widetilde{X}}$ and $U_{\widetilde{Y}}$ be 2ρ-neighbourhoods of $\widetilde{X} = (1,0,0) \in U_P$ and of $\widetilde{Y} = (1,0,1) \in U_Q$ such that $U_{\widetilde{X}} \subseteq U_P$, $P \notin \overline{U}_{\widetilde{X}}$, and $U_{\widetilde{Y}} \subseteq U_Q$, and $Q \notin \overline{U}_{\widetilde{Y}}$. Here $\overline{U}_{\widetilde{X}}, \overline{U}_{\widetilde{Y}}$ denotes the closure of $U_{\widetilde{X}}, U_{\widetilde{Y}}$, respectively. The number $\rho > 0$ is taken small enough so that

$$f(U_{\widetilde{X}}) \cap U_{\widetilde{X}} = \emptyset \quad \text{and} \quad f(U_{\widetilde{Y}}) \cap U_{\widetilde{Y}} = \emptyset.$$

To define the local perturbations, we use a bump C^r-function $B : \mathbb{R}^3 \to \mathbb{R}$

$$B(x,y,z) = b(x)\, b(y)\, b(z),$$

where $b : \mathbb{R} \to \mathbb{R}$ is a C^r-function satisfying

$$\begin{cases} b(x) = 0, & \text{if} \quad 2\rho \geq |x|, \\ 0 < b(x) < 1, & \text{if} \quad \rho < |x| < 2\rho, \\ b(x) = 1, & \text{if} \quad |x| \leq \rho, \end{cases}$$

for some small $\rho > 0$.

We consider a family of C^r-maps

$$t_{\bar{\mu},\bar{\nu}} \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \bar{\mu} = (\mu_1, \mu_2, \mu_3), \; \bar{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3, \tag{5.3.12}$$

such that

- if $(1+x, y, 1+z) \in U_{\widetilde{Y}}$ then

$$t_{\bar{\mu},\bar{\nu}}(1+x, y, 1+z) = (1+x, y, 1+z) + B(x,y,z)(\mu_1, \mu_2, \mu_3),$$

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- if $(1+x,y,z) \in U_{\widetilde{X}}$ then

$$t_{\bar{\mu},\bar{\nu}}(1+x,y,z) = (1+x,y,z) + B(x,y,z)(\nu_1,\nu_2,\nu_3).$$

Note that, by construction, we can extend these maps to complement of $U_{\widetilde{X}} \cup U_{\widetilde{Y}}$ as the identity. In this way we get a family of diffeomorphisms.

To rescale the arguments of the eigenvalues of the saddles P and Q we argue as follows. For each $\alpha \in \mathbb{R}$ we consider the following rotations in $\mathbb{R}^3$

$$I_\alpha := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ 0 & \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix} \quad \text{and} \quad J_\alpha := \begin{pmatrix} \cos(2\pi\alpha) & 0 & -\sin(2\pi\alpha) \\ 0 & 1 & 0 \\ \sin(2\pi\alpha) & 0 & \cos(2\pi\alpha) \end{pmatrix}.$$

Observe that $I_0 = J_0 = Id_{\mathbb{R}^3}$.

We now consider a bump C^r-function $w : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} w(x) = 0, & \text{if} \quad 4 \geq |x|, \\ 0 < w(x) < 1, & \text{if} \quad 3 < |x| < 4, \\ w(x) = 1, & \text{if} \quad |x| \leq 3. \end{cases} \tag{5.3.13}$$

Using these maps, we consider a family of C^r-maps

$$s_{\tilde{\alpha},\alpha} \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \tilde{\alpha}, \alpha \in \mathbb{R}, \tag{5.3.14}$$

defined as follows:

- if $(x,y,z) \in U_P$, then

$$s_{\tilde{\alpha},\alpha}(x,y,z) = I_{\tilde{\alpha}\, w(||(x,y,z)||)} \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right),$$

- if $(x,y,z) \in U_Q$, then

$$s_{\tilde{\alpha},\alpha}(x,y,z) = J_{\alpha\, w(||(x,y,z)||)} \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right).$$

Note that, by construction, we can extend these maps to complement of $U_P \cup U_Q$ as the identity. In this way we get a family of diffeomorphisms.

We are now ready to define the C^r-family of diffeomorphisms unfolding the cycle

$$f_{\bar{v}} = t_{\bar{\mu},\bar{\nu}} \circ s_{\tilde{\alpha},\alpha} \circ f, \quad \bar{v} = (\bar{\mu}, \bar{\nu}, \tilde{\alpha}, \alpha) \in [-\epsilon, \epsilon]^8. \tag{5.3.15}$$

By construction, $f_{\bar{0}} = f$, see Figure 5.4. In the next remark we list some relevant properties satisfied by this family.

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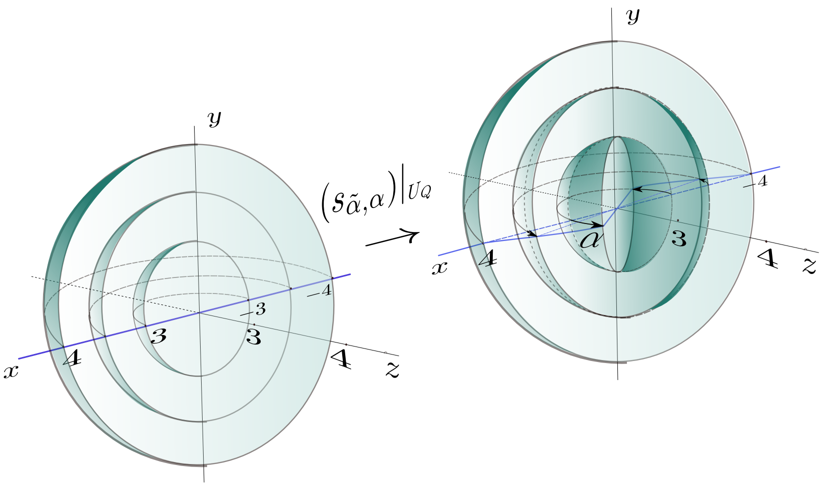

Figure 5.3: The perturbation $s_{\tilde{\alpha},\alpha}$ in a linear neighbourhood of Q.

Remark 5.3.4 (Properties of the unfolding family $f_{\bar{\upsilon}}$) The family $f_{\bar{\upsilon}}$ satisfies the following properties. Each map $f_{\bar{\upsilon}}$ has saddle periodic points $P_{\bar{\upsilon}} = P$ and $Q_{\bar{\upsilon}} = Q$.

1. The spectra of $P_{\bar{\upsilon}} = P$ and $Q_{\bar{\upsilon}} = Q$ for $f_{\bar{\upsilon}}$ are

$$\operatorname{spec}\Big(Df_{\bar{\upsilon}}^{\operatorname{per}(P)}(P)\Big) = \Big\{\lambda_P, \sigma_P\, e^{2\pi(\varphi_P+\tilde{\alpha})i}, \sigma_P\, e^{-2\pi(\varphi_P+\tilde{\alpha})i}\Big\},$$
$$\operatorname{spec}\Big(Df_{\bar{\upsilon}}^{\operatorname{per}(Q)}(Q)\Big) = \Big\{\sigma_Q, \lambda_Q\, e^{2\pi(\varphi_Q+\alpha)i}, \lambda e^{-2\pi(\varphi_Q+\alpha)i}\Big\}.$$

2. For every $(x, y, z) \in U_P \cap f_{\bar{\upsilon}}^{-1}(U_P)$, it holds $f_{\bar{\upsilon}}(x, y, z) = I_{\tilde{\alpha}} \circ f(x, y, z)$.

3. For every $(x, y, z) \in U_Q \cap f_{\bar{\upsilon}}^{-1}(U_Q)$, it holds $f_{\bar{\upsilon}}(x, y, z) = J_{\alpha} \circ f(x, y, z)$.

4. For every $(x, 1 + y, z)$ sufficiently close to $X = (0, 1, 0)$,

$$f_{\bar{\upsilon}}^{N_1}(x, 1 + y, z) = f^{N_1}(x, 1 + y, z) + (\nu_1, \nu_2, \nu_3).$$

5. For every $(x, 1 + y, 1 + z)$ sufficiently close to $Y = (0, 1, 1)$,

$$f_{\bar{\upsilon}}^{N_2}(x, 1 + y, z) = f^{N_2}(x, 1 + y, z) + (\mu_1, \mu_2, \mu_3).$$

Hereinafter we adopt the following notation.

Notation 5.3.5 Since all the relevant dynamics in our construction is contained in the neighborhoods $[-2, 2]^3 \subset U_P$ and $[-2, 2]^3 \subset U_Q$, we identify $[-2, 2]^3$ with these neighborhoods U_P and U_Q, see (5.3.1). Note also that $f_{\bar{\upsilon}}$ depends only on $\tilde{\alpha}$ in U_P, on α in U_Q, on $\bar{\nu}$ in U_X, and on $\bar{\mu}$ on U_Y. Motivated by this, we simply write

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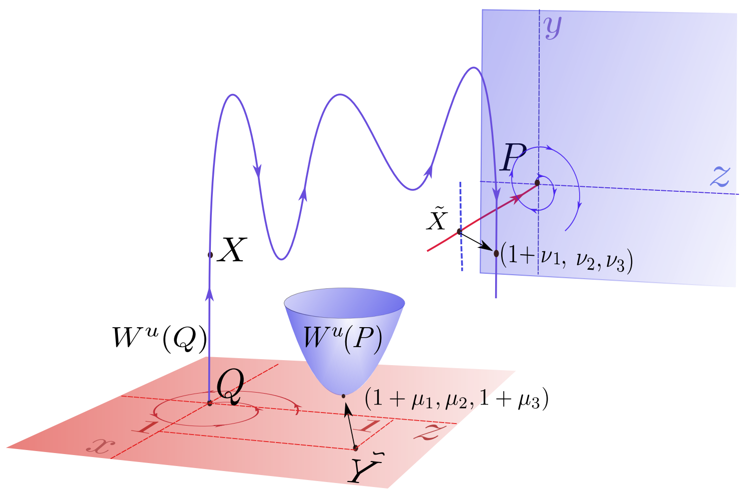

Figure 5.4: The Unfolding $\mathfrak{F}$ of the non-transverse heterodimensional cycle.

$$\begin{aligned} f_{\bar{v}}|_{U_P} &= f_{P,\varphi_P+\tilde{\alpha}}, \quad f_{\bar{v}}|_{U_Q} = f_{Q,\varphi_Q+\alpha}, \\ f_{\bar{v}}^{N_1}|_{U_X} &= f_{X,\bar{\nu}}^{N_1}, \quad f_{\bar{v}}^{N_2}|_{U_Y} = f_{Y,\bar{\mu}}^{N_2}. \end{aligned} \tag{5.3.16}$$

5.3.4 Renormalisation scheme and convergence

We now describe the elements of the renormalisation scheme associated to f in Theorem 2. This scheme involves compositions of the form

$$F_{\bar{v}}^{m,n} := f_{\bar{v}}^{N_2} \circ f_{\bar{v}}^{m} \circ f_{\bar{v}}^{N_1} \circ f_{\bar{v}}^{n}, \tag{5.3.17}$$

where N_1 and N_2 are the (fixed) transition times from Q to P and from P to Q given by the condition (T) of the cycle (see Section 5.3.2) and the adequately *sojourn times* m and n of the local dynamics of f in neighbourhoods U_P and U_Q, respectively, to be defined below.

The construction of the renormalisation scheme involves three main part. First, the choice of appropriate sojourn times m and n in Section 5.3.5. Second, we introduce a suitable (m, n)-sequence of unfolding parameters $\bar{v}_{m,n}$

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converging to $\mathbf{0} \in \mathbb{R}^8$ and a suitable (m,n)-sequence of parametrisations $\Psi_{m,n} : \mathbb{R}^3 \to U_Q$ converging to $\widetilde{Y}$ on the compact sets, see Section 5.3.6. The last part, consist in study the convergence of the *renormalised sequence* $\Psi^{-1}_{m,n} \circ F^{m,n}_{\bar{v}_{m,n}} \circ \Psi_{m,n}$ on compact sets in $\mathbb{R}^3$. This convergence is obtained in Section 5.3.8.

5.3.5 Adapted sojourn times

For selecting these sojourn times m, n, we use the following result.

Lemma 5.3.6 (Lemma 5.1 in (21)) *Consider the set* $\mathcal{Z} := (1,+\infty) \times (0,1)$*. There a residual subset* $\mathcal{R}$ *of* $\mathcal{Z}$ *consisting of points* $(\alpha,\beta) \in \mathcal{Z}$ *satisfying the following property:*

For every $\epsilon > 0$*,* $N_0 > 0$*,* $\omega > 0$ *and* $\xi > 0$ *satisfying* $\omega\,\xi^{-1} > 1$*, and* $\epsilon\,\xi^{-1} < 1$*, there exist integers* $m, n > N_0$ *such that*

$$|\omega\alpha^m\beta^n - \xi| < \epsilon, \quad |m - n\eta - \tilde{\eta}| < 1,$$

where $\eta = \log\beta^{-1}/\log\alpha$ *and* $\tilde{\eta} = \log(\omega\xi^{-1})/\log\alpha$*.*

In particular, there exits a subsequence $(m_k, n_k) \in \mathbb{N}^2$ *such that*

$$\omega\,\alpha^{m_k}\,\beta^{n_k} \to \xi, \quad k \to +\infty.$$

The sojourn times are defined by the sequence $(m_k, n_k) \in \mathbb{N}^2$ obtained by applying the previous lemma to:

- $(\sigma_P, \lambda_Q) \in \mathcal{Z}$, where λ_Q, σ_P are as in the spectral conditions (5.3.4)[1];
- $\omega = \omega(a_2, a_3) = \frac{a_3 - a_2}{\sqrt{2}} > 0$, where a_2 and a_3 are constants involved in the definition of the transitions f^{N_2} in (5.3.9),

– $\xi > 0$ is arbitrary but fixed[2].

As a consequence,

$$\sigma_P^{m_k}\,\lambda_Q^{n_k} \to \left(\frac{a_3 - a_2}{\sqrt{2}}\right)^{-1} \xi. \tag{5.3.18}$$

As in (21), the adapted sequence of sojourn times is used to guarantee the convergence of the renomalisation schemes, see (5.3.8).

[1]Note that as the spectral conditions (5.3.4) are open ones, we can suposse that $(\lambda, \tilde{\sigma}) \in \mathcal{R}$.

[2] The choice of this number will be important later, when we study the generation of blenders.

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5.3.6
Elements of the renormalisation scheme

Consider $\lambda_Q < 1 < \sigma_P$ satisfying the spectral condition in (5.3.4). Taking the $(\sigma_Q, \lambda_Q) \in \mathcal{Z}$ and $\xi > 0$, we get from Lemma 5.3.6 a adapted sequence of sojourn times (m_k, n_k). The renormalization scheme $\mathcal{R}(\xi, \mathfrak{F})$ as consisting in

- $m_k, n_k \geq 0$ sojourn times as above;
- $\Psi_{m_k,n_k} : \mathbb{R}^3 \to M$ a sequence of parameterizations on the manifold M;
- $\bar{\upsilon}_{m_k,n_k} : \mathbb{R}^3 \to \mathbb{R}^8$ a sequence of functions involved in the bifurcation parameter of the family $f_{\bar{\upsilon}}$, defined by

$$\bar{\upsilon}_{m_k,n_k}(\mu, \tilde{\alpha}, \alpha) = (\bar{\mu}_{m_k,n_k}(\mu), \nu_{m_k,n_k}, \bar{\alpha}_{m_k,n_k}(\tilde{\alpha}), , \bar{\alpha}_{m_k,n_k}(\alpha))$$

- $\mathcal{R}_{m_k,n_k}(f_{\bar{\upsilon}_{m_k,n_k}})$ the sequence in $\mathrm{Diff}^r(M)$ defined by

$$f^{N_2}_{\bar{\upsilon}_{m_k,n_k}} \circ f^{m_k}_{\bar{\upsilon}_{m_k,n_k}} \circ f^{N_1}_{\bar{\upsilon}_{m_k,n_k}} \circ f^{n_k}_{\bar{\upsilon}_{m_k,n_k}}.$$

  We will called to this composition *renormalised sequence* of f.
- $\varsigma_i : \mathbb{R} \times \mathbb{R}^9 \to \mathbb{R}$, $i = 1, 2, 3, 4, 5$ polynomial maps defining the coefficient of the limit endomorphism obtained by the convergence of renormalised sequence.

Remark 5.3.7 Recall the notations in (5.3.16), we get that the diffeomorphism $f_{\bar{\upsilon}_{m_k,n_k}}$ is a C^r-perturbation of f defined by the local perturbations

$$\begin{aligned}
f_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}|_{U_P} &= f_{P,\varphi_P + \bar{\alpha}_{m_k,n_k}(\alpha)}, \\
f_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}|_{U_Q} &= f_{Q,\varphi_Q + \bar{\alpha}_{m_k,n_k}(\tilde{\alpha})}, \\
f^{N_1}_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}|_{U_X} &= f^{N_1}_{X,\bar{\nu}_{m_k,n_k}}, \\
f^{N_2}_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}|_{U_Y} &= f^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu)}.
\end{aligned} \tag{5.3.19}$$

Next, we give the definition of each object in the renormalization scheme.

• For every compact set $K \subset \mathbb{R}^3$ containing the origin there is a $k_0 = k_0(K)$ such that for every $k \geq k_0$ we have

$$\Psi_{m_k,n_k} : K \to U_Q, \tag{5.3.20}$$

where

$$\Psi_{m_k,n_k}(x, y, z) := (1 + \sigma_P^{-m_k} \sigma_Q^{-n_k}\, x, \sigma_Q^{-n_k} + \sigma_P^{-2m_k} \sigma_Q^{-2n_k}\, y, 1 + \sigma_P^{-m_k} \sigma_Q^{-n_k}\, z).$$

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Note that $\Psi_{m_k,n_k}(K)$ converges (in the Hausdorff distance) to the point of heterodimensional tangency $\widetilde{Y}=(1,0,1)\in U_Q$ when $k\to+\infty$.

- The sequence of re-parameterizations $\bar{\mu}_{m_k,n_k}:\mathbb{R}\to\mathbb{R}^3$ is defined by

$$\bar{\mu}_{m_k,n_k}(\mu)=(-\lambda_P^{m_k}a_1,\sigma_Q^{-n_k}+\sigma_Q^{-2n_k}\sigma_P^{-2m_k}\mu-\lambda_P^{m_k}b_1,-\lambda_P^{m_k}c_1), \qquad (5.3.21)$$

where a_1,b_1,c_1 are given in (5.3.9).

Note that for every $\mu\in\mathbb{R}$ fixed, $\bar{\mu}_{m_k,n_k}(\mu)\to(0,0,0)$ as $k\to+\infty$.

- The sequence of re-parameterizations $\bar{\alpha}_{m_k,n_k,\varphi_P,\varphi_Q}:\mathbb{R}\to\mathbb{R}$ is defined by

$$\bar{\alpha}_{m_k,n_k\varphi_P,\varphi_Q}(\alpha)=\frac{1}{2\pi m_k}\left(\frac{\pi}{4}-2\pi m_k\varphi_P+2\pi[m_k\varphi_P]\right)\frac{\alpha-\varphi_Q}{\varphi_P-\varphi_Q}+$$
$$+\frac{1}{2\pi n_k}\left(\frac{\pi}{2}-2\pi n_k\varphi_Q+2\pi[n_k\varphi_Q]+\frac{\ln(n_k)}{n_k}\right)\frac{\alpha-\varphi_P}{\varphi_Q-\varphi_P}.$$

Note that for every $\alpha\in\mathbb{R}$, $\bar{\alpha}_{m_k,n_k\varphi_P,\varphi_Q}(\alpha)\to 0$ as $k\to+\infty$.

Remark 5.3.8 For notational simplicity, in what follows we omit the subscripts φ_P,φ_Q in $\bar{\alpha}_{m_k,n_k\varphi_P,\varphi_Q}(\cdot)$.

Remark 5.3.9 Since for every $x\in\mathbb{R}$ the sequence of rational numbers $\frac{[n\,x]}{n}$ converges to x and since for every n in $\mathbb{N}$, $\log(n)$ is a irrational number[3], we get that

$$\varphi_Q+\bar{\alpha}_{m_k,n_k\varphi_P,\varphi_Q}(\varphi_Q)=\frac{[n_k\varphi_Q]}{n_k}+\frac{1}{2\,\pi\,n_k}\left(\frac{\pi}{2}+\frac{\ln(n_k)}{n_k}\right),$$

is a sequence of irrational numbers converging to φ_Q.

To define the sequence of parameters ν_{m_k,n_k} we need to introduce some notation. Let

$$\begin{aligned}\tilde{\mathfrak{c}}_k &:= \cos\Big(2\pi m_k(\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P))\Big),\\ \tilde{\mathfrak{s}}_k &:= \sin\Big(2\pi m_k(\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P))\Big),\\ \mathfrak{c}_k &:= \cos\Big(2\pi n_k(\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q))),\\ \mathfrak{s}_k &:= \cos\Big(2\pi n_k(\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q)))\end{aligned} \qquad (5.3.22)$$

and

[3]This follow from Lindemann-Weierstrass theorem: e^a is *transcendental* for all *a algebraic* and non-zero. In particular if a is rational, e^a cannot be rational. Hence, for every $n\in\mathbb{N}$, $\ln(n)$ is irrational.

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$$
\begin{aligned}
\tilde{\rho}_{2,k} &:= \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{H}_2(\mathbf{0})(\mathfrak{c}_k - \mathfrak{s}_k)^2 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\tilde{H}_2(\mathbf{0})(\mathfrak{s}_k + \mathfrak{c}_k)^2; \\
\tilde{\rho}_{3,k} &:= \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{H}_3(\mathbf{0})(\mathfrak{s}_k - \mathfrak{c}_k)^2 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\tilde{H}_3(\mathbf{0})(\mathfrak{s}_k + \mathfrak{c}_k)^2.
\end{aligned} \tag{5.3.23}
$$

We note that for every k large enough $f_{P,\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P)}$ is a C^r-perturbation of f_{P,φ_P} and the coefficients of the rotation $\left(\begin{smallmatrix}\tilde{\mathfrak{c}}_k & -\tilde{\mathfrak{s}}_k \\ \tilde{\mathfrak{c}}_k & \tilde{\mathfrak{s}}_k\end{smallmatrix}\right)$ of $f^{m_k}_{P,\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P)}$ associated to an expansion satisfies:

$$
\begin{aligned}
\tilde{\mathfrak{c}}_k &= \cos\Big(2\pi m_k(\varphi_P + \bar{\alpha}_{m_k,n_k}(\varphi_P))\Big) = \cos\Big(\frac{\pi}{4} + 2\pi[m_k\varphi_P]\Big) = \frac{1}{\sqrt{2}}, \\
\tilde{\mathfrak{s}}_k &= \sin\Big(2\pi m_k(\varphi_P + \bar{\alpha}_{m_k,n_k}(\varphi_P))\Big) = \sin\Big(\frac{\pi}{4} + 2\pi[m_k\varphi_P]\Big) = \frac{1}{\sqrt{2}}.
\end{aligned}
$$

Arguing analogously we get that for every k large enough $f_{Q,\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q)}$ is a C^r-perturbation of f_{Q,φ_Q} and the coefficients of the rotation $\left(\begin{smallmatrix}\mathfrak{c}_k & -\mathfrak{s}_k \\ \mathfrak{c}_k & \mathfrak{s}_k\end{smallmatrix}\right)$ of $f^{n_k}_{Q,\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q)}$ associated to a contraction satisfies:

$$
\begin{aligned}
\mathfrak{c}_k &= \cos\Big(2\pi n_k(\varphi_Q + \bar{\alpha}_{m_k,n_k}(\varphi_Q))\Big) = \cos\left(\frac{\pi}{2} + 2\pi[n_k\varphi_Q] + \frac{\ln(n_k)}{n_k}\right) \to 0, \\
\mathfrak{s}_k &= \sin\Big(2\pi n_k(\varphi_Q + \bar{\alpha}_{m_k,n_k}(\varphi_Q))\Big) = \sin\left(\frac{\pi}{2} + 2\pi[n_k\varphi_Q] + \frac{\ln(n_k)}{n_k}\right) \to 1.
\end{aligned}
$$

In summary, we have the following observation.

Remark 5.3.10 It holds

$$
\tilde{\mathfrak{c}}_k = \frac{1}{\sqrt{2}}, \quad \tilde{\mathfrak{s}}_k = \frac{1}{\sqrt{2}}, \quad \mathfrak{c}_k \to 0, \quad \mathfrak{s}_k \to 1.
$$

In particular, for every big k

$$
\tilde{\rho}_{2,k} \to \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{H}_2(\mathbf{0}) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\tilde{H}_2(\mathbf{0}), \quad \tilde{\rho}_{3,k} \to \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{H}_3(\mathbf{0}) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\tilde{H}_3(\mathbf{0}).
$$

Now we continue with the description of element of renormalisation scheme.

- The sequence of parameter values $\nu_{m_k,n_k} \in \mathbb{R}^3$ is given by

$$
\Big(-\lambda_Q^{n_k}(\mathfrak{c}_k - \mathfrak{s}_k), \sqrt{2}\,\sigma_P^{-m_k} - \lambda_Q^{2n_k}\tilde{\rho}_{2,k}, -\lambda_Q^{n_k}(\mathfrak{c}_k + \mathfrak{s}_k) - \lambda_Q^{2n_k}\tilde{\rho}_{3,k}\Big). \tag{5.3.24}
$$

Note that $\nu_{m_k,n_k} \to (0,0,0) \in \mathbb{R}^3$ as $k \to +\infty$.

- the renormalized sequence of maps

$$
\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}\Big) \circ \Psi_{m_k,n_k} : \mathbb{R}^3 \to \mathbb{R}^3,
$$

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is define as follows:

For $\bar{X} \in K$ we let

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{\upsilon}_{m_k,n_k}(\mu,\tilde{\alpha},\alpha)}\Big) \circ \Psi_{m_k,n_k}(\bar{X}) =$$
$$= \Psi^{-1}_{m_k,n_k} \circ f^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu)} \circ f^{m_k}_{Q,\varphi_Q+\bar{\alpha}_{m_k,n_k}(\tilde{\alpha})} \circ f^{N_1}_{X,\nu_{m_k,n_k}} \circ f^{n_k}_{P,\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi)} \circ \Psi_{m_k,n_k}(\bar{X}).$$

- The rational maps $\varsigma_i : \mathrm{Dom}(\varsigma_i) \subset \mathbb{R} \times \mathbb{R}^9 \to \mathbb{R}$, $i = 1,2,3,4,5$, are given by

$$\varsigma_1(x,\mathbf{X}) = \frac{x_2+x_3}{\sqrt{2}}, \quad \varsigma_2(x,\mathbf{X}) = \frac{x_5+x_6+x_7}{\sqrt{2}}, \quad \varsigma_3(x,\mathbf{X}) = x^2\Big(\frac{x_5+x_6-x_7}{(x_3-x_2)^2}\Big)$$

$$\varsigma_4(x,\mathbf{X}) = x\sqrt{2}\Big(\frac{x_6-x_5}{x_3-x_2}\Big), \quad \varsigma_5(x,\mathbf{X}) = \Big(\frac{x_8+x_9}{\sqrt{2}}\Big),$$

where

$$\mathbf{X} = (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9).$$

This completes the description of the elements in the renomalisation scheme.

Remark 5.3.11 We now list some observations.

(i) Note that for any fixed μ and b_1 in $\mathbb{R}$, the y-coordinate of $\bar{\mu}_{m_k,n_k}(\mu)$ is positive for every $k \geq 1$ large enough. Indeed, by (5.3.43) we have that $\lambda_P^{m_k}\sigma_Q^{n_k} \to 0$ when $k \to +\infty$. Thus, it holds

$$1 + \sigma_Q^{-n_k}\sigma_P^{-2m_k}\mu - \lambda_P^{m_k}\sigma_Q^{n_k}b_1 \to 1, \quad k \to +\infty.$$

This fact will be relevant to study the dynamical properties of the family $E_{(\xi,\mu,\bar{\varsigma})}$ in (4.3.1), see Section 6.2.

(ii) For $i = 1,2,5$ it holds $\mathrm{Dom}(\varsigma_i) = \mathbb{R}\times\mathbb{R}^9$, and for $i = 3,4$, it holds $\mathrm{Dom}(\varsigma_i) = \mathbb{R}\times\Big\{X \in \mathbb{R}^9 : x_3 - x_2 \neq 0\Big\}$.

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5.3.7
The renormalised sequence of maps

We now provide step-by-step calculations to obtain entries of the return maps

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{\upsilon}_{m_k,n_k}}\Big) \circ \Psi_{m_k,n_k} : K \to \mathbb{R}^3,$$

where $\bar{\upsilon}_{m_k,n_k} = \bar{\upsilon}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)$ and K is a compact subset of $\mathbb{R}^3$. Note that the compact set K can be chosen arbitrarily large, for this it is enough to take k sufficiently large. The parametrization Ψ_{m_k,n_k} maps the point $\bar{X} = (x,y,z) \in K$ to

$$\bar{X}_k := (1 + \sigma_P^{-m_k}\sigma_Q{}^{-n_k}\,x, \sigma_Q^{-n_k} + \sigma_P^{-2m_k}\sigma_Q^{-2n_k}\,y, 1 + \sigma_P^{-m_k}\sigma_Q{}^{-n_k}\,z) \in U_Q.$$

By the compactness of the set K, $\bar{X}_k \to \widetilde{Y} = (1,0,1) \in U_Q$, as $k \to +\infty$.

After n_k iterations by the linear map $f_{Q,\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q)}$, the point $\bar{X}_k$ moves to

$$(x_{n_k}, y_{n_k}+1, z_{n_k}) := f^{n_k}_{Q,\varphi_Q+\bar{\alpha}_{m_k,n_k}(\varphi_Q)}(\bar{X}_k) \in U_Q, \tag{5.3.25}$$

where

$$\begin{aligned}
x_{n_k} &:= {\sigma_P}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \lambda_Q^{n_k}\,(\mathfrak{c}_k - \mathfrak{s}_k),\\
y_{n_k} &:= \sigma_P^{-2m_k}\,{\sigma_Q}^{-n_k}\,y,\\
z_{n_k} &:= {\sigma_P}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + \lambda_Q^{n_k}(\mathfrak{c}_k + \mathfrak{s}_k).
\end{aligned}$$

Since K is a compact set, the spectral conditions (5.3.4) imply that

$$x_{n_k} = O(\lambda_Q^{n_k}), \quad y_{n_k} = O(\sigma_P^{-2m_k}\,{\sigma_Q}^{-n_k}), \quad z_{n_k} = O(\lambda_Q^{n_k}), \tag{5.3.26}$$

where $O(\cdot)$ denotes the symbol of Landau.

This guarantees $(x_{n_k}, y_{n_k}+1, z_{n_k}) \to X = (0,1,0) \in U_Q$, when $k \to +\infty$. Thus, for k large enough, we can apply the transition $f^{N_1}_{\nu_{m_k,n_k}}$ (see equation (5.3.19)) to the point $(x_{n_k}, y_{n_k}+1, z_{n_k})$ in (5.3.25) obtaining

$$(1+\tilde{x}_{m_k}, \tilde{y}_{m_k}, \tilde{z}_{m_k}) := f^{N_1}_{X,\nu_{m_k,n_k}}(x_{n_k}, y_{n_k}+1, z_{n_k}) \in U_P, \tag{5.3.27}$$

where

$$\begin{aligned}
\tilde{x}_{m_k} &:= {\sigma_P}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \widetilde{H}_1\Big(\mathbf{x}_k\Big),\\
\tilde{y}_{m_k} &:= {\sigma_P}^{-2m_k}\,\sigma_Q^{-n_k}\,y + \sqrt{2}\,{\sigma_P}^{-m_k} + \widetilde{H}_2\Big(\mathbf{x}_k\Big) - \lambda_Q^{2n_k}\,\tilde{\rho}_{2,k},\\
\tilde{z}_{m_k} &:= {\sigma_P}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + \widetilde{H}_3\Big(\mathbf{x}_k\Big) - \lambda_Q^{2n_k}\,\tilde{\rho}_{3,k},
\end{aligned}$$

and

$$\mathbf{x}_k := (x_{n_k}, y_{n_k}, z_{n_k}). \tag{5.3.28}$$

By simplicity, in what follows we write

$$\widehat{H}_i\Big(\mathbf{x}_k\Big) := \widetilde{H}_i\Big(\mathbf{x}_k\Big) - \lambda_Q^{2n_k}\,\tilde{\rho}_{i,k}, \quad i = 2,3. \tag{5.3.29}$$

Next, we apply m_k iterations by the linear map $f_{P,\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P)}$. For k large enough, the point $(1+\tilde{x}_{m_k}, \tilde{y}_{m_k}, \tilde{z}_{m_k}) \in U_P$ it is mapped to

$$(\hat{x}_{m_k}, 1+\hat{y}_{m_k}, 1+\hat{z}_{m_k}) := f^{m_k}_{P,\varphi_P+\bar{\alpha}_{m_k,n_k}(\varphi_P)}(1+\tilde{x}_{m_k}, \tilde{y}_{m_k}, \tilde{z}_{m_k}) \in U_P, \tag{5.3.30}$$

where

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$$\begin{aligned}
\hat{x}_{m_k} &= \lambda_P^{m_k} + \lambda_P^{m_k}\, \sigma_P{}^{-m_k} \lambda_Q^{n_k}\, \sigma_Q^{-n_k}\, (\mathfrak{c}_k\, x - \mathfrak{s}_k\, z) + \lambda_P{}^{m_k}\, \widetilde{H}_1\big(\mathbf{x}_k\big), \\
\hat{y}_{m_k} &= \frac{\sigma_P^{-m_k}\, \sigma_Q{}^{-n_k}}{\sqrt{2}}\, y - \frac{\lambda_Q^{n_k}\, \sigma_Q^{-n_k}}{\sqrt{2}} (\mathfrak{s}_k\, x + \mathfrak{c}_k\, z) + \\
&\quad + \frac{\sigma_P^{m_k}}{\sqrt{2}} \Big(\widehat{H}_2\big(\mathbf{x}_k\big) - \widehat{H}_3\big(\mathbf{x}_k\big)\Big), \\
\hat{z}_{m_k} &= \frac{\sigma_P^{-m_k}\, \sigma_Q{}^{-n_k}}{\sqrt{2}}\, y + \frac{\lambda_Q^{n_k}\, \sigma_Q^{-n_k}}{\sqrt{2}} (\mathfrak{s}_k\, x + \mathfrak{c}_k\, z) + \\
&\quad + \frac{\sigma_P{}^{m_k}}{\sqrt{2}} \Big(\widehat{H}_2\big(\mathbf{x}_k\big) + \widehat{H}_3\big(\mathbf{x}_k\big)\Big).
\end{aligned} \tag{5.3.31}$$

We now estimate the Landau's symbols of $\hat{x}_{m_k}$, $\hat{y}_{m_k}$ and $\hat{z}_{m_k}$.

Informally speaking, the terms $\lambda_Q^{2n_k}\, \tilde{\rho}_{i,k}$ “correct" the quadratic terms of $\widetilde{H}_i\big(\mathbf{x}_k\big)$ that provide $O(\lambda_Q^{2n_k})$ of the symbol of Landau of $\widetilde{H}_i\big(\mathbf{x}_k\big)$. This kind of correction allows us to control the convergence of the renormalization scheme.

Lemma 5.3.12 *$\hat{x}_{m_k} = O(\lambda_P{}^{m_k})$ and $\hat{y}_{m_k} = \hat{z}_{m_k} = O(\sigma_P^{-m_k}\, \sigma_Q{}^{-n_k}) + O(\lambda_Q^{n_k}\, \sigma_Q^{-n_k})$.*

Proof. Clearly $\hat{y}_{m_k}$ and $\hat{z}_{m_k}$ have the same symbol of Landau.

From the Taylor expansion of the transition f^{N_1} in (5.3.7), we have that the higher order terms $\widetilde{H}_i$, $i = 1, 2, 3$ are are dominated by quadradic terms. This implies that

$$\begin{aligned}
\widetilde{H}_i\big(\mathbf{x}_k\big) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\widetilde{H}_i(\mathbf{0})\, x_{n_k}^2 + \frac{\partial^2}{\partial x \partial y}\widetilde{H}_i(\mathbf{0})\, x_{n_k}\, y_{n_k} + \\
&\quad + \frac{\partial^2}{\partial x \partial z}\widetilde{H}_i(\mathbf{0})\, x_{n_k}\, z_{n_k} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\widetilde{H}_i(\mathbf{0})\, y_{n_k}^2 + \\
&\quad + \frac{\partial^2}{\partial y \partial z}\widetilde{H}_i(\mathbf{0})\, y_{n_k}\, z_{n_k} + \frac{1}{2}\frac{\partial^2}{\partial z^2}\widetilde{H}_i(\mathbf{0})\, z_{n_k}^2 + \cdots
\end{aligned} \tag{5.3.32}$$

$i = 1, 2, 3$.

Since x_{n_k} and z_{n_k} have the same Landau symbol (see (5.3.26)), we have

$$\widetilde{H}_i\big(\mathbf{x}_k\big) = O(\lambda_Q^{2n_k}) + O(\sigma_P{}^{-2m_k} \lambda_Q^{n_k} \sigma_Q{}^{-n_k}) + O(\sigma_P{}^{-4m_k}\, \sigma_Q^{-2n_k}), \quad i = 1, 2, 3. \tag{5.3.33}$$

Recalling that the set K is compact it follows that $\hat{x}_{m_k} = O(\lambda_P{}^{m_k})$.

We now us estimate the symbol of $\hat{y}_{m_k}$. For this, we need to estimate the term $\sigma_P{}^{m_k}\Big(\widehat{H}_2\big(\mathbf{x}_k\big) - \widehat{H}_3\big(\mathbf{x}_k\big)\Big)$ in the definition of $\hat{y}_{m_k}$.

Using the Taylor formula in (5.3.32) and the definition of the coordinates (5.3.25), we obtained from (5.3.26) that

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$$\begin{aligned}\widetilde{H}_i\Big(\mathbf{x}_k\Big) - \lambda_Q^{2n_k}\tilde{\rho}_{i,k} &= O({\sigma_P}^{-m_k}\,\lambda_Q^{2n_k}\,\sigma_Q^{-n_k}) + O({\sigma_P}^{-2m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}) + \\ &+ O(\sigma_P^{-4m_k}\,{\sigma_Q}^{-2n_k}) \\ &= O({\sigma_P}^{-m_k}\,\lambda_Q^{2n_k}\sigma_Q^{-n_k}) + O(\sigma_P^{-4m_k}\,{\sigma_Q}^{-2n_k}), \quad i = 2,3.\end{aligned} \tag{5.3.34}$$

We observe that in this last equality we use the fact that ${\sigma_P}^{m_k}\,\lambda_Q^{n_k} > 1$ for every k big sufficient. Thus, the definition of $\hat{H}_i\Big(\mathbf{x}_k\Big)$ in (5.3.29) and its symbol of Landau given in (5.3.34), implies

$${\sigma_P}^{m_k}\widehat{H}_i\Big(\mathbf{x}_k\Big) = O(\lambda_Q^{2n_k}\sigma_Q^{-n_k}) + O(\sigma_P^{-3m_k}\,{\sigma_Q}^{-2n_k}), \quad i = 2,3.$$

Then

$$\hat{y}_{m_k} = O(\sigma_P^{-m_k}\,{\sigma_Q}^{-n_k}) + O(\lambda_Q^{n_k}\,\sigma_Q^{-n_k}).$$

This completes the prove of the lemma. ■

Remark 5.3.13 It follows from the calculations above that the Landau's symbols of $\hat{y}_{m_k}$ and $\hat{z}_{m_k}$ are not modified by the absence/presence of higher order terms $\tilde{\sigma}^{m_k}\widehat{H}_i\Big(\mathbf{x}_k\Big)$. In order of to have a more transparent calculations in the sequel we assume that $\widehat{H}_i\Big(\mathbf{x}_k\Big) = 0$, for $i = 2,3$ in (5.3.30). At the end of this section we will recover these expressions and we will study the effect of these terms in the convergence of the return map $\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}}\Big) \circ \Psi_{m_k,n_k}$.

Lemma 5.3.12 guarantees the convergence of $(\hat{x}_{m_k}, 1+\hat{y}_{m_k}, 1+\hat{z}_{m_k}) \to Y = (0,1,1) \in U_P$ as $k \to +\infty$. Thus, for k large enough, we can apply the transition $f^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu)}$ to the point $(\hat{x}_{m_k}, 1+\hat{y}_{m_k}, 1+\hat{z}_{m_k})$ obtaining

$$(1+\bar{x}_{m_k}, \bar{y}_{m_k}, 1+\bar{z}_{m_k}) := f^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu),\nu_{m_k,n_k}}(\hat{x}_{m_k}, 1+\hat{y}_{m_k}, 1+\hat{z}_{m_k}) \in U_Q$$

Bearing in mind the Remark 5.3.13, we can omit the higher order terms in the vector

$$\hat{\mathbf{x}}_k = (\hat{x}_{m_k}, \hat{y}_{m_k}, \hat{z}_{m_k}). \tag{5.3.35}$$

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In this way we obtain

$$\begin{aligned}
\bar{x}_{m_k} &= a_1\,\lambda_P{}^{m_k}\,\sigma_P{}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \left(\frac{a_2+a_3}{\sqrt{2}}\right)\sigma_P^{-m_k}\,\sigma_Q{}^{-n_k}\,y + \\
&\quad + \lambda_Q^{n_k}\,\sigma_Q^{-n_k}\left(\frac{a_3-a_2}{\sqrt{2}}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + a_1\,\lambda_P{}^{m_k}\widetilde{H}_1\big(\mathbf{x}_k\big) + H_1\big(\hat{\mathbf{x}}_k\big),\\
\bar{y}_{m_k} &= \sigma_Q^{-n_k} + \sigma_P^{-2m_k}\,\sigma_Q{}^{-2n_k}\mu + b_1\,\lambda_P{}^{m_k}\,\sigma_P{}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \\
&\quad + \sigma_P^{-2m_k}\,\sigma_Q{}^{-2n_k}\left(\frac{b_2+b_3+b_4}{2}\right)y^2 + \sigma_P^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-2n_k}\,\big(b_3-b_2\big)\,(\mathfrak{s}_k\,x\,y + \mathfrak{c}_k\,y\,z) + \\
&\quad + \lambda_Q^{2n_k}\sigma_Q^{-2n_k}\left(\frac{b_2+b_3-b_4}{2}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z)^2 + b_1\,\lambda_P{}^{m_k}\widetilde{H}_1\big(\mathbf{x}_k\big) + H_2\big(\hat{\mathbf{x}}_k\big),\\
\bar{z}_{m_k} &= c_1\,\lambda_P^{m_k}\,\sigma_P{}^{-m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{-n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \left(\frac{c_2+c_3}{\sqrt{2}}\right)\sigma_P^{-m_k}\,\sigma_Q{}^{-n_k}\,y + \\
&\quad + \lambda_Q^{n_k}\,\sigma_Q^{-n_k}\left(\frac{c_3-c_2}{\sqrt{2}}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + c_1\,\lambda_P{}^{m_k}\,\widetilde{H}_1\big(\mathbf{x}_k\big) + H_3\big(\hat{\mathbf{x}}_k\big).
\end{aligned}$$

Recalling the map Ψ_{m_k,n_k} in (5.3.20), we have that the inverse map

$$\Psi^{-1}_{m_k,n_k} : U_Q \to \mathbb{R}^3 \tag{5.3.36}$$

is given by

$$\Psi^{-1}_{m_k,n_k}(\bar{x},\bar{y},\bar{z}) = (\sigma_P^{m_k}\sigma_Q{}^{n_k}(\bar{x}-1), \sigma_P^{2m_k}\,\sigma_Q{}^{2n_k}(\bar{y}-\sigma_Q^{-n_k}), \sigma_P^{m_k}\,\sigma_Q{}^{m_k}(\bar{z}-1)).$$

Applying now $\Psi^{-1}_{m_k,n_k}$ to the point $(1+\bar{x}_{m_k}, \bar{y}_{m_k}, 1+\bar{z}_{m_k})$ we get the return map

$$\check{X}_k := \Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(f_{\bar{v}_{m_k,n_k}}) \circ \Psi_{m_k,n_k}(\bar{X}). \tag{5.3.37}$$

Performing the corresponding substitutions, the coordinates $(\check{x}_k, \check{y}_k, \check{z}_k) \in \mathbb{R}^3$ of $\check{X}_k$ are defined by

$$\begin{aligned}
\check{x}_k &= a_1\,\lambda_P^{m_k}\,\lambda_Q{}^{n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \varsigma_1\,y + \sigma_P{}^{m_k}\,\lambda_Q^{n_k}\left(\frac{a_3-a_2}{\sqrt{2}}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + \\
&\quad + a_1\,\lambda_P{}^{m_k}\,\sigma_P^{m_k}\,\sigma_Q{}^{n_k}\,\widetilde{H}_1\big(\mathbf{x}_k\big) + \sigma_P^{m_k}\,\sigma_Q{}^{n_k}H_1\big(\hat{\mathbf{x}}_k\big),\\
\check{y}_k &= \mu + b_1\,\lambda_P^{m_k}\,\sigma_P{}^{m_k}\,\lambda_Q{}^{n_k}\,\sigma_Q^{n_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \varsigma_2\,y^2 + \\
&\quad + \sigma_P{}^{m_k}\,\lambda_Q^{n_k}\,\big(b_3-b_2\big)\,(\mathfrak{s}_k\,x\,y + \mathfrak{c}_k\,y\,z) + \sigma_P{}^{2m_k}\,\lambda_Q^{2n_k}\left(\frac{b_2+b_3-b_4}{2}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z)^2 + \\
&\quad + b_1\,\lambda_P{}^{m_k}\,\sigma_P^{2m_k}\,\sigma_Q{}^{2n_k}\widetilde{H}_1\big(\mathbf{x}_k\big) + \sigma_P^{2m_k}\,\sigma_Q{}^{2n_k}\,H_2\big(\hat{\mathbf{x}}_k\big),\\
\check{z}_k &= c_1\,\lambda_P^{n_k}\,\lambda_Q{}^{m_k}\,(\mathfrak{c}_k\,x - \mathfrak{s}_k\,z) + \varsigma_5\,y + \sigma_P^{m_k}\,\lambda_Q^{n_k}\left(\frac{c_3-c_2}{\sqrt{2}}\right)(\mathfrak{s}_k\,x + \mathfrak{c}_k\,z) + \\
&\quad + c_1\,\lambda_P{}^{m_k}\,\sigma_P^{m_k}\,\sigma_Q{}^{n_k}\widetilde{H}_1\big(\mathbf{x}_k\big) + \sigma_P^{m_k}\,\sigma_Q{}^{n_k}\,H_3\big(\hat{\mathbf{x}}_k\big),
\end{aligned}$$

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where

$$\varsigma_1(\xi,v) := \left(\frac{a_2+a_3}{\sqrt{2}}\right), \quad \varsigma_2(\xi,v) := \left(\frac{b_2+b_3+b_4}{2}\right), \quad \varsigma_5(\xi,v) := \left(\frac{c_2+c_3}{\sqrt{2}}\right),$$

and

$$v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2).$$

This completes the calculations of the return map

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k}.$$

5.3.8
Convergence of the renormalised sequence

We now prove that as $k \to +\infty$ the sequence of maps $\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k}$ converges in the C^r-topology to a family of endomorphisms on any compact set of $\mathbb{R}^3$.

In order to make transparent our calculations, we recall below the convergence of some leading terms in the definition of the coordinates $(\check{x}_k, \check{y}_k, \check{z}_k)$:

- From the neutral dynamic conditions in (5.3.18), we have the convergence ${\sigma_P}^{m_k}\,\lambda_Q^{n_k} \to \xi(\frac{a_3-a_2}{\sqrt{2}})^{-1}$.
- From Remark 5.3.10, we get the limits $\tilde{\mathfrak{c}}_k = \tilde{\mathfrak{s}}_k = \frac{1}{\sqrt{2}}$, $\mathfrak{c}_k \to 0$, and $\mathfrak{s}_k \to 1$.

Using these facts and recalling the spectral conditions in (5.3.4) and the condition $c_2 = c_3$ in (5.3.10), we obtain the following convergence result: for every compact set K in $\mathbb{R}^3$ and $(x,y,z) \in K$ the coordinates

$$(\check{x}_k, \check{y}_k, \check{z}_k) := \Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k}(x,y,z),$$

satisfy

$$\begin{aligned}
&\check{x}_k - a_1{\lambda_P}^{m_k}\sigma_P^{m_k}\,{\sigma_Q}^{n_k}\tilde{H}_1\Big(\mathbf{x}_k\Big) - \sigma_P^{m_k}\,{\sigma_Q}^{n_k}H_1\Big(\hat{\mathbf{x}}_k\Big) \to \xi\,x + \varsigma_1\,y,\\
&\check{y}_k - b_1\,{\lambda_P}^{m_k}\sigma_P^{2m_k}\,{\sigma_Q}^{2n_k}\tilde{H}_1\Big(\mathbf{x}_k\Big) -\\
&\qquad - \sigma_P^{2m_k}\,{\sigma_Q}^{2n_k}H_2\Big(\hat{\mathbf{x}}_k\Big) \to \mu + \varsigma_2\,y^2 + \varsigma_3\,x^2 + \varsigma_4\,x\,y,\\
&\check{z}_k - c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}\,{\sigma_Q}^{n_k}\tilde{H}_1\Big(\mathbf{x}_k\Big) - \sigma_P^{m_k}\,{\sigma_Q}^{n_k}H_3\Big(\hat{\mathbf{x}}_k\Big) \to \varsigma_5\,y,
\end{aligned} \tag{5.3.38}$$

when $k \to +\infty$, where

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$$\begin{aligned}
\varsigma_1(\xi, v) &= \frac{a_2 + a_3}{\sqrt{2}}, \quad \varsigma_2(\xi, v) = \frac{b_2 + b_3 + b_4}{2}, \\
\varsigma_3(\xi, v) &= \xi^2 \left(\frac{b_2 + b_3 - b_4}{(a_3 - a_2)^2} \right), \\
\varsigma_4(\xi, v) &= \xi\sqrt{2} \left(\frac{b_3 - b_2}{a_3 - a_2} \right), \quad \varsigma_5(\xi, v) = \frac{c_2 + c_3}{\sqrt{2}},
\end{aligned} \tag{5.3.39}$$

and

$$v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2).$$

Let us denote the limit endomorphism in (5.3.38) by

$$E_{(\xi,\mu,\bar{\varsigma})}(x, y, z) = (\xi\, x + \varsigma_1\, y,\ \mu + \varsigma_2\, y^2 + \varsigma_3\, x^2 + \varsigma_4\, x\, y,\ \varsigma_5\, y), \tag{5.3.40}$$

where $\bar{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5)$ is a vector of coordinates $\varsigma_i = \varsigma_i(\xi, v)$ as in (5.3.39). We observe this vector depend on both the constant ξ and the transition map f^{N_2}.

We will prove the following:

Lemma 5.3.14 *When $k \to +\infty$ the sequence*

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\left(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\right) \circ \Psi_{m_k,n_k}$$

converges to $E_{(\xi,\mu,\bar{\varsigma})}$ in the C^r-topology on compact sets in $\mathbb{R}^3$.

The proof this lemma consist in to estimate the Landau's symbols of the high order terms of the difference

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\left(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\right) \circ \Psi_{m_k,n_k} - E_{(\xi,\mu,\bar{\varsigma})}$$

and we check the C^r-convergence to zero on compact set in $\mathbb{R}^3$. In this difference, there are two types of high order terms associated to terms $\widetilde{H}_i(\cdot)$ and $H_i(\cdot)$, $i = 1, 2, 3$, in the definition of the transitions maps, see (5.3.7) and (5.3.9). We observe that from (5.3.38) the higher order terms of

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\left(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\right) \circ \Psi_{m_k,n_k} - E_{(\xi,\mu,\bar{\varsigma})}$$

containing the (higher order) terms H_i, $i = 1, 2, 3$, are

$$\sigma_P^{m_k}\, {\sigma_Q}^{n_k} H_1(\hat{\mathbf{x}}_k), \quad \sigma_P^{2m_k}\, {\sigma_Q}^{2n_k} H_2(\hat{\mathbf{x}}_k), \quad \sigma_P^{m_k}\, {\sigma_Q}^{n_k} H_3(\hat{\mathbf{x}}_k). \tag{5.3.41}$$

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On the other hand, to study the higher order terms in

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k} - E_{(\xi,\mu,\bar{\varsigma})}$$

associated to the terms $\widetilde{H}_i$, $i = 1,2,3$, we need to estimate the Landau's symbols in

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k}$$

of those (higher order) term omitted in the coordinates (5.3.31), see Remark 5.3.13. This higher order term also converges to zero in the C^r-topology on compact set in $\mathbb{R}^3$. This completes the proof of lemma.

Before going to the prove of lemma we point out the following.

Remark 5.3.15 We recall the sequences $\mathbf{x}_k = (x_{n_k}, y_{n_k}, z_{n_k})$ and $\hat{\mathbf{x}}_k = (\hat{x}_{m_k}, \hat{y}_{m_k}, \hat{z}_{m_k})$, whose coordinates are given in (5.3.25) and (5.3.31), respectively.

We note that by definition:

– the higher order terms $\widetilde{H}_i\Big(\mathbf{x}_k\Big)$, $i = 1,2,3$, associates to the transition from Q to P (see (5.3.7)), are dominated by the quadratic terms

$$x^2_{n_k}, y^2_{n_k}, z^2_{n_k}, x_{n_k}\, y_{n_k}, x_{n_k}\, z_{n_k}, y_{n_k}\, z_{n_k}.$$

Recalling the expansions (5.3.32) and the symbols of Landau of x_{n_k}, y_{n_k} and z_{n_k} in (5.3.26) we have

$$\widetilde{H}_i\Big(\mathbf{x}_k\Big) = O(\lambda_Q^{2n_k}) + O(\sigma_P{}^{-2m_k}\, \lambda_Q^{n_k} \sigma_Q^{-n_k}) + O(\sigma_P^{-4m_k}\, \sigma_Q{}^{-2n_k}).$$

Note that x_{n_k} and z_{n_k} have the same symbol of Landau.

– the higher order terms $H_1\Big(\hat{\mathbf{x}}_k\Big)$ and $H_3\Big(\hat{\mathbf{x}}_k\Big)$, associates to the transition from P to Q (see (5.3.9)), are dominated by quadratic terms

$$\hat{x}^2_{m_k}, \hat{y}^2_{m_k}, \hat{z}^2_{m_k}, \hat{x}_{m_k}\hat{y}_{m_k}, \hat{x}_{m_k}\hat{z}_{m_k}, \hat{y}_{m_k}\hat{z}_{m_k},$$

and the higher order term $H_2\Big(\hat{\mathbf{x}}_k\Big)$ is are dominated by the quadratic terms

$$\hat{x}^2_{m_k}, \hat{x}_{m_k}\hat{y}_{m_k}, \hat{x}_{m_k}\hat{z}_{m_k}.$$

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Recalling the symbols of Landau of $\hat{x}_{m_k}$,$\hat{y}_{m_k}$ and $\hat{z}_{m_k}$ given in (5.3.12), we have

$$H_i\left(\hat{\mathbf{x}}_k\right) = O({\lambda_P}^{2m_k}) + O({\lambda_P}^{m_k}\sigma_P^{-m_k}{\sigma_Q}^{-n_k}) + O({\lambda_P}^{m_k}\lambda_Q^{n_k}\sigma_Q^{-n_k}) + \\ + O(\sigma_P^{-2m_k}{\sigma_Q}^{-2n_k}) + O({\sigma_P}^{-m_k}\lambda_Q^{n_k}\sigma_Q^{-2n_k}) + O(\lambda_Q^{2n_k}\sigma - Q^{-2n_k}), \quad i = 1,3;$$

and

$$H_2\left(\hat{\mathbf{x}}_k\right) = O({\lambda_P}^{2m_k}) + O({\lambda_P}^{m_k}\sigma_P^{-m_k}{\sigma_Q}^{-n_k}) + O(\lambda_P^{m_k}{\lambda_Q}^{n_k}\sigma_Q^{-n_k}).$$

Note that $\hat{y}_{m_k}$ and $\hat{z}_{m_k}$ have the same symbol of Landau.

In what follows we proceed to complete the proof of the lemma.

Proof.[Proof of Lemma 5.3.14] In the proof of the lemma we need to control in a separated way to types of higher order terms (associated to H_i and $\widetilde{H}_i$).

Higher order terms containing H_1, H_2, H_3. From the estimates in Remark 5.3.15 and the expressions in (5.3.38) we have

$$\begin{aligned}
\sigma_P^{m_k}{\sigma_Q}^{n_k}H_i\left(\hat{\mathbf{x}}_k\right) &= O({\lambda_P}^{2m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}) + O({\lambda_P}^{m_k}) + \\
&\quad + O({\lambda_P}^{m_k}{\sigma_P}^{m_k}\lambda_Q^{n_k}) + O(\sigma_P^{-m_k}{\sigma_Q}^{-n_k}) + \\
&\quad + O(\lambda_Q^{n_k}\sigma_Q^{-n_k}) + O({\sigma_P}^{m_k}\lambda_Q^{2n_k}\sigma_Q^{-n_k}), \quad i = 1,3;\\
\sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\left(\hat{\mathbf{x}}_k\right) &= O({\lambda_P}^{2m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}) + O({\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}) + \\
&\quad + O(\lambda_P^{m_k}{\sigma_P}^{2m_k}{\lambda_Q}^{n_k}\sigma_Q^{n_k}).
\end{aligned} \tag{5.3.42}$$

By Lemma (5.3.6) there exist a constant $C > 0$ such that

$$\lambda_P{}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k} = \left({\lambda_P}^{\frac{m_k}{2}}{\sigma_P}^{m_k}\sigma_Q^{n_k}\right)^2 < C\left(\left(({\lambda_P}^{\frac{1}{2}}\sigma_P)^{\eta}\sigma_Q\right)^{n_k}\right)^2. \tag{5.3.43}$$

The spectral conditions (5.3.4) implies that the right-hand term in this last inequality tends to zero when $k \to +\infty$. Thus, we have that

$$\lambda_P^{m_k}{\sigma_P}^{2m_k}{\lambda_Q}^{n_k}\sigma_Q^{n_k},\ {\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k} \to 0$$

Moreover, it is easy to see also that the convergence of (5.3.42) it holds for the derivatives of order $1 \le k \le r$.

Higher order terms containing $\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3$. Here we need to study the higher order terms of $\widetilde{H}_i$, $i = 1,2,3$, in the coordinates $(\check{x}_k, \check{y}_k, \check{z}_k)$ which are explicit in (5.3.38) as well as the omitted terms are higher order, see Remark 5.3.13. For this last type of terms, we need to estimate the effect of $\Psi^{-1}_{m_k,n_k} \circ f^{N_2}_{\bar{\mu}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}$ on the Landau's symbols of higher order terms that were omitted in the coordinates $\hat{y}_{m_k}$ and $\hat{z}_{m_k}$ in (5.3.31), see also Remark (5.3.13). We recall that

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the omitted terms are

$$\widehat{H}_i\big(\mathbf{x}_k\big) = \widetilde{H}_i\big(\mathbf{x}_k\big) - \lambda_Q^{2n_k}\tilde{\rho}_{i,k}, \quad i = 2, 3,$$

with corresponding symbols of Landau

$$\widehat{H}_i\big(\mathbf{x}_k\big) = O({\sigma_P}^{-m_k}\lambda_Q^{2n_k}{\sigma_Q}^{-n_k}) + O(\sigma_P^{-4m_k}{\sigma_Q}^{-2n_k}), \quad i = 2, 3,$$

see (5.3.34).

We start by studying the coordinates $\check{x}_k$ and $\check{z}_k$. It is easy to see that these coordinates contains the following higher order terms

$$\lambda_P^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\widetilde{H}_1\big(\mathbf{x}_k\big), \quad \sigma_P^{2m_k}{\sigma_Q}^{n_k}\widehat{H}_2\big(\mathbf{x}_k\big), \quad \sigma_P^{2m_k}{\sigma_Q}^{n_k}\widehat{H}_3\big(\mathbf{x}_k\big). \tag{5.3.44}$$

Using the estimates in Remark 5.3.15, we have

$$\begin{aligned}
{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\widetilde{H}_1\big(\mathbf{x}_k\big) &= O(\lambda_P^{m_k}{\sigma_P}^{m_k}{\lambda_Q}^{2n_k}\sigma_Q^{n_k}) + O({\lambda_P}^{m_k}{\sigma_P}^{-m_k}\lambda_Q^{n_k}) + \\
&\quad + O({\lambda_P}^{m_k}\sigma_P^{-3m_k}{\sigma_Q}^{-n_k}),\\
\sigma_P^{2m_k}{\sigma_Q}^{n_k}\widehat{H}_i\big(\mathbf{x}_k\big) &= O({\sigma_P}^{m_k}\lambda_Q^{2n_k}) + O(\sigma_P^{-2m_k}{\sigma_Q}^{-n_k}), \quad i = 2, 3.
\end{aligned}$$

The spectral conditions (5.3.4), the convergence in (5.3.18), and the previous arguments imply the C^r-convergence to zero of (5.3.44) in compact sets of $\mathbb{R}^3$.

We now study the convergence of the coordinate $\check{y}_k$ in

$$\Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}\Big(f_{\bar{\upsilon}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}\Big) \circ \Psi_{m_k,n_k}.$$

Keeping in mind Remark 5.3.13, is not hard to see that the higher order terms in $\check{y}_k$ are

$$\begin{aligned}
&{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\widetilde{H}_1\big(\mathbf{x}_k\big),\\
&\sigma_P^{2m_k}{\sigma_Q}^{n_k}\widehat{H}_i\big(\mathbf{x}_k\big), \quad {\sigma_P}^{2m_k}\lambda_Q^{n_k}\sigma_Q^{n_k}\widehat{H}_i\big(\mathbf{x}_k\big), \quad i = 2, 3;\\
&\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\Big(\tilde{\sigma}^{m_k}\widehat{H}_i\big(\mathbf{x}_k\big)\Big)^2, \quad i = 2, 3;\\
&\sigma_P^{4m_k}{\sigma_Q}^{2n_k}\widehat{H}_2\big(\mathbf{x}_k\big)\,\widehat{H}_3\big(\mathbf{x}_k\big).
\end{aligned} \tag{5.3.45}$$

Noting that the terms $\widehat{H}_2\big(\mathbf{x}_k\big)$ and $\widehat{H}_3\big(\mathbf{x}_k\big)$ have the same symbol of Landau (see Remark 5.3.15), and using the spectral conditions in (5.3.4), the convergence in (5.3.45) can be reduced to the convergence of the terms ${\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\widetilde{H}_1\big(\mathbf{x}_k\big)$ and $\sigma_P^{2m_k}{\sigma_Q}^{n_k}\widehat{H}_i\big(\mathbf{x}_k\big)$, $i = 2, 3$. Observe that these two last expressions were estimated in the analysis of the coordinates $\check{x}_k$ and $\check{z}_k$ above.

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Therefore it follows from the estimates in Remark 5.3.15 that

$$\lambda_P{}^{m_k}\sigma_P^{2m_k}\sigma_Q{}^{2n_k}\widetilde{H}_1\Big(\mathbf{x}_k\Big) = O(\lambda_P{}^{m_k}\sigma_P{}^{2m_k}\lambda_Q{}^{2n_k}\sigma_Q^{2n_k}) + O(\lambda_P{}^{m_k}\lambda_Q{}^{n_k}\sigma_Q{}^{n_k}) + \\ + O(\lambda_P{}^{m_k}\sigma_P{}^{-2m_k}).$$

The only term whose convergence to zero is not obvious is $\lambda_P{}^{m_k}\sigma_P^{2m_k}\lambda_Q^{2n_k}\sigma_Q{}^{2n_k}$.

Applying same arguments that in the study of the convergence of the equations in (5.3.45) we conclude that if $k \to +\infty$ then

$$\lambda_P{}^{m_k}\sigma_P^{2m_k}\lambda_Q^{2n_k}\sigma_Q{}^{2n_k} \to 0.$$

This completes the proof of Lemma 5.3.14. ■

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6
Laminations of the parameter space corresponding to diffeomorphisms with blender-horseshoes: Proof of item (I) in Theorem 4

Recall the definitions of the space $\mathcal{T}_{\rm quad}$ in Definition 4.2.1, of the set $\mathcal{N}^r_{P,Q}(\mathcal{T})$, with $\mathcal{T} \subset \mathcal{T}_{\rm quad}$ in Definition 4.2.2 and of the renormalised sequence of diffeomorphisms $\mathcal{R}_{m_k,n_k}(f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)})$ of a diffeomorphism $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}_{\rm quad})$ in Theorem 2. In what follows, the arguments φ_P and φ_Q in the notation $f_{\bar{v}_{m_k,n_k}(\mu,\varphi_P,\varphi_Q)}$ are irrelevant, thus they will be omitted. Recall also that $\mathrm{RV}(h)$ denotes the set of regular values of a map $h \in C^r(M,N)$ and that if $y \in \mathrm{RV}(h)$ then $h^{-1}(y)$ is a sub-manifold in M of dimension $\dim(M) - \dim(N) > 0$.

The main goal in this section is the next theorem above. We observe that this theorem implies item (I) in Theorem 4.

Theorem 6.0.1 *Given $r \geq 2$ there are:*

- *an open subset $\mathcal{B} := I \times \mathcal{V} \subset \mathbb{R} \times \mathbb{R}^2$ and an open interval $J \subset \mathbb{R}$,*
- *a family $x \in \mathbb{R} \to \gamma_x : \mathrm{Dom}(\gamma_x) \subset \mathbb{R}^9 \to \mathbb{R}^2$ of C^∞-maps such that for each x it holds $\mathrm{RV}(\gamma_x) = \mathbb{R}^2$, and*
- *a projection $\Pi : \mathcal{T}_{\rm quad} \to \mathbb{R}^9$*

such that every sub-manifold of the family $\mathcal{T}_{\mathcal{B}} \colon = \left\{\mathcal{T}_{\bar{b}} : \bar{b} \in \mathcal{B}\right\}$ where

$$\mathcal{T}_{\bar{b}} := \Pi^{-1}\Big(\gamma_\xi^{-1}(w)\Big), \quad \bar{b} := (\xi, w) \in \mathcal{B},$$

satisfying the following property: If $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}_{\bar{b}})$ then for every $\mu \in J$ the renormalised sequence $\mathcal{R}_{m_k,n_k}\Big(f_{\bar{v}_{m_k,n_k}(\mu)}\Big)$ has a blender-horseshoe near the heterodimensional tangency point of f.

The proof Theorem 6.0.1 it is divided three parts and it is organized as follows.

In the first part, Section 6.1, we split the manifold $\mathcal{T}_{\rm quad}$ into different parts according algebraic and geometric properties.

In the second part, Section 6.2, we state Theorems 6.2.2 and 6.2.2, that contains dynamics properties of the family $E_{(\xi,\mu,\bar{\varsigma})}$ in accordance with the

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choice of the parameters $(\xi, \mu, \bar{\varsigma})$ (blender-horseshoe and strong homoclinic intersection associated to saddle-node). In particular, Theorem 6.0.1 follows from Theorem 6.2.2.

In Sections 6.3 and 6.6, we prove Theorems 6.2.2 and 6.2.4, respectively.

6.1 Splitting the space of quadratic transition maps

Recall the definition of the manifold $\mathcal{T}_{\text{quad}}$ in Definition 4.2.1. We split this manifold into different parts according algebraic (polynomial form of the transition) and geometric (relative positions and shape of the invariant manifold at the tangency) conditions that we proceed to describe. This partition will play a key in the proof of the second part of Theorem 4. The set $\mathcal{T}_{\text{quad}}$ consists of the quadratic polynomials $q = q_v$ of the form

$$q_v \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1x + a_2y + a_3z \\ b_1x + b_2y^2 + b_3z^2 + b_4yz \\ c_1x + c_2y + c_3z \end{pmatrix}, \tag{6.1.1}$$

where $v := (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2)$ belongs to the set

$$\mathrm{Q} := \Big\{(a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathbb{R}^9 : (a_3 - a_2)\, b_1\, (b_2 + b_3 + b_4)\, c_2 \neq 0\Big\}.$$

Consider the diffeomorphism given by the projection

$$\Pi : \mathcal{T}_{\text{quad}} \to \mathrm{Q}, \quad \Pi(q_v) = v. \tag{6.1.2}$$

Remark 6.1.1 It is not hard to see that the set Q is a union of sixteen open sets in $\mathbb{R}^9$ whose closure is whole $\mathbb{R}^9$. We detail breafly this last claim. Each of the four conditions deferent of zero defining the set Q in $\mathbb{R}^9$ providing two inequality. This splits Q in 2^4 open sets. We can describe these open sets as follows. We splits $\mathbb{R}^9$ as $\mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}^2$ with coordinates $(\bar{a}, \bar{b}, \bar{c})$, where $\bar{a} = (a_1, a_2, a_3)$, $\bar{b} = (b_1, b_2, b_3, b_4)$ and $\bar{c} = (c_1, c_2)$. Thus, the set of vectors $\bar{a} \in \mathbb{R}^3$ satisfying $a_2 - a_3 \neq 0$ consist of whole $\mathbb{R}^3$ minus one plane. Analogously, the set of vectors $\bar{b} \in \mathbb{R}^4$ satisfying $b_1 \neq 0$ and $b_2 + b_3 + b_3 \neq 0$ consist of whole $\mathbb{R}^4$ minus the two hyperplanes (three-dimensional subspaces) $\{0\} \times \mathbb{R}^3$ and $\mathbb{R} \times \mathrm{P}$ where P is the plane in $\mathbb{R}^3$ generated by the condition $b_2 + b_3 + b_3 = 0$. Finally, the set of vectors $\bar{c} \in \mathbb{R}^2$ satisfying $c_2 \neq 0$ consist of whole $\mathbb{R}^2$ least one line.

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6.1.1 The algebraic splitting

Consider the following subsets of Q,

$$\mathrm{Q}^+ := \Big\{(a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q} : b_2 + b_3 + b_4 > 0,\ c_2(a_2 - a_3) \neq 0\Big\};$$
$$\mathrm{Q}^- := \Big\{(a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q} : b_2 + b_3 + b_4 < 0,\ c_2(a_2 - a_3) \neq 0\Big\};$$

We consider the partition of $\mathcal{T}_{\mathrm{quad}}$ in open regions:

$$\mathcal{T}_{\mathrm{quad}} = \mathcal{T}^+_{\mathrm{quad}} \cup \mathcal{T}^-_{\mathrm{quad}} \quad \mathcal{T}^{\mp}_{\mathrm{quad}} := \Pi^{-1}(\mathrm{Q}^{\mp}). \tag{6.1.3}$$

Remark 6.1.2 Note that any set $\mathrm{Q}^{\pm,\mp}$ is a union of eight open set in $\mathbb{R}^9$. See Remark 6.1.1.

Consider the following subsets of Q:

$$\begin{aligned} \mathrm{Q}^I &:= \Big\{(a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q} : a_2 + a_3 \neq 0\Big\}; \\ \mathrm{Q}^{II} &:= \Big\{(a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q} : a_2 + a_3 = 0\Big\}; \end{aligned} \tag{6.1.4}$$

We consider the partition of $\mathcal{T}_{\mathrm{quad}}$ in open regions:

$$\mathcal{T}_{\mathrm{quad}} = \mathcal{T}^I_{\mathrm{quad}} \cup \mathcal{T}^{II}_{\mathrm{quad}}, \quad \text{where} \quad \mathcal{T}^{\ell}_{\mathrm{quad}} := \Pi^{-1}(\mathrm{Q}^{\ell}), \quad \ell = I, II. \tag{6.1.5}$$

Remark 6.1.3 Note that the submanifold Q^{II} has dimension eight.

6.1.2 The geometrical splitting

This splitting depends on the behaviour of $v \in \mathrm{Q} \to q_v \in \mathcal{T}_{\mathrm{quad}}$ in a neighbourhood of $(0,0,0) \in \mathbb{R}^3$. Consider the neighbourhood $U_{\mathbf{0}} = [-\delta,\delta]^3$ of $(0,0,0)$ for some small $\delta > 0$ and write

$$U_{\mathbf{0}} := \bigcup_{x \in [-\delta,\delta]} B_x, \quad \text{where } B_x = \{x\} \times [-\delta,\delta]^2.$$

Let

$$v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q}.$$

Note that for every $x \in [-\delta,\delta]$ the leaf $q_v(B_x)$ is the translation of $q_v(B_0)$ by the vector $x\bar{w}_v \in \mathbb{R}^3$, where $\bar{w}_v = (a_1, b_1, c_1)$. Thus in what follows we restrict

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our analysis to the leave $q_v(B_0)$ and translate this analysis to $q_v(B_x)$ using the translation $x\bar{w}_v$.

Writing $\varphi(y,z) = b_2y^2 + b_3z^2 + b_4yz$ we have that

$$q_v(0,y,z) = (a_2\, y + a_3 z, \varphi(y,z), c_2\, y + c_2 z) = (\bar{x}, \varphi \circ A_v^{-1}(\bar{x},\bar{z}), \bar{z}), \qquad (6.1.6)$$

where

$$(\bar{x},\bar{z}) = A_v(y,z) \quad \text{and} \quad A_v = \begin{pmatrix} a_2 & a_3 \\ c_2 & c_2 \end{pmatrix}.$$

Note that the definition of $\mathcal{T}_{\mathrm{quad}}$ implies that $\det A_v \neq 0$ and hence A_v^{-1} is well defined. Thus $q_v(B_0)$ is locally the graph of the function

$$(\bar{x},\bar{z}) \to \varphi_{A_v}(\bar{x},\bar{z}), \quad \text{where } \varphi_{A_v} :\, = \varphi \circ A_v^{-1},$$

see Figure 6.1.

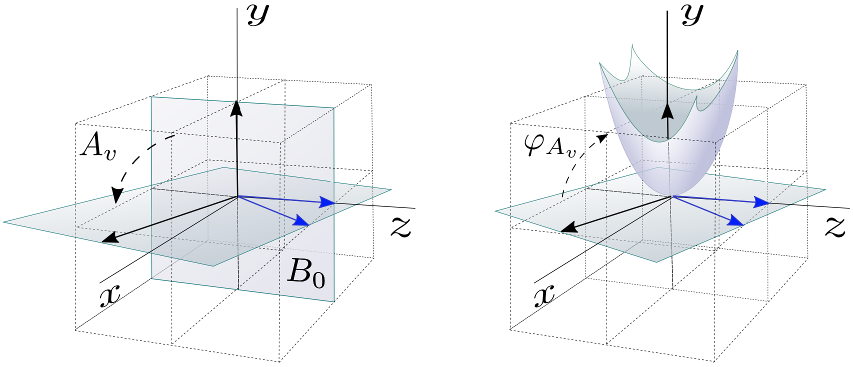

Figure 6.1: The linear map A_v and the function φ_{A_v}.

We observe that $\varphi_{A_v}(0,0) = 0$ and that $(0,0)$ is a critical point of φ_{A_v}, i.e., $D\varphi_{A_v}(0,0) = (0,0)$. We now study the type of this criticality.

- *The Hessian matrix of φ_{A_v} at the* $(0,0)$. This matrix is given by

$$H\varphi_{A_v}(0,0) = \begin{pmatrix} \frac{\partial^2}{\partial x^2}\varphi_A(0,0) & \frac{\partial^2}{\partial x \partial z}\varphi_A(0,0) \\ \frac{\partial^2}{\partial x \partial z}\varphi_A(0,0) & \frac{\partial^2}{\partial z^2}\phi_A(0,0) \end{pmatrix}. \qquad (6.1.7)$$

The entries of $H\varphi_{A_v}(0,0)$ depend on the vector $v \in \mathrm{Q}$. A straightforward calculation gives

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\varphi_{A_v}(0,0) &= \frac{2(b_2+b_3-b_4)}{(a_2-a_3)^2} :\, = c_1(v), \\ \frac{\partial^2}{\partial z^2}\varphi_{A_v}(0,0) &= \frac{2(b_2\, a_3^2 + b_3\, a_2^2 - b_4\, a_2\, a_3)}{c_2^2\,(a_2-a_3)^2} :\, = c_2(v), \qquad (6.1.8) \\ \frac{\partial^2}{\partial x \partial z}\varphi_{A_v}(0,0) &= \frac{b_4\,(a_2-a_3) - 2(b_2\, a_3 + b_3\, a_2)}{c_2\,(a_2-a_3)^2} :\, = c_3(v). \end{aligned}$$

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Thus, if $\det\left(D^2\phi_A(0,0)\right) > 0$ then $\widetilde{Y}$ is of elliptic type while if $\det\left(D^2\phi_A(0,0)\right) < 0$ then $\widetilde{Y}$ is of hyperbolic type. Thus,

$$H(v) :\ = \det\left(H\varphi_{A_v}(0,0)\right) = c_1(v)c_3(v) - c_2(v)^2.$$

There are the following possibilities for the critical point:

(i) If $H(v) > 0$ and $c_1(v) > 0$ then $(0,0)$ is a local minimum,

(ii) If $H(v) > 0$ and $c_1(v) < 0$ then $(0,0)$ is a local maximum,

(iii) If $H(v) < 0$ then $(0,0)$ is a saddle point,

(iv) If $H(v) = 0$ there is no information about the criticality.

In cases (i) and (ii) we say that $(0,0)$ is of *elliptic type* and the case (iii) we say that $(0,0)$ is of *hyperbolic type*.

Remark 6.1.4 The parameters corresponding to a Hessian equal to zero (case (iv) above) will be discarded. We note that the map $v \in \mathrm{Q} \to H(v)$ is smooth one and $dH(v) \neq 0$ for every $v \in \mathrm{Q}$ (this last assertions follows from fact that $dH(v) = 0$ iff $b_2 = b_3 = b_4 = 0$). From here we get that for every $c \in \mathbb{R}$, $H^{-1}(c)$ is a eight-sub-manifold in Q, see (25, Theorem 3.2).

Under the condition $H(v) > 0$, the surface $q_v(B_0)$ is a tangent to $C_0 := [-\delta,\delta] \times \{0\} \times [-\delta,\delta]$ in $(0,0,0)$ this follows from cases (i) and (ii). When $H(v) < 0$, the set $q_v(B_0)$ meet transversely to C_0. This is the case (iii) in that $q_v(B_0)$ has horse-saddle shape. From the comment above the leaf $q_v^{-1}(C_0)$ satisfies identical properties in relation to B_0, see Figure 6.2.

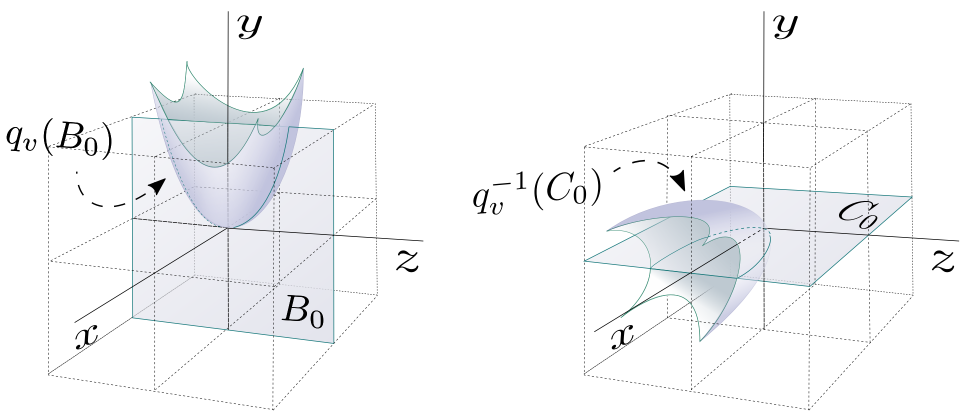

Figure 6.2: The parabolic configuration in (P1).

• *Relative positions.* We now to study the local relative position of the surfaces $q_v(B_0)$ and $q_v^{-1}(C_0)$ at $(0,0,0)$. The relative position of $q_v(B_0)$ is determined by conditions (i),(ii) and (iii), while the relative position of $q_v^{-1}(C_0)$ is determined

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by the orientation of the linear maps $(y,z) \to A_v(y,z) = (\bar{x},\bar{z})$. More precisely, consider the sets $U_0^{(\pm,\cdot)}$ and $U_0^{(\cdot,\pm)}$ given by

$$U_0^{(+,\cdot)} :\ = [0,\delta] \times [-\delta,\delta]^2, \qquad U_0^{(-,\cdot)} :\ = [-\delta,0] \times [-\delta,\delta]^2,$$

$$U_0^{(\cdot,+)} :\ = [-\delta,\delta] \times [0,\delta] \times [-\delta,\delta], \qquad U_0^{(\cdot,-)} :\ = [-\delta,\delta] \times [-\delta,0] \times [-\delta,\delta].$$

If $q_v(B_0)$ and $q_v^{-1}(C_0)$ are paraboloids (cases (i) and (ii) above) there are the following configurations (P1)-(P4) and (H) given by:

(P1) if $H(v) > 0$, $c_1(v) > 0$ and $\det A_v > 0$ then

$$q_v(B_0) \subset U_0^{(\cdot,+)} \quad \text{and} \quad q_v^{-1}(C_0) \subset U_0^{(+,\cdot)};$$

(P2) if $H(v) > 0$, $c_1(v) < 0$ and $\det A_v > 0$

$$q_v(B_0) \subset U_0^{(\cdot,-)} \quad \text{and} \quad q_v^{-1}(C_0) \subset U_0^{(-,\cdot)};$$

(P3) if $H(v) > 0$, $c_1(v) > 0$ and $\det A_v < 0$

$$q_v(B_0) \subset U_0^{(\cdot,+)} \quad \text{and} \quad q_v^{-1}(C_0) \subset U_0^{(-,\cdot)};$$

(P4) if $H(v) > 0$, $c_1(v) < 0$ and $\det A_v < 0$

$$q_v(B_0) \subset U_0^{(\cdot,-)} \quad \text{and} \quad q_v^{-1}(C_0) \subset U_0^{(+,\cdot)}.$$

If $q_v(B_0)$ and $q_v^{-1}(C_0)$ are of horse-saddle type (i.e., $H(v) < 0$) it holds that

(H) $q_v(B_0) \pitchfork C_0 \neq \emptyset$.

This condition implies that $q_v^{-1}(C_0) \pitchfork B_0 \neq \emptyset$ and

$$q_v(B_0) \cap U_0^{(\cdot,\pm)} \neq \emptyset \quad \text{and} \quad q_v^{-1}(B_0) \cap U_0^{(\pm,\cdot)} \neq \emptyset. \tag{6.1.9}$$

6.1.3
The split

We are now ready to define the splitting of $\mathcal{T}_{\text{quad}}$ in terms of the conditions above. Consider the sets

$$\mathrm{Q}^p := \Big\{ v \in \mathrm{Q} : H(v) > 0 \Big\} \quad \text{and} \quad \mathrm{Q}^h := \Big\{ v \in \mathrm{Q} : H(v) < 0 \Big\} \tag{6.1.10}$$

and the subsets $\mathrm{Q}^{p,\pm,\mp}$ of Q^p defined by:

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$$\mathrm{Q}^{p,+,+} := \Big\{ v \in \mathrm{Q}^{p} \colon v \text{ satisfying (P1)} \Big\},\ \ \mathrm{Q}^{p,-,+} := \Big\{ v \in \mathrm{Q}^{p} \colon v \text{ satisfying (P2)} \Big\};$$
$$\mathrm{Q}^{p,+,-} := \Big\{ v \in \mathrm{Q}^{p} \colon v \text{ satisfying (P3)} \Big\},\ \ \mathrm{Q}^{p,-,-} := \Big\{ v \in \mathrm{Q}^{p} \colon v \text{ satisfying (P4)} \Big\}.$$

Consider the corresponding subsets in $\mathcal{T}_{\mathrm{quad}}$ given by

$$\mathcal{T}^{h}_{\mathrm{quad}} := \Pi^{-1}(\mathrm{Q}^{h}) \quad \text{and} \quad \mathcal{T}^{p,\pm,\mp}_{\mathrm{quad}} := \Pi^{-1}(\mathrm{Q}^{p,\pm,\mp}). \tag{6.1.11}$$

Remark 6.1.5 We observe that Q^{p} and Q^{h} are open sets in Q such that $\overline{\mathrm{Q}^{p} \cup \mathrm{Q}^{h}} = \mathrm{Q}$. Indeed, by Remark 6.1.4, $\mathrm{Q}^{p} \cup \mathrm{Q}^{h}$ is equal to Q minus a eight-sub-manifolds (given by $H^{-1}(0)$).

6.2 Dynamics of the quadratic family: Theorems 6.2.2 and 6.2.4

We study some dynamical properties of the quadratic family of endomorphisms

$$E_{(\xi,\mu,\bar{\varsigma})}(x,y,z) = (\xi\, x + \varsigma_1\, y,\ \mu + \varsigma_2\, y^2 + \varsigma_3\, x^2 + \varsigma_4\, x\, y,\ \varsigma_5\, y), \tag{6.2.1}$$

agree to parameters $(\xi,\mu,\bar{\varsigma})$, where

$$\bar{\varsigma} \colon \mathbb{R} \times \mathbb{R}^9 \to \mathbb{R}^5, \quad \bar{\varsigma}(\xi,v) = \Big(\varsigma_1(\xi,v),\ldots,\varsigma_5(\xi,v)\Big). \tag{6.2.2}$$

Our main result state that there exist an open set $\mathcal{B}$ in $\mathbb{R}^3$, an open interval J, a family of submanifolds $\Big\{\mathrm{Q}_{\bar{b}} : \bar{b} \in \mathcal{B}\Big\}$ in $\mathbb{R}^9$, and a submanifold Q' in $\mathbb{R}^9$ satisfying the following properties:

(i) Let $\bar{b} \in \mathcal{B}$. Then there exist a compact set $K \subset \mathbb{R}^3$ such that if $(\xi,\mu,v) \in I \times J \times \mathrm{Q}_{\bar{b}}$ then $E_{(\xi,\mu,\bar{\varsigma}(\xi,v))}|_K$ has horseshoes-blenders.

(ii) Let $(\mu,v) \in J \times \mathrm{Q}'$. Then there are a compact set $K \subset \mathbb{R}^3$ such that $E_{(1,\mu,\bar{\varsigma}(1,v))}|_K$ has a two partially hyperbolic saddle-node whose strong invariant manifolds meets cyclically and quasi-transversely.

Properties above allows translate/generate some dynamical properties to diffemorphisms nearby enough to such endomorphisms. A immediate consequence from item (i) is that every (local) diffeomorphism $F|_K$ sufficiently C^1-close to $E_{(\xi,\mu,\bar{\varsigma}(\xi,v))}|_K$ has blender-horseshoe. This fact will be used in the proof of item (I) of Theorem 4.

On the other hand, we will see later (Theorem 6.2.4) that the item (ii) leads to the following assertion. If $(\mu,v) \in J \times \mathrm{Q}'$ then every sequence in $\mathrm{Diff}^r(\mathbb{R}^3)$, C^r-converging to $E_{(1,\mu,\bar{\varsigma}(1,v))}$ on compact sets in $\mathbb{R}^3$, may be slightly C^r-perturbed so that this new sequence display *strong homoclinic intersections*

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associated to a partially saddle-node (roughly, a saddle with a neutral direction whose strong stable and unstable manifolds meets quasi-transversely). This type of saddle-node are a key ingredient to generate robust (heterodimensional) cycles, see (9). We now formulate in precise form our assertions.

Definition 6.2.1 (Saddle-nodes and strong homoclinic intersections) Let S be a periodic point of period $\pi(S)$ of a diffeomorphism $f: M \to M$.

- We say that S is a *partially hyperbolic saddle-node* of f if the derivative $Df^{\pi(S)}(S)$ has exactly one eigenvalue equal to 1 and all other eigenvalues of $Df^{\pi(S)}(S)$, are all different from 1 in modulus.
- Consider the strong unstable (resp. stable) invariant direction E^{uu} (resp. E^{ss}) corresponding to the eigenvalues κ of $Df^{\pi(S)}(S)$ with $|\kappa| > 1$ (resp. $|\kappa| < 1$). The strong unstable manifold $W^{\mathrm{uu}}(S, f)$ of S is the unique f-invariant manifold tangent to E^{uu} of the same dimension as E^{uu}. The strong stable manifold $W^{\mathrm{ss}}(S, f)$ of S is defined similarly considering E^{ss}.
- We say that S has a *strong homoclinic intersection* at a point $X \neq S$ if $X \in W^{\mathrm{ss}}(S, f) \cap W^{\mathrm{uu}}(S, f)$ and

$$\dim\Big(T_X W^{\mathrm{ss}}(S, f) \oplus T_X W^{\mathrm{uu}}(S, f)\Big) = \dim M - 1.$$

To state the two main results of this section recall the definitions of the Henón-like family $G_{(\xi,\mu,\kappa,\eta)} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ in (3.5.1), the sub-manifolds Q^I and Q^{II} in (6.1.4), and the map $\bar{\varsigma}$ in (6.2.2) with components $\varsigma_1, \dots, \varsigma_5$.

Theorem 6.2.2 (Blender-horseshoes for $E_{(\xi,\mu,\bar{\varsigma})}$) *There exist*

- *an open subset $\mathcal{B} = I \times \mathcal{V}$ in $\mathbb{R} \times \mathbb{R}^2$ and an open interval J,*
- *a family of seven-sub-manifolds $\Big(\mathrm{Q}_{\mathbf{w}}\Big)_{\mathbf{w} \in \mathbb{R}^3} \subset \mathrm{Q}^I$,*
- *a family of coordinate change $\Theta_{\boldsymbol{w}}: \mathbb{R}^4 \to \mathbb{R}^4$, $\boldsymbol{w} \in \mathbb{R}^3$, and*
- *rational maps $\kappa : \mathrm{Dom}(\kappa) \to \mathbb{R}$ and $\eta : \mathrm{Dom}(\eta) \to \mathbb{R}$ whose domains are contained in $\mathbb{R} \times \mathbb{R}^9$*

such that

(I) *For every $\xi, \mu \in \mathbb{R}$ and every $v \in \mathrm{Q}^I$ the endomorphisms $(\mu, E_{(\xi,\mu,\bar{\varsigma}(\xi,v))})$ and $(\mu, G_{(\xi,\mu,\kappa(\xi,v),\eta(\xi,v))})$ are $\Theta_{(\varsigma_1(\xi,v),\varsigma_2(\xi,v),\varsigma_5(\xi,v))}$-conjugate.*

(II) *For every $(\xi, \mu, w) \in I \times J \times \mathcal{V}$ and every $v \in \mathrm{Q}_{(\xi,w)}$ the endomorphism $G_{(\xi,\mu,\kappa(\xi,v),\eta(\xi,v))}$ has a blender-horseshoe.*

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Remark 6.2.3 The proof of item (I) of Theorem 4 follows inmediatly from Theorem 6.2.2. Indeed, this theorem jointly with Theorem 2 imply that $f \in \mathcal{N}^r_{P,Q}(\Pi^{-1}\big(\mathrm{Q}_{\bar{b}}\big))$, $\bar{b} \in \mathcal{B}$, then the sequence $\{f_k\}_k$ of C^r-diffemorphisms accumulating f obtained by its renomalisation display horseshoes-blenders for every k large enough.

Theorem 6.2.4 (Strong homoclinic intersections for $E_{(\xi,\mu,\varsigma)}$) *There is a seven-dimensional sub-manifold $\mathrm{Q}^{RC} \subset \mathrm{Q}^{II}$ such that for every $(\mu, v) \in J \times \mathrm{Q}^{RC}$ and every sequence of diffeomeorphisms $(F_k)_k$ in $\mathbb{R}^3$ converging on compact sets to $G_{(1,\mu,\kappa(1,v),\eta(1,v))}$ in the C^r-topology, there are $\varepsilon_k \to 0^+$ and ε_k-C^r-perturbations G_k of F_k such that G_k has a strong homoclinic intersection associated to a partially hyperbolic saddle-node for every large k.*

6.3 Blender-horseshoes for quadratic family: Proof of Theorem 6.2.2

6.4 Proof of item (I)

Let $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, with $w_1\, w_2\, w_3 \neq 0$. Consider the linear change of coordinates $\Theta_{\mathbf{w}} : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\Theta_{\mathbf{w}}(\mu, x, y, z) : = (w_2^{-1}\,\mu, w_1\, w_2^{-1}\, x, w_2^{-1}\, y, w_3\, w_2^{-1}\, z) \tag{6.4.1}$$

and the rational maps

$$\kappa, \eta : \mathbb{R} \times \mathrm{D} \to \mathbb{R}, \quad \mathrm{D} \subset \mathbb{R}^9,$$

defined by

$$\begin{aligned} \kappa(x, \mathbf{X}) &= x^2 \Big(\frac{x_2 + x_3}{x_3 - x_2}\Big)^2 \Big(\frac{x_5 + x_6 - x_7}{x_5 + x_6 + x_7}\Big), \\ \eta(x, \mathbf{X}) &= 2x \Big(\frac{x_2 + x_3}{x_3 - x_2}\Big)\Big(\frac{x_6 - x_5}{x_5 + x_6 + x_7}\Big), \end{aligned} \tag{6.4.2}$$

where

$$\mathbf{X} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathbb{R}^9$$

and

$$\mathrm{D} := \Big\{\mathbf{X} \in \mathbb{R}^9 : (x_6 - x_5)(x_5 + x_6 + x_7) \neq 0\Big\}$$

Recall the definition of Q^I in (6.1.4). The first item of the theorem follows immediately from the next lemma.

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Lemma 6.4.1 *For every $\xi, \mu \in \mathbb{R}$ and $v \in \mathrm{Q}^I$, the endomorphisms $\big(\mu, E_{(\xi,\mu,\bar{\varsigma}(\xi,v))}\big)$ and $\big(\mu, G_{(\xi,\mu,\kappa(\xi,v),\eta(\xi,v))}\big)$ are $\Theta_{(\varsigma_1(\xi,v),\varsigma_2(\xi,v),\varsigma_5(\xi,v))}$-conjugate.*

Proof. Note that if $v \in \mathrm{Q}^I$ then for any $\xi \in \mathbb{R}$ it holds $\varsigma_1(\xi, v) \neq 0$, $\varsigma_2(\xi, v) \neq 0$ and $\varsigma_5(\xi, v) \neq 0$, see (5.3.39). Thus the map $\Theta_{(\varsigma_1,\varsigma_2,\varsigma_5)} : \mathbb{R}^4 \to \mathbb{R}^4$, with $\varsigma_i = \varsigma_i(\xi, v)$, is well defined. A straightforward calculation shows that $\Theta_{(\varsigma_1(\xi,v),\varsigma_2(\xi,v),\varsigma_5(\xi,v))}(\mu, x, y, z)$ is a conjugation between the families of endomorphisms $\big(\mu, E_{(\xi,\mu,\bar{\varsigma}(\xi,v))}(x, y, z)\big)$ and $\big(\mu, G_{(\xi,\mu,\kappa(\xi,v),\eta(\xi,v))}(x, y, z)\big)$, where κ and η are as in (6.4.2). This completes the proof of lemma. ■

Remark 6.4.2 Note that for every $(\xi, v) \in \mathbb{R} \times \mathrm{Q}^I$ it holds

$$\kappa(\xi, v) = \varsigma_1(\xi, v)^2\, \varsigma_2(\xi, v)^{-1}\, \varsigma_3(\xi, v), \quad \eta(\xi, v) = \varsigma_1(\xi, v)\, \varsigma_2(\xi, v)^{-1}\, \varsigma_4(\xi, v).$$

6.5 Proof of item (II)

To define the sub-manifolds $\big\{\mathrm{Q}_{\mathbf{w}}\big\}_{\mathbf{w}\in\mathbb{R}^3}$ in Q^I we recall the following result. If $y \in N$ is a regular value of a C^r-map $h : M \to N$ then $h^{-1}(y)$ is a sub-manifold of class C^r and dimension $\dim(M) - \dim(N) > 0$, see for instance (25, Theorem 3.2).

We observe that domain $\mathbb{R} \times \mathrm{D}$ of the maps κ and η in (6.4.2) contain the open set $\mathbb{R} \times \mathrm{Q}^I$. We consider the restrictions of κ and η to this last open set. For every $\xi \in \mathbb{R}$ we define the C^∞-map

$$\gamma_\xi : \mathrm{Q}^I \to \mathbb{R}^2, \quad \gamma_\xi(v) = \big(\kappa(\xi, v), \eta(\xi, v)\big).$$

Putting $v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2)$ we have that

$$\begin{aligned} \kappa(\xi, v) &= \xi^2 \Big(\frac{a_2 + a_3}{a_3 - a_2}\Big)^2 \Big(\frac{b_2 + b_3 - b_4}{b_2 + b_3 + b_4}\Big); \\ \eta(\xi, v) &:= 2\xi \Big(\frac{a_2 + a_3}{a_3 - a_2}\Big) \Big(\frac{b_3 - b_2}{b_2 + b_3 + b_4}\Big). \end{aligned} \tag{6.5.1}$$

Lemma 6.5.1 *For every $\xi \in \mathbb{R}$, it holds that $\mathrm{RV}(\gamma_\xi) = \mathbb{R}^2$. In particular,*

$$\Big\{\mathrm{Q}_{(\xi,\kappa_0,\eta_0)} := \gamma_\xi^{-1}(\kappa_0, \eta_0) : (\xi, \kappa_0, \eta_0) \in \mathbb{R}^3\Big\},$$

is a family of sub-manifold in Q^I.

Proof.[Proof of Lemma 6.5.1] We adopt the following notation

$$\bar{a} = (a_1, a_2, a_3), \quad \bar{b} = (b_1, b_2, b_3, b_4), \quad \text{and} \quad \bar{c} = (c_1, c_2).$$

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Given $\xi \in \mathbb{R}$ and $(\kappa_0, \eta_0) \in \mathbb{R}^2$ we consider the set

$$\mathrm{Q}_{(\xi,\kappa_0,\eta_0)} = \Big\{(\bar{a}, \bar{b}, \bar{c}) \in \mathrm{Q}^I : \kappa(\xi, \bar{a}, \bar{b}, \bar{c}) = \kappa_0,\ \eta(\xi, \bar{a}, \bar{b}, \bar{c}) = \eta_0\Big\}. \tag{6.5.2}$$

To see that $(\kappa_0, \eta_0) \in \mathrm{RV}(\gamma_\xi)$ it is enough to see that if $(\bar{a}, \bar{b}, \bar{c}) \in \mathrm{Q}_{(\xi,\kappa_0,\eta_0)}$ then the vectors

$$d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi, \bar{a}, \bar{b}, \bar{c}) \quad \text{and} \quad d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi, \bar{a}, \bar{b}, \bar{c}),$$

(here $d_{(\bar{a},\bar{b},\bar{c})}\vartheta(v)$ stands for the differential of the map ϑ at $(\bar{a}, \bar{b}, \bar{c})$ applied to the vector v) are linearly independent. For that we rewrite

$$\kappa(\xi, \bar{a}, \bar{b}, \bar{c}) = \xi^2\, R_1(\bar{a})^2\, R_2(\bar{b}), \quad \eta(\xi, \bar{a}, \bar{b}, \bar{c}) = \xi\, R_1(\bar{a})\, R_3(\bar{b}),$$

where

$$\begin{aligned} R_1(\bar{a}) &= \frac{a_2 + a_3}{a_3 - a_2}, \\ R_2(\bar{b}) &= \frac{b_2 + b_3 - b_4}{b_2 + b_3 + b_4}, \\ R_3(\bar{b}) &= \frac{b_3 - b_2}{b_2 + b_3 + b_4}. \end{aligned} \tag{6.5.3}$$

Thus,

$$\begin{aligned} d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi, \bar{a}, \bar{b}, \bar{c}) &= \Big(2\,\xi^2\, d_{\bar{a}}\, R_1(\bar{a})\, R_1(\bar{a})\, R_2(\bar{b}), \xi^2\, R_1(\bar{a})^2\, d_{\bar{b}}\, R_2(\bar{b}), \mathbf{0}\Big), \\ d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi, \bar{a}, \bar{b}, \bar{c}) &= \Big(\xi\, d_{\bar{a}}\, R_1(\bar{a})\, R_3(\bar{b}), \xi\, R_1(\bar{a})\, d_{\bar{b}}\, R_3(\bar{b}), \mathbf{0}\Big), \end{aligned} \tag{6.5.4}$$

where $\mathbf{0} = (0, 0)$.

We note that if $v = (\bar{a}, \bar{b}, \bar{c}) \in \mathrm{Q}$ then $R_1(\bar{a}) \neq 0$.

We the study the set $\mathrm{Q}_{(\xi,\kappa_0,\eta_0)}$ in (6.5.2) to following four cases:

(i) Suppose $\kappa_0 \neq 0, \eta_0 \neq 0$. In this case $R_2(\bar{b})$ and $R_3(\bar{b})$ are different of zero and the derivate in (6.5.4) are given by:

$$\begin{aligned} d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi, \bar{a}, \bar{b}, \bar{c}) &= \left(2\,\kappa_0\, \frac{d_{\bar{a}}\, R_1(\bar{a})}{R_1(\bar{a})}, \kappa_0\, \frac{d_{\bar{b}}\, R_2(\bar{b})}{R_2(\bar{b})}, \mathbf{0}\right), \\ d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi, \bar{a}, \bar{b}, \bar{c}) &= \left(\eta_0\, \frac{d_{\bar{a}}\, R_1(\bar{a})}{R_1(\bar{a})}, \eta_0\, \frac{d_{\bar{b}}\, R_3(\bar{b})}{R_3(\bar{b})}, \mathbf{0}\right); \end{aligned} \tag{6.5.5}$$

Analogously,

(ii) If $\kappa_0 \neq 0, \eta_0 = 0$, then $R_3(\bar{b}) = 0$ (i.e. $b_2 = b_3$) and

$$\begin{aligned} d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi, \bar{a}, \bar{b}, \bar{c}) &= \left(2\,\kappa_0\, \frac{d_{\bar{a}}\, R_1(\bar{a})}{R_1(\bar{a})}, \kappa_0\, \frac{d_{\bar{b}}\, R_2(\bar{b})}{R_2(\bar{b})}, \mathbf{0}\right), \\ d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi, \bar{a}, \bar{b}, \bar{c}) &= \Big(\mathbf{0}, \xi\, R_1(\bar{a}) d_{\bar{b}}\, R_3(\bar{b}), \mathbf{0}\Big); \end{aligned} \tag{6.5.6}$$

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(iii) If $\kappa_0 = 0, \eta_0 \neq 0$ then $R_3(\bar{b}) = 0$ (i.e. $b_2 + b_3 = b_4$) and

$$\begin{aligned} d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi,\bar{a},\bar{b},\bar{c}) &= \left(\mathbf{0}, \xi^2\, R_1(\bar{a})^2 d_{\bar{b}}\, R_2(\bar{b}), \mathbf{0}\right), \\ d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi,\bar{a},\bar{b},\bar{c}) &= \left(\eta_0\, \frac{d_{\bar{a}}\, R_1(\bar{a})}{R_1(\bar{a})}, \eta_0\, \frac{d_{\bar{b}}\, R_3(\bar{b})}{R_3(\bar{b})}, \mathbf{0}\right); \end{aligned} \tag{6.5.7}$$

(iv) If $\kappa_0 = \eta_0 = 0$ then $R_2(\bar{b}) = R_3(\bar{b}) = 0$ (i.e. $b_2 = b_3$ and $b_2 + b_3 = b_4$) and

$$\begin{aligned} d_{(\bar{a},\bar{b},\bar{c})}\kappa(\xi,\bar{a},\bar{b},\bar{c}) &= \left(\mathbf{0}, \xi^2\, R_1(\bar{a})^2 d_{\bar{b}}\, R_2(\bar{b}), \mathbf{0}\right), \\ d_{(\bar{a},\bar{b},\bar{c})}\eta(\xi,\bar{a},\bar{b},\bar{c}) &= \left(\mathbf{0}, \xi\, R_1(\bar{a}) d_{\bar{b}}\, R_3(\bar{b}), \mathbf{0}\right). \end{aligned} \tag{6.5.8}$$

We observe that

$$\begin{aligned} d_{\bar{a}}\, R_1(\bar{a}) &= \left(0, \frac{2\,a_3}{(a_3-a_2)^2}, -\frac{2\,a_2}{(a_3-a_2)^2}\right) \\ d_{\bar{b}}\, R_2(\bar{b}) &= \left(0, \frac{2\,b_4}{(b_2+b_3+b_4)^2}, \frac{2\,b_4}{(b_2+b_3+b_4)^2}, -\frac{2\,(b_2+b_3)}{(b_2+b_3+b_4)^2}\right) \\ d_{\bar{b}}\, R_3(\bar{b}) &= \left(0, \frac{-2b_3-b_4}{(b_2+b_3+b_4)^2}, \frac{2\,b_2+b_4}{(b_2+b_3+b_4)^2}, \frac{b_2-b_3}{(b_2+b_3+b_4)^2}\right). \end{aligned} \tag{6.5.9}$$

Thus, in the case (i) the vectors (6.5.5) are linearly dependent if there exist $\lambda = \lambda_{\bar{a},\bar{b}} \in \mathbb{R} \setminus \{0\}$ such that

$$\kappa_0\, \frac{d_{\bar{b}}\, R_2(\bar{b})}{R_2(\bar{b})} = \lambda\, \eta_0\, \frac{d_{\bar{b}}\, R_3(\bar{b})}{R_3(\bar{b})}.$$

However, the vectors $d_{\bar{b}}\, R_2(\bar{b})$ and $d_{\bar{b}}\, R_3(\bar{b})$ are linearly independent. This follows from next claim.

Claim 6.5.2 *The vectors*

$$U(\bar{b}) := (0, b_4, b_4, -2\,b_2 - 2\,b_3), \quad \textit{and} \quad V(\bar{b}) = (0, -2b_3 - b_4, 2\,b_2 + b_4, b_2 - b_3),$$

are linearly independent.

Proof. Taking into account that $b_2 + b_3 + b_4 \neq 0$, the proof of this claim follows easily studying separately the cases $b_4 = 0$ and $b_4 \neq 0$. ∎

For case (ii) we note that $d_{\bar{a}}\, R_1(\bar{a})$ and $d_{\bar{b}}\, R_3(\bar{b})$ are not zero vector. This imply that the vectors (6.5.6) are linearly independent. The same argument shows that the vectors (6.5.7) in the case (iii) are linearly independent. For the case (iv), the linear independence follows intermediately evaluating $b_2 = b_3$ and $b_2 + b_3 = b_4$ in (6.5.8). This completes the proof of Lemma. ∎

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Lemma 6.5.3 *For every* $(\xi, \kappa_0, \eta_0) \in \mathbb{R}^3$ *the sub-manifolds* $\gamma_\xi^{-1}(\kappa_0, \eta_0) \subset \mathbb{R}^9$ *has dimension seven.*

Proof. Observe that the maps $\gamma_\xi(v)$ depends only of the coordinates a_2, a_3, b_2, b_3, b_4 of v. Recall the description of Q in Remark 6.1.1 in term of the coordinates $v = (\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}^2$. Let $\Pi^* : \mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^5$, $\Pi^*(\bar{a}, \bar{b}, \bar{c}) = (a_2, a_3, b_2, b_3, b_4)$, and $\tilde{\gamma}_\xi : \mathbb{R}^5 \to \mathbb{R}^2$ defined by $\tilde{\gamma}_\xi \circ \Pi^* = \gamma_\xi$. By the proof Lemma 6.5.1, for every (ξ, w) in $\mathbb{R} \times \mathbb{R}^2$ the set $\tilde{\gamma}_\xi^{-1}(w)$ is a three-dimensional sub-manifold of $\mathbb{R}^5$ (see (25, Theorem 3.2)). Thus, $\gamma_\xi^{-1}(w) = (\Pi^*)^{-1}\big(\tilde{\gamma}_\xi^{-1}(w)\big) \subset \mathbb{R}^9$ is a seven-sub-manifold diffeomorphic to $\tilde{\gamma}_\xi^{-1}(w) \times \mathbb{R}^4$. This completes the lemma. ■

We now proceed the proof of item (II) from Theorem 6.2.2.

Recall the family $G_{(\xi,\mu,\kappa,\eta)}$ and the open set of parameters $\mathcal{O} = (1.18, 1.19) \times (-10, -9) \times (-\varepsilon, \varepsilon)^2$ in Theorem 1. We observe that family $G_{(\xi,\mu,\kappa,\eta)}$ and the family $\tilde{G}_{(\xi,\mu,\kappa,\eta)}$ in (3.5.1) are conjugated:

$$G_{(\xi,\mu,\kappa,\eta)}(x, y, z) = \tilde{\Theta}^{-1} \circ \tilde{G}_{(\xi,\mu,\kappa,\eta)} \circ \tilde{\Theta}(x, y, z), \quad (x, y, z) \in \mathbb{R}^3, \tag{6.5.10}$$

where $\tilde{\Theta}(x, y, z) = (z, y, x)$, see Remark 4.1.1. Consider the family of sub-manifolds in Lemma 6.5.1 and the open set $\mathcal{B}{:} = I \times \mathcal{V}$ in $\mathbb{R}^3$ defined the subsets

$$I := (1.18, 1.19) \quad \text{and} \quad \mathcal{V} := (-\varepsilon, \varepsilon)^2. \tag{6.5.11}$$

By construction, the family of sub-manifolds

$$\mathrm{Q}_{\bar{b}} = \gamma_\xi^{-1}(\kappa_0, \eta_0), \quad \bar{b} = (\xi, \kappa_0, \eta_0) \in \mathcal{B} \tag{6.5.12}$$

satisfies item (II) of the theorem. This completes the proof of the theorem ■.

6.6 Strong homoclinic intersections associated to saddle-node of quadratic family: Proof of Theorem 6.2.4

We discusse some preliminaries facts.

6.6.1 Preliminaries

Recall the rational maps κ and η in (6.4.2). We observe the domains of these maps contain to the sub-manifold $\mathbb{R} \times \mathrm{Q}^{II}$. If $v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2) \in \mathrm{Q}^{II}$ then $a_2 + a_3 = 0$. Thus, we have that

$$\kappa(\xi, v) = \eta(\xi, v) = 0, \quad \text{for every } \xi \in \mathbb{R}.$$

Consider the sub-manifold Q^{RC} of Q^{II} given by

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$$\Big\{(a_1,a_2,a_3,b_1,b_2,b_3,b_4,c_1,c_2)\in \mathrm{Q}^{II} : b_2=b_3, \quad b_2+b_3-b_4=0\Big\}. \tag{6.6.1}$$

We observe that to the rational maps ς_i, $i=1,\dots,5$, in (5.3.39), we get that if $v\in \mathrm{Q}^{RC}$ then

$$\varsigma_1(\xi,v)=\varsigma_3(\xi,v)=\varsigma_4(\xi,v)=0, \quad \varsigma_2(\xi,v)\neq 0, \text{ and } \varsigma_5(\xi,v)\neq 0, \tag{6.6.2}$$

for every $\xi\in\mathbb{R}$. This leads to the following fact:

Lemma 6.6.1 *For every $v\in \mathrm{Q}^{RC}$ the endomorphisms*

$$\Big(\mu, E_{(1,\mu,\varsigma_2(1,v),\varsigma_5(1,v))}(x,y,z)\Big) \quad \textit{and} \quad \Big(\mu, G_{(1,\mu,0,0)}(x,y,z)\Big).$$

are conjugate by the map

$$\Theta_{(\varsigma_2,\varsigma_5)}(\mu,x,y,z)=(\varsigma_2^{-1}\,\mu,\varsigma_2^{-1}\,z,\varsigma_2^{-1}\,y,\varsigma_5\,\varsigma_2^{-1}\,x), \quad \varsigma_i=\varsigma_i(1,v).$$

Remark 6.6.2 Using Remark 6.1.1 and the analysis of the proof of Lemma 6.5.3 follows easily that Q^{RC} is a sub-manifold of dimension seven.

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6.6.1.1 Strong homoclinic intersections

We study the invariant manifolds of the saddle-node in the family $G_{(1,\mu,0,0)}$.

Lemma 6.6.3 *For every $\mu\in(-10,-9)$ and any $z_0\in\mathbb{R}$ fixed, the endomorphism $G_{(1,\mu,0,0)}$ has two partially hyperbolic saddle-nodes $P^+_\mu(z_0)$ and $P^-_\mu(z_0)$ in the plane $\big\{z=z_0\big\}$ of $\mathbb{R}^3$ such that*

- $W^{\mathrm{ss}}\Big(P^+_\mu(z_0),G_{(1,\mu,0,0)}\Big)$ *meets quasi-transversely* $W^{\mathrm{uu}}\Big(P^-_\mu(z_0),G_{(1,\mu,0,0)}\Big)$;
- $W^{\mathrm{uu}}\Big(P^+_\mu(z_0),G_{(1,\mu,0,0)}\Big)$ *meets quasi-transversely* $W^{\mathrm{ss}}\Big(P^-_\mu(z_0),G_{(1,\mu,0,0)}\Big)$.

Proof. Note that the fixed points of $G_{(1,\mu,0,0)}(x,y,z)=(y,\mu+y^2,z)$, are the from (P_μ,z) where P_μ is a fixed point from endomorphisms $g_\mu(x,y):=(y,\mu+y^2)$. The map g_μ, $\mu\in(-10,-9)$, has a two fixed points in $[-4,4]^2$ of the form $P^\pm_\mu=(y^\pm_\mu,y^\pm_\mu)$, where

$$y^\pm_\mu=\frac{1\pm\sqrt{1-4\mu}}{2}.$$

As was observed in (5.1.2), the condition $\mu\in(-10,-9)$ imply that

$$-2.7<y^-_\mu<-2.5 \quad \text{and} \quad 3.5<y^+_\mu<3.71.$$

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We now to describe the invariant manifolds of $P^{\pm}_{\mu}$ in $[-4,4]^2$. Consider the following local strong stable manifold of $P^{\pm}_{\mu}$ given by

$$W^{\mathrm{ss}}_{\mathrm{loc}}(P^{\pm}_{\mu}, g_{\mu}) = \Big\{(x, y^{\pm}_{\mu}) : |x| \leq 4\Big\} \subset [-4,4]^2.$$

The fact that this set is contained in the strong stable manifold follows from $g_{\mu}\Big(W^{\mathrm{ss}}_{\mathrm{loc}}(P^{\pm}_{\mu}, g_{\mu})\Big) = \{P^{\pm}_{\mu}\}$.

Similarly, consider the following local strong unstable manifold of $P^{\pm}_{\mu}$ given by

$$W^{\mathrm{uu}}_{\mathrm{loc}}(P^{+}_{\mu}, g_{\mu}) := \Big\{(y, \mu + y^2) : \sqrt{-4-\mu} \leq y \leq \sqrt{4-\mu}\Big\} \subset [-4,4]^2;$$
$$W^{\mathrm{uu}}_{\mathrm{loc}}(P^{-}_{\mu}, g_{\mu}) := \Big\{(y, \mu + y^2) : -\sqrt{4-\mu} \leq y \leq -\sqrt{-4-\mu}\Big\} \subset [-4,4]^2.$$

To see, for instance, that the first set is contained in the strong unstable manifold of P^{+}_{μ} consider the curve

$$\ell^{+}_{\mu,1} := \Big\{(y, \mu + y^2) : \sqrt{-\mu + \sqrt{-4-\mu}} \leq y \leq \sqrt{-\mu + \sqrt{4-\mu}}\Big\}.$$

It is easy to see that $P^{+}_{\mu} \in \ell^{+}_{\mu,1}$ and that $g_{\mu}(\ell^{+}_{\mu,1}) = W^{\mathrm{uu}}_{\mathrm{loc}}(P^{+}_{\mu}, g_{\mu})$, see Figure 6.3.

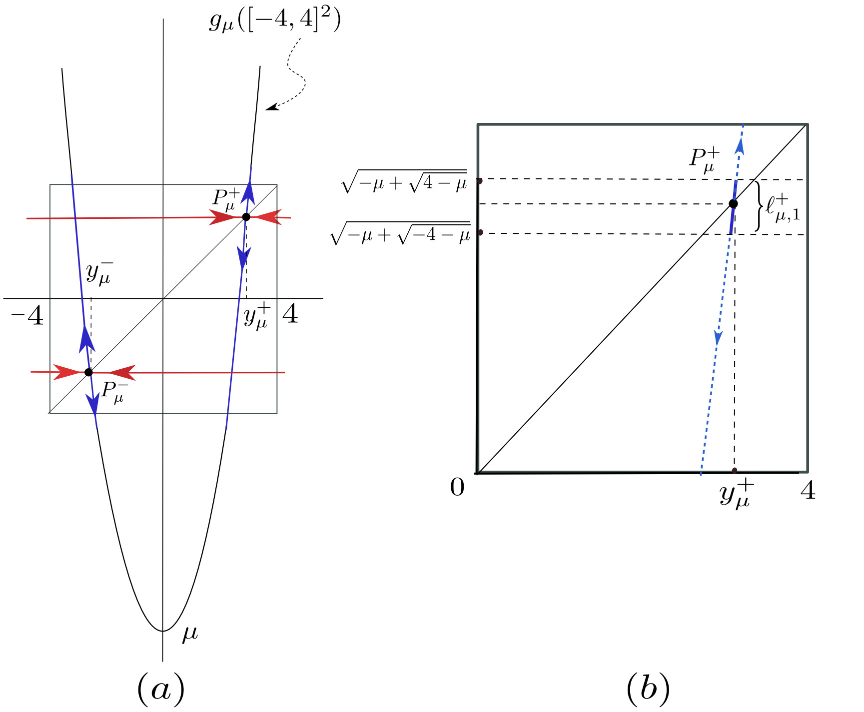

Figure 6.3: (a) Fixed points $P^{\pm}_{\mu}$ of g_{μ}. (b) Construction of the unstable manifold of P^{+}_{μ}.

Proceeding inductively we construct a nested sequence of discs

$$\ell^+_{\mu,n} \subset W^{\mathrm{uu}}_{\mathrm{loc}}(P^+_\mu, g_\mu) \tag{6.6.3}$$

such that for every $n \geq 1$ it holds $P^+_\mu \in \ell^+_{\mu,n}$ and $g_\mu(\ell^+_{\mu,n+1}) = \ell^+_{\mu,n}$. This imply that $W^{\mathrm{uu}}_{\mathrm{loc}}(P^+_\mu, g_\mu)$ is contained in the unstable manifold[1] of P^+_μ in $[-4,4]^2$.

Now the intersections below follow immediately

$$W^{\mathrm{uu}}_{\mathrm{loc}}(P^+_\mu, g_\mu) \pitchfork W^{\mathrm{ss}}_{\mathrm{loc}}(P^-_\mu, g_\mu) \neq \emptyset, \quad W^{\mathrm{ss}}_{\mathrm{loc}}(P^+_\mu, g_\mu) \pitchfork W^{\mathrm{uu}}_{\mathrm{loc}}(P^-_\mu, g_\mu) \neq \emptyset.$$

The proof of lemma follows noting that:

$$\begin{aligned} W^{\mathrm{uu}}_{\mathrm{loc}}\Big(P^\pm_\mu(z_0), G_{(1,\mu,0,0)}\Big) &= W^{\mathrm{uu}}_{\mathrm{loc}}(P^\pm_\mu, g_\mu) \times \{z_0\}, \\ W^{\mathrm{ss}}_{\mathrm{loc}}\Big(P^\pm_\mu(z_0), G_{(1,\mu,0,0)}\Big) &= W^{\mathrm{ss}}_{\mathrm{loc}}(P^\pm_\mu, g_\mu) \times \{z_0\}. \end{aligned} \tag{6.6.4}$$

■

6.6.1.2
Proof of Theorem 6.2.4

The first step in the proof of this theorem is the following result.

Proposition 6.6.4 *Let $r \geq 1$. Consider $\{F_k\}_k$ a sequence of C^r-diffeomorphisms in $\mathbb{R}^3$ converging on compact sets to $G_{(1,\mu,0,0)}$ in the C^r-topology. Then there are a sequence of positive numbers $\varepsilon_k \to 0$ and a local ε_k-C^r-perturbation G_k of F_k such that G_k has a strong homoclinic intersections associated to a partially hyperbolic saddle-node fixed point for every k large enough.*

Proof. Let $F_k : \mathbb{R}^3 \to \mathbb{R}^3$ be any sequence of diffeomorphisms converging to $G_{(1,\mu,0,0)}$ as above. We write

$$F_k(x,y,z) = (F^1_k(x,y,z), F^2_k(x,y,z), F^3_k(x,y,z)).$$

For every $M > 0$, consider the compact set $\Delta_M := [-4,4]^2 \times [-M,M]$. We let

$$\begin{aligned} \epsilon^1_k &:= \|\Big(F^1_k(x,y,z) - y\Big))|_{\Delta_M}\|_r, \\ \epsilon^2_k &:= \|\Big(F^2_k(x,y,z) - \mu + y^2)|_{\Delta_M}\|_r, \\ \epsilon^3_k &:= \|\Big(F^3_k(x,y,z) - z)|_{\Delta_M}\|_r, \\ \epsilon_k &:= \|\Big(F_k - G_{(1,\mu,0,0)}\Big)|_{\Delta_M}\|_r. \end{aligned} \tag{6.6.5}$$

Note that $\epsilon^i_k \leq \epsilon_k$, $i = 1,2,3$, and by hypothesis $\epsilon_k \to 0$ as $k \to +\infty$.

[1]Recall that the unstable manifold of a hyperbolic fixed (periodic) point p of a endomorphism h in a neighbourhood U consists of those x_0 for which there is a sequence $(x_n)_n$ in U with $h(x_{n+1}) = x_n$ and $x_n \to p$, see for instance (46, Theorem 6.1).

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The shape of $G_{(1,\mu,0,0)}|_{\Delta_M}$ implies that for every $z \in [-M, M]$ and k large enough there are $y_k^-(z) < 0 < y_k^+(z)$, such that

$$F_k^1\Big(y_k^\pm(z), y_k^\pm(z), z\Big) = F_k^2\Big(y_k^\pm(z), y_k^\pm(z), z\Big) = y_k^\pm(z).$$

Lemma 6.6.5 *For every $z_0 \in [-M, M]$ there exist $\varepsilon_k \to 0$ and a local C^r-ε_k-perturbation $\widetilde{F}_k^{z_0}$ of F_k such that every k sufficiently large it holds*

$$\widetilde{F}_k^{z_0}\Big(y_k^\pm(z_0), y_k^\pm(z_0), z_0\Big) = \Big(y_k^\pm(z_0), y_k^\pm(z_0), z_0\Big).$$

Proof. Without loss of generality we can assume that $z_0 = 0$. Fix a small $\rho > 0$ and consider a C^r-function $b = b_\rho : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} b(x) = 0, & \text{if} \quad 2\rho \geq |x|, \\ 0 < b(x) < 1, & \text{if} \quad \rho < |x| < 2\rho, \\ b(x) = 1, & \text{if} \quad |x| \leq \rho. \end{cases} \tag{6.6.6}$$

Consider the perturbation of the identity θ_k^+ supported in $P_k^+ := F_k\Big(y_k^+(0), y_k^+(0), 0\Big)$ defined (in local coordinates around P_k^+) by

$$\theta_k^+\Big((x, y, z) + P_k^+\Big) = (x, y, z) + P_k^+ - \Big(F_k^3(y_k^+(0), y_k^+(0), 0)\Big)\,(0, 0, b(z)). \tag{6.6.7}$$

Note that

$$\|\Big(\theta_k^+ - \mathrm{id}\Big)|_{\Delta_M}\|_r \leq \|\Big(F_k^3(x, y, z) - z\Big)|_{\Delta_M}\|_r\, \|b\|_r = \epsilon_k^3\|b\|_r \to 0,$$

where the convergence follows from (6.6.5). Thus the diffeomorphism $\widetilde{F}_k^+ := \theta_k^+ \circ F_k$ is a C^r-perturbation of F_k of size $\epsilon_k^3\|b\|_r$ satisfying

$$\widetilde{F}_k(y_k^+(0), y_k^+(0), 0) = \theta_k^+ \circ F_k(y_k^+(0), y_k^+(0), 0) = \theta_k^+(P_k^+) = \Big(y_k^+(0), y_k^+(0), 0\Big).$$

See Figure 6.4. Changing "+" by "−" in the construction above, we get a C^r-perturbation θ_k^- of the identity (of the same size as θ_k^+) supported in P_k^- (and whose support is disjoint from $\mathrm{supp}(\theta_k^+)$) such that the perturbation $\widetilde{F}_k := \theta_k^- \circ \widetilde{F}_k^+$ of F_k satisfies

$$\widetilde{F}_k(y_k^\pm(0), y_k^\pm(0), 0) = \Big(y_k^\pm(0), y_k^\pm(0), 0\Big).$$

To complete the proof of the lemma it is enough to take $\widetilde{F}_k^0 = \widetilde{F}_k$ and $\varepsilon_k = \epsilon_k^3\|b\|_r$. ∎

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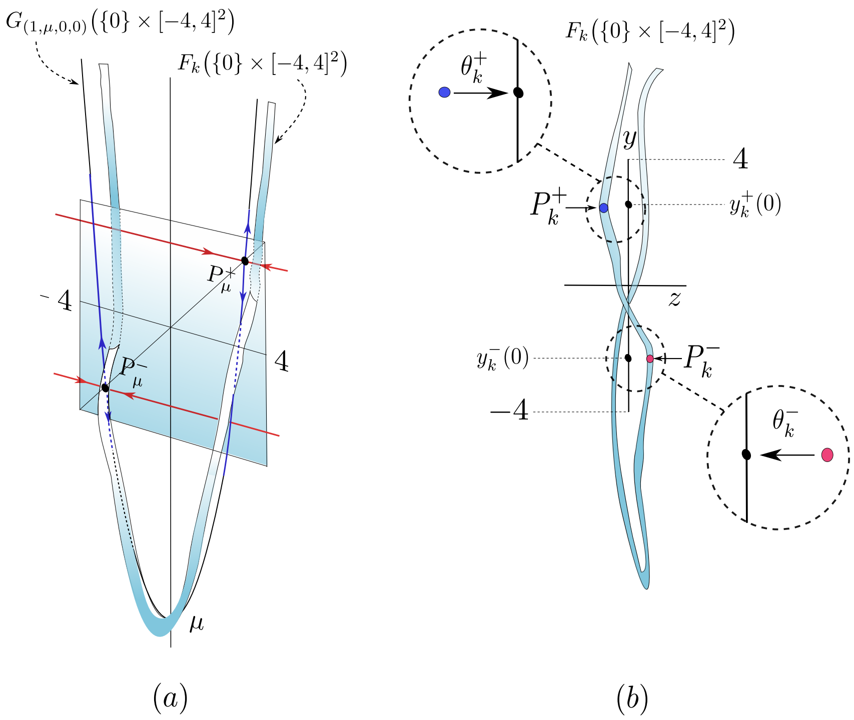

Figure 6.4: (a) Images of the square $[-4,4]^2\times\{0\}$ by $G_{(1,\mu,0,0)}$ and F_k. (b) Projection of the perturbations $\theta_k^{\pm}$ on $F_k([-4,4]^2\times\{0\})$.

Lemma 6.6.6 *Let $\widetilde{F}_k=\widetilde{F}_k^0$ be as in Lemma 6.6.5. There exist $\varepsilon_k\to 0$ and a ε_k-C^r-perturbation $\widehat{F}_k$ of $\widetilde{F}_k$ such that $\widehat{F}_k$ has a two partially hyperbolic saddle-node for every k large enough.*

Proof. Given $(x,y,z)\in\mathbb{R}^3$ and k large enough we can write

$$D\widetilde{F}_k(x,y,z):=\begin{pmatrix}\delta_{1,1}^k & 1+\delta_{1,2}^k & \delta_{1,3}^k\\ \delta_{2,1}^k & 2\,y+\delta_{2,2}^k & \delta_{2,3}^k\\ \delta_{3,1}^k & \delta_{3,2}^k & 1+\delta_{3,3}^k\end{pmatrix}, \tag{6.6.8}$$

where the entries $\delta_{i,j}^k=\delta_{i,j}^k(x,y,z)$ are functions of C^{r-1}-class converging to 0 on the compact sets.

Recall the fixed point $P_k^{\pm}=(y^{\pm}(0),y^{\pm}(0),0)$ of $\widetilde{F}_k(x,y,z)$. Consider the numbers $\gamma_{i,j}^{k,\pm}:=\delta_{i,j}^{k,\pm}(P_k^{\pm})$. In what follows we consider perturbations at P_k^+ (the perturbations at P_k^- are analogous and hence omitted). For simplicity we simply write $P_k=P_k^+$ (where $y_k(0)=y_k^+(0)$) and $\gamma_{i,j}^k:=\gamma_{i,j}^{k,+}$.

Using the map $b(z)$ in (6.6.6) we define the local perturbation of identity at P_k given by

$$\widehat{\theta}_k\Big((x,y,z)+P_k\Big)=\Big(x,y,\frac{z}{1+\gamma_{3,3}^k}\Big)+P_k-zb(z)\bar{w}_k,$$

where

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$$\bar{w}_k := \left(\frac{\gamma^k_{1,3}}{1+\gamma^k_{3,3}}, \frac{\gamma^k_{2,3}}{1+\gamma^k_{3,3}}, 0\right). \tag{6.6.9}$$

Note that $\bar{w}_k \to (0,0,0)$ and that $\|\widehat{\theta}_k - \mathrm{id}\|_r \leq \varepsilon_k$, where

$$\varepsilon_k : = M\left(\left|\frac{1}{1+\gamma^k_{3,3}} - 1\right| + \|b\|_r \, ||\bar{w}_k||\right) \to 0, \quad k \to +\infty.$$

Let

$$\widehat{F}_k(x,y,z) = \widehat{\theta}_k \circ \widetilde{F}_k(x,y,z).$$

Then, for large $k \geq 1$, it holds $\widehat{F}_k\big(P_k\big) = P_k$ and hence

$$D\widehat{F}_k\big(P_k\big) = D\widehat{\theta}_k\big(P_k\big) \circ D\widetilde{F}_k\big(P_k\big).$$

Noting that

$$D\widehat{\theta}_k\big(P_k\big) = \begin{pmatrix} 1 & 0 & -\dfrac{\gamma^k_{1,3}}{1+\gamma^k_{3,3}} \\ 0 & 1 & -\dfrac{\gamma^k_{2,3}}{1+\gamma^k_{3,3}} \\ 0 & 0 & \dfrac{1}{1+\gamma^k_{3,3}} \end{pmatrix}$$

and recalling equation (6.6.8) we get that

$$D\widehat{F}_k(P_k) = \begin{pmatrix} \gamma^k_{1,1} - \dfrac{\gamma^k_{1,3}\,\gamma^k_{3,1}}{1+\gamma^k_{3,3}} & 1+\gamma^k_{1,2} - \dfrac{\gamma^k_{1,3}\,\gamma^k_{3,2}}{1+\gamma^k_{3,3}} & 0 \\ \gamma^k_{2,1} - \dfrac{\gamma^k_{3,1}\,\gamma^k_{2,3}}{1+\gamma^k_{3,3}} & 2\,y_k(0) + \gamma^k_{2,2} - \dfrac{\gamma^k_{3,2}\,\gamma^k_{2,3}}{1+\gamma^k_{3,3}} & 0 \\ \dfrac{\gamma^k_{3,1}}{1+\gamma^k_{3,3}} & \dfrac{\gamma^k_{3,1}}{1+\gamma^k_{3,3}} & 1 \end{pmatrix} \tag{6.6.10}$$

Therefore $\lambda_k = 1$ is eigenvalue of $D\widehat{F}_k(P_k)$, hence P_k is a saddle-node fixed of $\widehat{F}_k$ for every k large enough [2]. This complete the proof of lemma. ∎

We define the local strong stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(P^{\pm}_k, \widehat{F}_k)$ as the connected component of $W^{\mathrm{ss}}(P^{\pm}_k, \widehat{F}_k) \cap \Delta_M$ containing $P^{\pm}_k$. Similarly, we define $W^{\mathrm{uu}}_{\mathrm{loc}}(P^{\pm}_k, \widehat{F}_k)$.

The end of the proof of the proposition has two steps. We first obtain a pair os saddle-node whose strong invariant manifolds meet cyclically and quasi-transversely. In the second step consist in turn one of these saddle-node in a saddle of index one. The λ-lemma leads to the existence of a strong homoclinic intersection associated to remaininig saddle-node. Observe now that as $k \to +\infty$ we get the following C^r-convergence

[2]The other two directions of P_k are hyperbolic $(\mathrm{Spec}\big(D\widehat{F}_k(P_k)\big) \setminus \{\lambda_k\}$ does not intersect $\mathbb{S}^1)$. This follows from $D\widehat{F}_k(P_k) \to DG_{(1,\mu,0,0)}(P_\mu, 0)$, where P_μ is a fixed point of g_μ, recall Section 6.6.1.

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$$W^{**}_{\mathrm{loc}}(P_k^{\pm},\widehat{F}_k)\to W^{**}_{\mathrm{loc}}(P_\mu^{\pm}(0),G_{(1,\mu,0,0)}),\quad *=\mathrm{s},\mathrm{u}, \tag{6.6.11}$$

where $W^{\mathrm{ss}}_{\mathrm{loc}}(P_\mu^{\pm}(z_0),G_{(1,\mu,0,0)})$ and $W^{\mathrm{uu}}_{\mathrm{loc}}(P_\mu^{\pm}(z_0),G_{(1,\mu,0,0)})$ are definied as in (6.6.4).

Recall that the strong invariant manifolds of $P_\mu^+(0)$ and $P_\mu^-(0)$ meets cyclically and quasi-tranversely, see Lemma 6.6.3.

Lemma 6.6.7 *There exist $\varepsilon_k\to 0$ and a ε_k-C^r-perturbation $\widetilde{G}_k$ of $\widehat{F}_k$ such that $\widetilde{G}_k$ has a pair of partially hyperbolic saddles whose strong invariant manifolds meet cyclically and quasi-transversely*

Proof. From convergence in (6.6.11) and Lemma 6.6.3 we can consider $\varepsilon_k\to 0$ such that the distance between the manifolds $W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^-,\widehat{F}_k)$ and $W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^+,\widehat{F}_k)$ is ε_k. Consider $X_k^+\in W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^+,\widehat{F}_k)$ and $\bar{w}_{k,+}\in\mathbb{R}^3$, with $||\bar{w}_{k,+}||=1$, be such that

$$X_k^-:\ =X_k^+ +\varepsilon_k\, w_{k,+}\in W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^-,\widehat{F}_k).$$

Using the map $b(x)$ in (6.6.6) we define a local perturbation of identity at X_k^+ given by:

$$\theta_k\Big((x,y,z)+X_k^+\Big)=(x,y,z)+X_k^+ +\varepsilon_k\, B(x,y,z)\,\bar{w}_{k,+}. \tag{6.6.12}$$

Then, for every k large enough, the diffeomorphism $\widetilde{G}_k:\ =\theta_k\circ\widehat{F}_k$ is a ε_k-C^r-perturbation of $\widehat{F}_k$ such that

$$X_k^-=\widetilde{G}_k(X_k^+)\in W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^-,\widetilde{G}_k)\cap W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^+,\widetilde{G}_k).$$

See Figure 6.5. We can generate the same type of intersection between the manifolds $W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^-,\widetilde{G}_k)$ and $W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^+,\widetilde{G}_k)$ keeping the size $\varepsilon_k||b||_r^3$ of the C^r-perturbation. Therefore the diffeomorphism $\widetilde{G}_k$ has a pair of partially hyperbolic saddles P_k^- and P_k^+ whose strong invariant manifolds meet cyclically and quasi-transversely. This completes the first step. ■

Through a new local C^r-perturbation of $\widetilde{G}_k$ supported in P_k^-, that we denote $\widehat{G}_k$, we can turn the saddle-node P_k^- into a hyperbolic saddle of index one[3]. Note that this perturbation we can be taking arbitrarily close to $\widetilde{G}_k$ and preserving the strong heteroclinic intersections obtained in Lemma 6.6.7.

Claim 6.6.8 *There exist an arbitrarily small C^r-perturbation G_k of $\widehat{G}_k$ such that P_k^+ is a partially hyperbolic saddle-node with a strong homoclinic intersection.*

[3]This is done by identifying $W^{\mathrm{c}}_{\mathrm{loc}}(P_k^-,\widetilde{G}_k)$ with a small interval I centered in zero, and perturbing $\widehat{G}_k|_{W^{\mathrm{c}}_{\mathrm{loc}}(P_k^-,\widetilde{G}_k)}=\mathrm{id}_I$ by a linear map with slope $0<\lambda_k<1$, such that $\lambda_k\to 1$.

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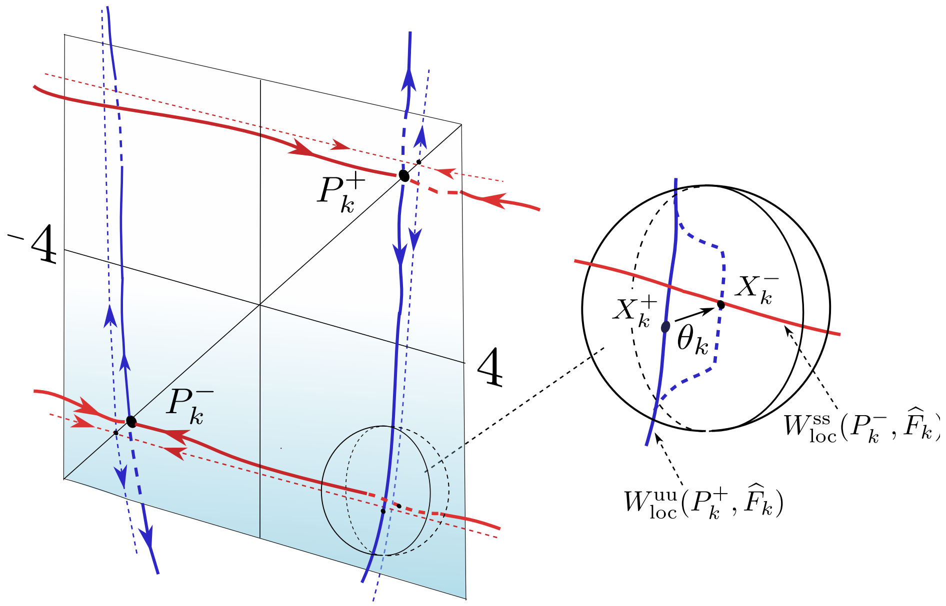

Figure 6.5: Perturbation θ_k.

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Proof. Consider ℓ^{u} a (small) segment of $W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^+, \widehat{G}_k)$ containing the intersection point $W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^-, \widehat{G}_k) \cap W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^+, \widehat{G}_k)$. For every big sufficient $n \geq 1$, the segment $\widehat{G}_k^n(\ell^{\mathrm{u}})$ transversely intersects the two dimensional manifold $W^{\mathrm{s}}_{\mathrm{loc}}(P_k^-, \widehat{G}_k)$. By the λ-lemma, for every $n \geq 1$ large enough, there exist $\ell^{\mathrm{u}}_n \subset \ell^{\mathrm{u}}$ such that $\widehat{G}_k^n(\ell^{\mathrm{u}}_n)$ is C^r-close to $W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^-, \widehat{G}_k)$. Since $W^{\mathrm{uu}}_{\mathrm{loc}}(P_k^-, \widehat{G}_k)$ meets quasi-transversely to $W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^+, \widehat{G}_k)$, we can modify by a small C^r-perturbation θ_k^* (like θ_k in 6.6.12), the strong unstable manifold of P_k^+ so that $\widehat{G}_k^n(\ell^{\mathrm{u}}_n)$ and $W^{\mathrm{ss}}_{\mathrm{loc}}(P_k^+, \widehat{G}_k)$ meets quasi-transversely, see Figure 6.6. We put G_k this last C^r-diffeomorphism. This ends the proof of claim. ■

The proof of Proposition 6.6.4 is now complete. ■

6.7 Generation of C^r-robust cycles: Proof of Theorem 3

Theorem 3 is a consequence of Theorem 6.2.4. Recall the definitions of the set Q^{RC} in (6.6.1) and of the projection Π in (6.1.2). Consider $\mathcal{T}^{RC} := \Pi^{-1}\big(\mathrm{Q}^{RC}\big)$. Theorem 3 follows immediately from the next proposition.

Proposition 6.7.1 *Let $r \geq 2$. Given any diffeomorphism $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^{RC})$ there is sequence $(g_k)_k$ of C^r-diffeomorphisms converging to f in the C^r-topology such that every g_k has a partially hyperbolic saddle-node P_k with a strong homoclinic intersection.*

Remark 6.7.2 In this paper we begin by considering a diffeomorphism f with a heterodimensional cycle and a heterodimensional tangency associated

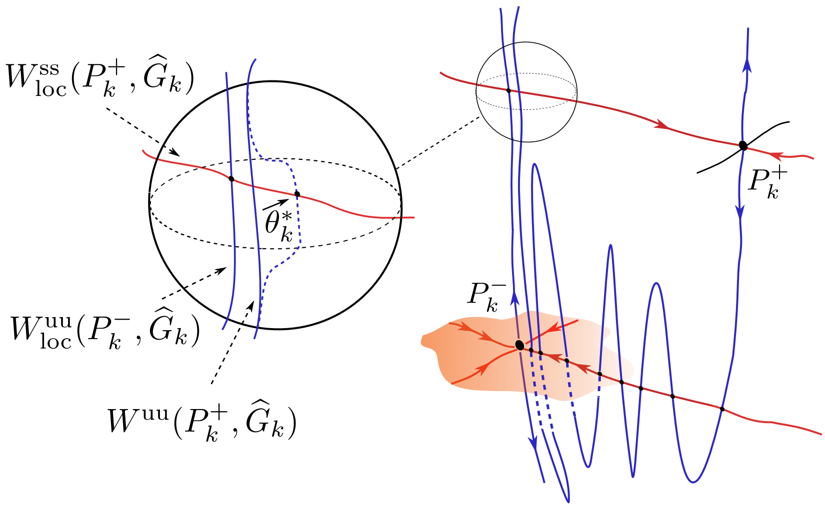

Figure 6.6: Perturbation θ_k^*.

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to a pair of saddles P and Q. Under appropriate conditions, we have that $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^{RC})$. Theorem 3 gives diffeomorphisms arbitrarily C^r-close to f having robust cycles. However, a priori, these robust cycles may not involve the (continuations of) saddles in the initial cycle. This is related to the stabilisation problem in Section 7.

Note that, by (9, Theorem 4.1), Proposition 6.7.1 guarantees the existence of diffeomorphisms exhibiting C^1-robust cycles arbitrarily C^1-close to $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^{RC})$.

We will see in Proposition 6.7.3 that these approximations also hold in the C^r-setting, for $r > 1$. Indeed the construction in (13) have two parts. The first part, that is genuinely C^1: heterodimensional cycles lead to the existence of strong homoclinic intersections associated to partially hyperbolic saddle-node. The second part consist in the passage from this saddle-node to robust cycles. This is obtained by local C^r-perturbations, for $r \geq 1$ (see for instance the construction in (13) which provides C^∞-families of diffeomorphisms having blenders).

Proof.[Proof of Proposition 6.7.1] Consider the renormalisation scheme of $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^{RC})$ in Theorem 2 associated to $\xi = 1$ (recall also the main ingredients of this scheme). Since $\mathrm{Quad}(f) := q_v \in \mathcal{T}^{RC}$ it holds

$$\varsigma_1(1,v) = \varsigma_3(1,v) = \varsigma_4(1,v) = 0, \quad \varsigma_2(1,v) \neq 0, \text{ and } \varsigma_5(1,v) \neq 0.$$

By Theorem 2 the corresponding renormalised sequence $\mathcal{R}_{m_k,n_k}(f_k)$, (see Section (5.3.6)), where f_k denote the sequence (5.3.19), generates a sequence of global diffeomorphisms

$$F_k : \mathbb{R}^3 \to \mathbb{R}^3, \quad F_k := \Psi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(f_k) \circ \Psi_{m_k,n_k}.$$

converging to $E_{(1,\mu,0,\varsigma_2,0,0,\varsigma_5)}$ on compact sets of $\mathbb{R}^3$. Recall that $E_{(1,\mu,0,\varsigma_2,0,0,\varsigma_5)}$ is C^∞-conjugate to $G_{(1,\mu,0,0)}$, see Lemma 6.6.1.

Applying Proposition 6.6.4 to $F_k|_{\Delta_M}$ we obtain a small C^r-perturbation G_k of $F_k|_{\Delta_M}$ such that G_k has a strong homoclinic intersection associated a saddle-node fixed point. Let g_k be now the C^r-perturbation of f_k given by the composition

$$\mathcal{R}_{m_k,n_k}(g_k) := \Psi_{m_k,n_k} \circ G_k \circ \Psi^{-1}_{m_k,n_k} \in \mathrm{Diff}^r(M).$$

By construction, $g_k \to f$ and g_k has a strong homoclinic intersection associated a saddle-node fixed point, completing the proof of the proposition. ■

Proposition 6.7.3 (C^r-version of Theorem 4.1 in (9)) *Let $r \geq 1$ and $f : M \to M$ be a C^r-diffeomorphism having a partially hyperbolic saddle-node S with a strong homoclinic intersection. Then there is a diffeomorphism h arbitrarily C^r-close to f with a robust heterodimensional cycle.*

Proof. We follow closely and modify accordingly the construction in (12, Proposition 3.4). For simplicity, let us assume that S has period one. After a C^r-perturbation, we can suppose that for the resulting diffeomorphism g the saddle-node S splits into two hyperbolic fixed points S^-_g (contracting in the central direction) and S^+_g (expanding in the central direction)[4]. The saddles S^-_g and S^+_g have different indices and the manifolds $W^{\mathrm{s}}(S^-_g, g)$ and $W^{\mathrm{u}}(S^+_g, g)$ have a transverse intersection containing (the interior of) a central curve joining S^-_g and S^+_g. Note that this transverse intersection is C^r-robust. The next step is to unfold the quasi-transverse strong homoclinic point between the strong unstable manifold and the stable manifold of S^+_g and S^-_g (exactly as in (13)). In this way we have that

(i) Using S^+_g and a strong homoclinic intersection we generate a partially hyperbolic horseshoe of u-index two. A small C^r-perturbation of g, unfolding the strong homoclinic intersection, produces a blender-horseshoe Γ_g having S^+_g reference fixed point.

(ii) A small C^r-perturbation of g, unfolding (some point of the orbit of) the strong homoclinic intersection associated to saddle S^-_g, generates

[4]The diffeomorphism g is obtained identifying $W^{\mathrm{c}}_{\mathrm{loc}}$ with a small interval I centred at zero and perturbing $f|_{W^{\mathrm{c}}_{\mathrm{loc}}(S)} = \mathrm{id}_I$ to get a (locally) Morse-Smale diffeomorphism having exactly a contracting and an expanding fixed points.

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a uu-disc Δ in the superposition region D of the blender-horseshoe Γ_g. Thus, by the definition of blender-horseshoe, $W^{\mathrm{s}}(\Gamma_g, g)$ intersects $W^{\mathrm{u}}(S_g^-, g)$. Hence, as $S_g^+ \in \Gamma_g$ and $W^{\mathrm{s}}(S_g^-, g) \pitchfork W^{\mathrm{u}}(S_g^+, g) \neq \emptyset$, there is a heterodimensional cycle associated to Γ_g and S_g^-.

(iii) Finally, the following three properties are C^r-open ones: 1) the continuation of the hyperbolic set Γ_g to be a blender (the elements in the definition of a blender depend continuously on g, see Remark 3.2); 2) $W^{\mathrm{u}}(S_g^-, g)$ to contain a vertical disk in the superposition region D of the blender; 3) $W^{\mathrm{s}}(S_g^-, g) \pitchfork W^{\mathrm{u}}(S_g^+, g) \neq \emptyset$.

Therefore, every diffeomorphism h that is C^r-close to g has a heterodimensional cycle associated to S_h^- and Γ_h. Since g can be taken arbitrarily close to f this concludes the proof of the proposition. ■

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7
Stabilisation of cycles: Proof of item II of Theorem 4

In this section we prove the second part of Theorem 4. This result follows from the result below, where $\mathcal{T}_{\bar{b}}$ is a leaf and $\mathcal{T}^*_{\bar{b}_0}$ a suitable open subset of it (for the precise definitions see Section 7.0.1). Here $\bar{b}_0$ belong to a subset $\mathcal{B}'$ of $\mathcal{B}$.

Theorem 7.0.1 (Stabilisation of cycles) *Let $2 \leq r < \infty$. Given $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$ there exists a sequence $\{g_k\}_k$ of diffeomorphisms converging to f in the C^r-topology such that every g_k has a blender-horseshoe Λ_{g_k} satisfying:*

(i) Λ_{g_k} is related to the saddle Q_{g_k} by a C^r-robust cycle and

(ii) Λ_{g_k} is homoclinically related to the saddle P_{g_k}.

Moreover, the homoclinic classes $H(P_{g_k}, g_k)$ and $H(Q_{g_k}, g_k)$ are both non-trivial and is intermingled C^r-persistently.

Note that the second item in the theorem implies that there is a transitive hyperbolic set Σ_{g_k} containing Λ_{g_k} and $\{P_{g_k}\}$ and $H(P_{g_k}, g_k)$ is non-trivial. This also means that the initial cycle can be C^r-stabilised.

This Section is organised as follows. In Sub-section 7.0.1 We construct the leaf $\mathcal{T}_{\bar{b}_0}$ and the subset $\mathcal{T}^*_{\bar{b}_0}$. We state some auxiliary results in Sub-section 7.0.2 and using these, we construct new heteroclinic orbits in Sub-section 7.0.3. We study the transition maps associated to these new intersections in Sub-section 7.0.4. Finally, with these ingredients in hand we start the proof of Theorem 7.0.1 in Sub-section 8 with the definition of the sequence g_k. Concerning the robust cycles of these maps, the robust intersections between the one dimensional invariant manifolds are obtained in Sub-section 8.1. Transverse intersections between the two dimensional invariant manifolds are obtained in Sub-section 8.2.

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7.0.1 Restrictions of the bifurcation setting

Recall the subsets $\mathcal{B} \subset \mathbb{R}^3$ and the family of sub-manifolds $\left\{\mathrm{Q}_{\bar{b}} : \bar{b} \in \mathcal{B}\right\}$ in (6.5.12). Consider the family $\mathcal{T}_{\text{quad}}$ given by

$$\mathcal{T}_{\bar{b}} := \Pi^{-1}\Big(\mathrm{Q}_{\bar{b}}\Big), \quad \bar{b} \in \mathcal{B}. \tag{7.0.1}$$

Recall the follows open subsets of $\mathcal{T}_{\text{quad}}$: $\mathcal{T}^{\pm}_{\text{quad}} := \Pi^{-1}(\mathrm{Q}^{\pm})$ in (6.1.3) and $\mathcal{T}^{h}_{\text{quad}} := \Pi^{-1}(\mathrm{Q}^{h})$, $\mathcal{T}^{p,\pm,\mp}_{\text{quad}} := \Pi^{-1}(\mathrm{Q}^{p,\pm,\mp})$ in 6.1.11. The bifurcation setting it is defined by the sets:

$$\mathcal{T}^{*}_{\bar{b}_0} := \mathcal{T}_{\bar{b}_0} \cap (\mathcal{T}^{-}_{\text{quad}} \cap \mathcal{T}^{h}_{\text{quad}} \cup \mathcal{T}^{p,+,-}_{\text{quad}}), \tag{7.0.2}$$

where $\bar{b}_0 = (\xi, \mu, 0, 0) \in \mathcal{B}' := \mathcal{P} \times \{(0,0)\}$ with $\mathcal{P} := (1.18, 1.19) \times (-10, -9)$.

Recall q_v in (6.1.1) and $\kappa(\xi, v), \eta(\xi, v)$ in (6.5.1). Note that if $q_v \in \mathcal{T}_{\bar{b}_0}$ then $v \in \mathrm{Q}_{\bar{b}_0}$ and thus $\kappa(\xi, v) = \eta(\xi, v) = 0$.

We now see that the family of open sets $\mathcal{T}^{*}_{\bar{b}_0}$, with $\bar{b}_0 \in \mathcal{B}'$, is not empty in $\mathcal{T}_{\bar{b}_0}$.

Claim 7.0.2 *For every $\bar{b}_0 = (\xi, \mu) \in \mathcal{P} = (1.18, 1.19) \times (-10, -9)$ the set $\mathcal{T}^{*}_{\bar{b}_0}$ is not empty in $\mathcal{T}_{\bar{b}_0}$.*

Proof. To see that is not empty, for instance, consider the roots of $b^{\pm}$ from $b^2 - 4b_2 b - 3(b_2)^2 = 0$ and consider the set

$$\Big\{v_h = (a_1, a, -a, b_1, b_2, -b_2, b, c_1, c_2) : a\, b_1\, b_2\, c_2 \neq 0, b \in (b^-, b^+)\Big\} \subset \mathbb{R}^9,$$

is contained in the set $\mathrm{Q}^{-} \cap \mathrm{Q}^{h} \cap \mathrm{Q}_{(\xi,\mu,0,0)}$ and the set

$$\Big\{v_p = (a_1, a, -a, b_1, -2, -1, -2, c_1, c_2) : a\, b_1\, c_2 \neq 0,\; a\, c_1 > 0\Big\} \subset \mathbb{R}^9,$$

is contained in the set $\mathrm{Q}^{-} \cap \mathrm{Q}^{p,+,-} \cap \mathrm{Q}_{(\xi,\mu,0,0)}$, for every $(\xi, \mu) \in (1.18, 1.19) \times (-10, -9)$. This completes the proof. ∎

Let $f \in \mathcal{N}^{r}_{P,Q}(\mathcal{T}_{\text{quad}})$ and consider the following sets associated to f:

$$\mathcal{P}^{\mathrm{s}}_{f} \subset U_P \text{ be the connected components of } W^{\mathrm{s}}(Q, f) \text{ containing } Y, \tag{7.0.3}$$

$$\mathcal{P}^{\mathrm{u}}_{f} \subset U_Q \text{ be the connected components of } W^{\mathrm{u}}(P, f) \text{ containing } \widetilde{Y}. \tag{7.0.4}$$

Consider the subsets of U_P and U_Q

$$\begin{aligned} U^{+}_{P} &= U_P \cap \Big\{x_P \geq 0\Big\}, \quad U^{-}_{P} = U_P \cap \Big\{x_P \leq 0\Big\}; \\ U^{+}_{Q} &= U_Q \cap \Big\{y_Q \geq 0\Big\}, \quad U^{-}_{Q} = U_Q \cap \Big\{y_Q \leq 0\Big\}. \end{aligned} \tag{7.0.5}$$

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We observe that since $\mathrm{Quad}(f) \in \mathcal{T}^{h}_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}}$ it holds that

$$\mathcal{P}^{\mathrm{s}}_{f} \cap U^{+}_{P} \neq \emptyset \quad \text{and} \quad \mathcal{P}^{\mathrm{u}}_{f} \cap U^{-}_{Q} \neq \emptyset.$$

In the next remark we list some properties satisfied by the diffeomorphisms f in $\mathcal{N}^{r}_{P,Q}(\mathcal{T}^{*}_{\bar{b}_0})$, that will be relevant in the remainder of this work. Besides we give some aditional flat conditions on the higher order terms of the transition map f^{N_2} (see (5.3.9)).

Remark 7.0.3 Let $\bar{b}_0 = (\xi, \mu) \in \mathcal{P}$. Consider $f \in \mathcal{N}^{r}_{P,Q}(\mathcal{T}^{*}_{\bar{b}_0})$, with $\mathrm{Quad}(f) = q_v$ and $v = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2)$. Then

(1) From $q_v \in \mathcal{T}^{-}_{\mathrm{quad}}$, the parameter $\varsigma_2(\xi, v) = \frac{b_2+b_3+b_4}{2}$ in (5.3.39) is negative.

(2) From $q_v \in \mathcal{T}_{\bar{b}_0}$, the parameters $\kappa(\xi, v)$ and $\eta(\xi, v)$ (6.5.1) are zero (this is equivalent to the conditions $b_2 + b_3 - b_4 = b_3 - b_2 = 0$) and there exist a sequence of diffeomorphisms $\{f_k\}_k$ C^r-converging to f (obtained in Theorem 2) such that the renormalised sequence of f_k converge (in suitable charts) to $G_{(\xi,\mu,0,0)}$ and hence f_k has a blender-horseshoe for every big sufficient k.

(3) Recall the sets $\mathcal{P}^{\mathrm{s}}_{f}$ and $\mathcal{P}^{\mathrm{u}}_{f}$ in (7.0.3) and (7.0.4), respectively. From $q_v \in \mathcal{T}^{h}_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}}$ and from Remark 5.3.11, these sets and its continuations $\mathcal{P}^{\mathrm{u}}_{f_k}$, $\mathcal{P}^{\mathrm{s}}_{f_k}$ satisfies:

$$\mathcal{P}^{\mathrm{s}}_{f} \cap U^{+}_{P} \neq \emptyset, \quad \mathcal{P}^{\mathrm{u}}_{f} \cap U^{-}_{Q} \neq \emptyset, \quad \text{and} \quad \mathcal{P}^{\mathrm{s}}_{f_k} \cap U^{\pm}_{P} \neq \emptyset, \quad \mathcal{P}^{\mathrm{u}}_{f_k} \cap U^{\pm}_{Q} \neq \emptyset.$$

The intersection associated to the continuations above it is equivalent to

$$\mathcal{P}^{s}_{f_k} \pitchfork W^{u}(P, f_k) \neq \emptyset, \quad \mathcal{P}^{u}_{f_k} \pitchfork W^{s}(Q, f_k) \neq \emptyset.$$

Recall the higher order terms $H_i(\cdot)$, $i = 1, 2, 3$, in (5.3.11). We assume the following conditions:

$$\frac{\partial^{p+q}}{\partial y^p\, \partial z^q} H_1(\mathbf{0}) = \frac{\partial^{p+q}}{\partial y^p\, \partial z^q} H_2(\mathbf{0}) = \frac{\partial^{p+q}}{\partial y^p\, \partial z^q} H_3(\mathbf{0}) = 0, \quad 1 \leq p+q \leq r. \quad (7.0.6)$$

Before going to the proof of Theorem 7.0.1 we need some preliminary results.

7.0.2
Preliminary technical results

In this section we prove the auxiliary Lemmas 7.0.5 and 7.0.7 used to generate new quasi-transverse intersections between the one dimensional invariant manifolds of the saddles in the cycle. The first lemma provides

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perturbations that modify the arguments of the irrational eigenvalues of $Df(P)$ and $Df(Q)$. The second lemma asserts that in the inicial cycle the closure of the one-dimensional manifold $W^{\mathrm{s}}(P,f)$ contains $W^{\mathrm{s}}_{\mathrm{loc}}(Q,f)$ (the same density holds for $W^{\mathrm{u}}(Q,f)$ in $W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$).

Recall that M denote a compact manifold of dimension 3. Let P be a saddle fixed of a C^r-diffeomorphism $f: M \to M$. Suppose that the index of P is two and the spectrum of $Df(P)$ is given by

$$\mathrm{Spec}\Big(Df(P)\Big) = \Big\{\lambda, \sigma\, e^{\pm 2\,\pi i \varphi}\Big\}, \quad \text{where } 0 < \lambda < 1 < \sigma \text{ and } \varphi \in \mathbb{Q}^c.$$

Suppose that there is a C^r-linearising chart $U_P \simeq [-3,3]^3$ at P such that

$$f|_{U_P} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma\cos(2\pi\varphi) & -\sigma\sin(2\pi\varphi) \\ 0 & \sigma\sin(2\pi\varphi) & \sigma\cos(2\pi\varphi) \end{pmatrix} \tag{7.0.7}$$

and

$$W^{\mathrm{s}}_{\mathrm{loc}}(P,f) = [-3,3] \times \{(0,0)\} \subset U_P, \quad W^{\mathrm{u}}_{\mathrm{loc}}(P,f) = \{0\} \times [-3,3]^2 \subset U_P.$$

To emphasize the argument φ we write $f_\varphi := f|_{U_P}$. Consider the canonical projections $\Pi^* : U_P \to W^*_{\mathrm{loc}}(P, f_\varphi)$, $* = \mathrm{s}, \mathrm{u}$, induced by the decomposition $U_P = W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi) \oplus W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$. Consider the unitary circle

$$\mathbb{S}^1_P := \Big\{(0,y,z) \in U_P : y^2 + z^2 = 1\Big\} \subset W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$$

and the radial projection

$$\Pi^P_{\mathrm{rad}} : W^{\mathrm{u}}_{\mathrm{loc}}(P,f) \to \mathbb{S}^1_P. \tag{7.0.8}$$

Definition 7.0.4 Let $S \subset M$ be a two-dimensional disc intersecting transversely $W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$. We say that S has *positive radial projection on* $W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$ if $\Pi^P_{\mathrm{rad}}(S \pitchfork W^{\mathrm{u}}_{\mathrm{loc}}(P,f))$ contains some interval in $\mathbb{S}^1_P$.

In what follows, $\ell_0, \ell_1 \subset U_P$ are two one-dimensional C^1-discs such that ℓ_0 is quasi-transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$ and ℓ_1 is transverse to $W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$ and $S \subset U_P$ is a two-dimensional C^1-disc with positive radial projection in $W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$, see Figure 7.1

Lemma 7.0.5 (Accelerating perturbation) *For every $m_0 \in \mathbb{N}$ there are an arbitrarily small local C^r-perturbation $\tilde f$ of f in U_P and $m > m_0$ such that $\tilde f^{-m}(\ell_1)$ and ℓ_0 meet quasi-transversely.*

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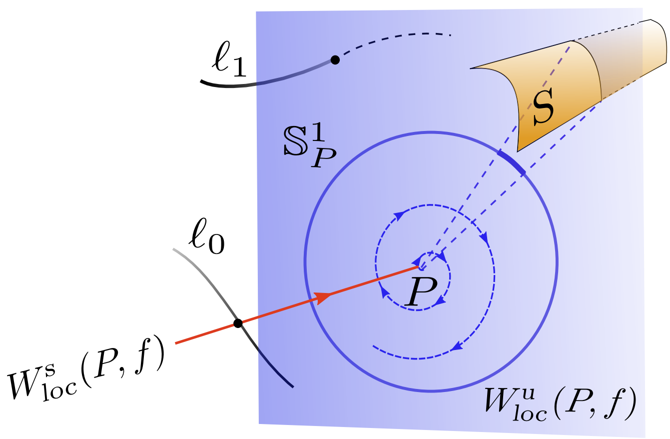

Figure 7.1: ℓ_0 is quasi-transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$, ℓ_1 is transverse to $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$ and S has positive radial projection in $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$.

Proof. Without loss of generality we can assume that

$$X = (1, 0, 0) \in \ell_0 \cap W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$$

is a quasi-transverse intersection point. The λ-lemma (see (38, Theorem 2)) implies that for every $\epsilon > 0$ there exists $m_0 = m_0(\epsilon)$ such that for every $m \geq m_0$ there is a one-disc ℓ_1^m (strictly) contained in ℓ_1 such that $f_\varphi^{-m}(\ell_1^m)$ is ϵ-C^r-close to $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$.

For every m large enough, we consider an angle $\theta = \theta(m) \in [0, 1]$ (modulus $2k\pi$) such that the rotation of the segment $f_\varphi^{-m}(\ell_1^m)$ (around $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$) by $-\theta$, intersect quasi-transversely ℓ_0. The rotation of this segment is given by $R_{-\theta}(f_\varphi^{-m}(\ell_1^m))$, where

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi\theta) & -\sin(2\pi\theta) \\ 0 & \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}.$$

Thus we have

$$R_{-\theta}(f_\varphi^{-m}(\ell_1^m)) \cap \ell_0 \neq \emptyset. \tag{7.0.9}$$

Recall now the C^r-perturbation $s_{\alpha,\tilde{\alpha}}$ of identity (with $\alpha, \tilde{\alpha}$ small enough) defined in (5.3.14). Let $S_\omega := s_{\omega,0}$

The perturbation $\tilde{f}$ of f is defined by

$$\tilde{f} = f_m := S_{\frac{\theta(m)}{m}} \circ f_\varphi.$$

Note if m is sufficiently big then $\tilde{f} = f_m$ is a small C^r-perturbation of f_φ. We now check that the $\tilde{f}^{-m}(\ell_1)$ meets quasi-transversely ℓ_0. Indeed, as

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$S_{\frac{\theta(m)}{m}} \circ f_\varphi = f_{\varphi + \frac{\theta(m)}{m}}$ in U_P (see (5.3.14)), we have that

$$\tilde{f}^{-m}(\ell_1^m) = f^{-m}_{\varphi+\frac{\theta(m)}{m}}(\ell_1^m) = R_{-\theta}\Big(f_\varphi^{-m}(\ell_1^m)\Big).$$

In view of (7.0.9) this ends the proof of the lemma. ■

Remark 7.0.6 Denoting by $\mathrm{LocDyn}_P(f|_{U_P}) = (\lambda, \sigma, \varphi)$ the parameters defining the local dynamic of f in U_P (see (7.0.7)) we have that $\mathrm{LocDyn}(\tilde{f}|_{U_P}) = (\lambda, \sigma, \varphi + \frac{\theta(m)}{m})$ and say that $\tilde{f}|_{U_P}$ is obtained *accelerating the argument* of $f|_{U_P}$. After an arbitrarily small perturbation of $f_\varphi^{-m}(\ell_1^m)$ if necessary, we can assume that $\theta(m)$ is rational and hence $\varphi + \frac{\theta(m)}{m}$ is a irrational number. Finally, taking m sufficiently large, we can assume that the quasi-transverse intersection $\Big\{X_m\Big\} = \tilde{f}^{-m}(\ell_1^m) \cap \ell$ is arbitrarily close to X.

We now see that the closure of the $\Big\{f_\varphi^i(\ell) : i \in \mathbb{N}\Big\}$ contains $W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$. We consider a small sub-disc $\tilde{\ell}$ of ℓ containing $X = (1,0,0) \in \tilde{\ell}$ parameterised as follows,

$$\tilde{\ell} := \Big\{\Big(1 + t\,v_1 + \rho_1(t), t\,v_2 + \rho_2(t), t\,v_3 + \rho_3(t)\Big) : |t| < \delta\Big\}, \tag{7.0.10}$$

where $v = (v_1, v_2, v_3)$ is a unitary vector in $T_X\ell$ and $\rho_i : \mathbb{R} \to \mathbb{R}$ are C^1-maps satisfying

$$\rho_i'(0) = \rho(0) = 0, \quad i = 1, 2, 3.$$

Note that since ℓ is quasi-transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varphi)$ we have that $(v_2, v_3) \neq (0,0)$. Note also that (at the origin)

$$\rho_i(t) := O(t^2). \tag{7.0.11}$$

Finally, consider the segment tangent to $\tilde{\ell}$ at X

$$\widehat{\ell} := \Big\{\Big(1 + t\,v_1, t\,v_2 +, t\,v_3\Big) : |t| < \delta\Big\}.$$

Lemma 7.0.7 *The closure of* $\Big\{f_\varphi^n(\tilde{\ell}) : n \in \mathbb{N}\Big\}$ *contains* $W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$.

Proof. Fix any point $Z \in W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$ and any ϵ-ball $B_\epsilon(Z)$ of Z, it is enough to see that there is n such that $f_\varphi^n(\tilde{\ell}) \cap B_\epsilon(Z) \neq \emptyset$.

We need a preparatory step. First, the irrationality of φ straightforwardly implies that the lemma holds for forward iterations of $\widehat{\ell}$. Indeed, as the argument φ is irrational, there is a sequence $(i_j)_j$ such that $f_\varphi^{i_j}(\widehat{\ell}) \cap B_\epsilon(Z) \neq \emptyset$ for every j. Consider sufficiently big j_0 such that $\sigma^{-i_j}\Big(||Z|| + \epsilon\Big) < \delta$ for every $j \geq j_0$. For $j \geq j_0$, consider the segment $\widehat{\ell}_j \subset \widehat{\ell}$ defined by

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$$\widehat{\ell}_j := \left\{ \Big(1 + \sigma^{-i_j}\, t\, v_1, \sigma^{-i_j}\, t\, v_2, \sigma^{-i_j}\, t\, v_3\Big), \quad |t| \le t^* := \frac{||Z|| + \epsilon'}{||(v_2, v_3)||} \right\}; \quad (7.0.12)$$

for some $0 < \epsilon' < \epsilon$. The extremes of $f_\varphi^{i_j}(\widehat{\ell}_{i_j})$ are given by (with a slight abuse of notation)

$$\bar{\mathbf{e}}_j^{\pm} := \Big(\lambda^{i_j} \pm \lambda^{i_j} \sigma^{-i_j} t^* v_1, \pm t^*\, V_j\Big) \quad (7.0.13)$$

with

$$V_j := \begin{pmatrix} \cos(2\pi i_j \varphi) & -\sin(2\pi i_j \varphi) \\ \sin(2\pi i_j \varphi) & \cos(2\pi i_j \varphi) \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}.$$

Note that the arguments of the vectors V_j tend to the argument of the vector Z. Thus, if j it is large enough, then $\lambda^{i_j} \pm \lambda^{i_j} \sigma^{-i_j} t^* v_1 < \epsilon$ and $||t^*\, V_j|| = ||Z|| + \epsilon'$. Then either $\bar{\mathbf{e}}_j^+$ or $\bar{\mathbf{e}}_j^-$ belongs to $B_\epsilon(Z)$, suppose that the first case holds.

We are now ready show that $f_\varphi^{i_j}(\tilde{\ell}) \cap B_\epsilon(Z) \neq \emptyset$. Let

$$\widetilde{V}_j(t) := \begin{pmatrix} \cos(2\pi i_j \varphi) & -\sin(2\pi i_j \varphi) \\ \sin(2\pi i_j \varphi) & \cos(2\pi i_j \varphi) \end{pmatrix} \begin{pmatrix} \rho_2\Big(\sigma^{-i_j} t\Big) \\ \rho_3\Big(\sigma^{-i_j} t\Big) \end{pmatrix}$$

and consider the curve $\tilde{\ell}_j \subset \ell$ defined by

$$\tilde{\ell}_j := \left\{ \Big(1 + \sigma^{-i_j} t\, v_1 + \rho_1(\sigma^{-i_j} t), t\, \sigma^{-i_j}\, V_j + \widetilde{V}_j(t) : \quad |t| \le t^* \right\}.$$

We will see that one of the extremes of $f^{i_j}(\tilde{\ell}_j)$ is ϵ_j close to $\bar{\mathbf{e}}_j^+$ where $\epsilon_j \to 0$ as $j \to \infty$. Therefore this extreme also belongs to $B_\epsilon(Z)$ and the proof of the lemma follows.

Consider the extreme $\tilde{\mathbf{e}}_j^+$ of the segment $f_\varphi^{i_j}(\tilde{\ell}_j)$ given by $\bar{\mathbf{e}}_j^+ + \Big(\lambda^{i_j} \rho_1\Big(\sigma^{-i_j} t^*\Big), \widetilde{V}_j(t^*)\Big)$. Let

$$||\tilde{\mathbf{e}}_j^+ - \bar{\mathbf{e}}_j^+|| = \|\Big(\lambda^{i_j} \rho_1\Big(\sigma^{-i_j} t^*\Big), \widetilde{V}_j\Big)\| = \epsilon_j.$$

We claim that $\epsilon_j \to 0$. As $|\lambda| < 1$ it is enough to see that $\sigma^{i_j} \rho_k\Big(\sigma^{-i_j} t^*\Big) \to 0$, for $k = 2, 3$. Since the condition (7.0.11) implies that

$$\rho_k\Big(\sigma^{-i_j} t^*\Big) = O(\sigma^{-2i_j}), \quad k = 1, 2, 3.$$

follows that

$$\sigma^{i_j} \rho_k\Big(\sigma^{-i_j} t^*\Big) = O(\sigma^{-i_j}) \to 0, \quad j \to +\infty.$$

Hence $||\tilde{\mathbf{e}}_j^+ - \bar{\mathbf{e}}_j^+|| \to 0$, proving the lemma. ∎

An immediate consequence of Lemma 7.0.7 is the following result:

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Lemma 7.0.8 *Let $S \subset U_P$ be a two-dimensional disc with positive radial projection in $W^{\mathrm{u}}_{\mathrm{loc}}(P, f_\varphi)$. Then the forward f_φ-orbit of $\tilde{\ell}$ meets transversely the disc S.*

There is a similar statement for the case of a saddle-focus Q of index one taken backwards iterates.

Suppose that f has a saddle Q of index one such that $W^{\mathrm{u}}(Q, f)$ and $W^{\mathrm{s}}(P, f)$ meets quasi-transversely. Lemmas 7.0.7 and 7.0.8 imply the properties:

(i) $W^{\mathrm{u}}_{\mathrm{loc}}(P, f) \subset \overline{W^{\mathrm{u}}(Q, f)}$, and

(ii) $W^{\mathrm{u}}(Q, f)$ intersects transversely every two-disc S with positive radial projection in $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$.

Properties (i) and (ii) above will be crucial in the subsequent applications. Note that these properties are not robust since the quasi-transverse intersections are not. In the following scholium we explain how using blenders these properties can be turn robust. For that recall the terminology of blenders in Section 3.4.

We observe that in similar way we define the same elements (replacing stable by unstable directions and vice-versa) for a saddle Q of index one.

Scholium 7.0.9 (Mixing superposition directions) Suppose that f has a blender-horseshoes Λ with reference P which is activated by an irrational saddle-focus of Q of index one. Then every diffeomorphism g sufficiently C^r-close to f the stable manifold $W^{\mathrm{s}}(P_g, g)$ intersect transversely every two-disc S with positive radial projection in $W^{\mathrm{u}}_{\mathrm{loc}}(Q_g, g)$. In particular $W^{\mathrm{s}}_{\mathrm{loc}}(Q_g, g) \subset \overline{W^{\mathrm{s}}(P_g, g)}$.

Proof. Since Q activates the blender Λ, for every diffeomorphism g sufficiently C^1-close to f it holds that $W^{\mathrm{s}}_{\mathrm{loc}}(X, g) \cap W^{\mathrm{u}}(Q_g, g) \neq \emptyset$ for some $X \in \Lambda_g$. The irrationality of the argument of the contrative eigenvalue of $Dg(Q_g)$ imply that $W^{\mathrm{s}}(X, g) \cap S \neq \emptyset$ (Lemma 7.0.8). Thus, there are N^* in $\mathbb{N}$ such that $W^{\mathrm{s}}_{\mathrm{loc}}(X, g) \cap g^{N^*}(S) \neq \emptyset$. Consider $P^*_g \in \Lambda_g$ homoclinically related to P_g and sufficiently close to X such that $W^{\mathrm{s}}_{\mathrm{loc}}(P^*_g, g) \cap g^{N^*}(S) \neq \emptyset$. The λ-lemma imply that $W^{\mathrm{s}}_{\mathrm{loc}}(P_g, g) \cap g^{N^*}(S) \neq \emptyset$, ending the proof. ■

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7.0.3 New quasi-transverse heteroclinic orbits

We show that there exist a C^r-dense subset $\mathcal{L}^r$ in $\mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ such that every diffeomorphism in $\mathcal{L}^r$ has a non-transverse cycle and two quasi-transverse heteroclinic orbits between the one-dimensional invariant manifolds. We now formalize this statement.

Let $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}_{\mathrm{quad}})$. Recall the C^r-linearising neighbourhoods $U_{P_f}, U_{Q_f} \simeq [-3,3]^3$ of the saddles P_f and Q_f described in Section 5.3.1. Note that in these coordinates we have

$$
\begin{aligned}
W^{\mathrm{s}}_{\mathrm{loc}}(P_f, f) &= [-3,3] \times \{(0,0)\},\\
W^{\mathrm{u}}_{\mathrm{loc}}(P_f, f) &= \{0\} \times [-3,3]^2\\
W^{\mathrm{s}}_{\mathrm{loc}}(Q_f, f) &= [-3,3] \times \{0\} \times [-3,3],\\
W^{\mathrm{u}}_{\mathrm{loc}}(Q_f, f) &= \{0\} \times [-3,3] \times \{0\}.
\end{aligned}
\tag{7.0.14}
$$

Recall the choice of the heteroclinic points associated to the transitions maps in the initial cycle:

– the quasi-transverse intersection points: $X = X_f \in W^{\mathrm{s}}(P_f, f) \cap W^{\mathrm{u}}(Q_f, f)$ and $X = \widetilde{X}_f \in \mathcal{O}_f(X)$ where

$$X = (0,1,0) \in W^{\mathrm{u}}_{\mathrm{loc}}(Q_f, f) \quad \text{and} \quad \widetilde{X} = (1,0,0) \in W^{\mathrm{s}}_{\mathrm{loc}}(P_f, f),$$

– the non-transverse intersection points: $Y = Y_f \in W^{\mathrm{u}}(P_f, f) \cap W^{\mathrm{s}}(Q_f, f)$ and $\widetilde{Y} = \widetilde{Y}_f \in \mathcal{O}(Y)$ where

$$Y = (0,1,1) \in W^{\mathrm{u}}_{\mathrm{loc}}(P_f, f) \quad \text{and} \quad \widetilde{Y} = (1,0,1) \in W^{\mathrm{s}}_{\mathrm{loc}}(Q_f, f).$$

Consider the canonical projection in the neighbourhood U_P defined by

$$\Pi_1, \Pi_2, \Pi_3 : U_{P_f} \to [-3,3] \tag{7.0.15}$$

$$\Pi_1(x,y,z) := x, \quad \Pi_2(x,y,z) := y, \quad \Pi_3(x,y,z) := z.$$

In similar way, we define the projections $\Pi_1, \Pi_2, \Pi_3 : U_{Q_f} \to [-3,3]$.

Recall the definition of the sets in (7.0.3) and (7.0.4),

$$\mathcal{P}^{\mathrm{s}} = \mathcal{P}^{\mathrm{s}}_f \subset W^{\mathrm{s}}(Q_f, f) \cap U_{P_f}, \quad \mathcal{P}^{\mathrm{u}}_f = \mathcal{P}^{\mathrm{u}}_f \subset W^{\mathrm{u}}(Q_f, f) \cap U_{Q_f},$$

and the neighbourhoods $U^{\pm}_{P_f}$ and $U^{\pm}_{Q_f}$ in (7.0.5).

By Remark 7.0.3 our bifurcation setting implies that

(i) $\mathcal{P}^{\mathrm{s}}_f \cap U^+_P \neq \emptyset$ and $\mathcal{P}^{\mathrm{u}}_f \cap U^-_P \neq \emptyset$; and

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(ii) for any k large enough it holds

$$\mathcal{P}^{\mathrm{s}}_{f_k} \pitchfork W^{\mathrm{u}}(P, f_k) \neq \emptyset, \quad \mathcal{P}^{\mathrm{u}}_{f_k} \pitchfork W^{\mathrm{s}}(Q, f_k) \neq \emptyset$$

where f_k is the sequence converging to f obtained via its renomalisation.

Finally, recall the notation in (5.3.16)

$$f|_{U_P} := f_{P,\varphi_P} \quad \text{and} \quad f|_{U_Q} := f_{Q,\varphi_Q}.$$

In what follows, for notational simplicity, we omit the subscript f in the dependence of the saddles P_f and Q_f.

Lemma 7.0.10 *Let $r \geq 2$. There is a C^r-dense subset $\mathcal{M}^r$ of $\mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ such that if $f \in \mathcal{M}^r$ then*

– *$H(Q, f)$ is non-trivial,*

– *there is Z close to X such that $Z \notin \mathcal{O}_f(X)$ and such that the one-dimensional manifolds $W^{\mathrm{s}}(P, f)$ and $W^{\mathrm{u}}(Q, f)$ meet quasi-transversely along the orbit of Z.*

Proof. Fix $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ and consider the quasi-transverse intersection points $\widetilde{X} = (1, 0, 0) \in U_P$ and $X = (0, 1, 0) \in U_Q$ above. Let $\ell^{\mathrm{u}}_0 \subset W^{\mathrm{u}}(Q, f)$ be a small one-disc such that $\widetilde{X}$ is in the interior of ℓ^{u}_0.

Using an accelerating (local) perturbation as in Lemma 7.0.5 (see also Remark 7.0.6), we get diffeomorphism $\tilde{f}|_{U_P}$, C^r-close to $f|_{U_P}$ and large $i \geq 0$ such that $\tilde{f}^i(\ell^{\mathrm{u}}_0)$ intersects transversely $\mathcal{P}^s_{\tilde{f}} = \mathcal{P}^s_f$.[1] For simplicity, let us denote the perturbed diffeomorphism $\tilde{f}$ also by f.

Fix $i_1 \geq 1$ such that $f^{i+i_1}(\ell^{\mathrm{u}}_0)$ meets transversely to $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f)$.

Consider a small segment $\ell^{\mathrm{s}}_0 \subset W^{\mathrm{s}}(P, f)$ containing X in its interior and contained in U_Q. A new accelerating (local) perturbation gives a diffeomorphism $\tilde{f}|_{U_Q}$, C^r-close to $f|_{U_Q}$, and large $j \geq 1$ such that $\tilde{f}^j\left(f^{i+i_1}(\ell^{\mathrm{u}}_0)\right)$ meets quasi-transversely ℓ^{s}_0 in a point $Z = Z_{\tilde{f}}$ close to X. The lemma follows observing that the orbits of Z and X are different. ■

Remark 7.0.11 The perturbation $\tilde{f}$ of $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ in Lemma 7.0.10 can be written as follows. Recall the parameters defining the local dynamics of f in the neighbourhoods of P and Q (see Section 5.3.1)

$$\mathrm{LocDyn}(f) = (\lambda_P, \sigma_P, \varphi_P, \lambda_Q, \sigma_Q, \varphi_Q),$$

[1] To be more precise, we consider a one-disc ℓ_1 transverse to $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$ in Y such that ℓ_1 is "interior" of $\mathcal{P}^s_{\tilde{f}}$. Then, the aplication of Lemma 7.0.5 provides a such traverse intersection between $\mathcal{P}^s_{\tilde{f}}$ and forward iterated of ℓ^{u}_0 .

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we have that $\tilde{f}$ it is the form $f_{i,j}$, where $f_{i,j}$ satisfy the (local) conditions

$$\mathrm{LocDyn}(f_{i,j}) = (\lambda_P, \sigma_P, \tilde{\varphi}_i, \lambda_Q, \sigma_Q, \hat{\varphi}_j),$$

where $\tilde{\varphi}_i \to \varphi_P$ and $\hat{\varphi}_j \to \varphi_Q$. Finally, we observe that by small C^r-perturbations defined in a small neighbourhood of the (transverse and quasi-transverse) intersections obtained in Lemma allowed us modify slightly the intersection points so that we can consider the new arguments $\tilde{\varphi}_i$ and $\hat{\varphi}_j$ as irrational numbers.

Scholium 7.0.12 Let $f_{i,j}$ be the sequence converging to f in Remark 7.0.11. By construction, there exists a segment $\ell_0^{\mathrm{u}} \subset W^{\mathrm{u}}(Q, f_{i,j})$ that intersects transversely $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{i,j})$ such that $f^j_{i,j}(\ell_0^{\mathrm{u}})$ meets quasi-transversely to $W^{\mathrm{s}}(P, f_{i,j})$ in a point Z_j close to X. By the λ-lemma, there exists a sequence of one-discs $\ell^{\mathrm{u}}_{0,j} \subset \ell_0^{\mathrm{u}}$ such that $f^j_{i,j}(\ell^{\mathrm{u}}_{0,j})$ C^r-converges to $W^{\mathrm{u}}_{\mathrm{loc}}(Q, f_{i,j})$ as $j \to +\infty$. Note that $f^j_{i,j}(\ell_0^{\mathrm{u}}) \cap W^{\mathrm{s}}(P, f_{i,j}) = \{Z_j\}$.

The previous comment imply that the perturbations $f_{i,j}$ of $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ satisfy the following properties: Given any small one-disc ℓ^{u} in $W^{\mathrm{u}}_{\mathrm{loc}}(Q, f_{i,j})$ containing X in its interior there exist a sequence of sub-discs $\ell^{\mathrm{u}}_{*,j}$ of ℓ_0^{u} such that:

– the one-disc $f^j_{i,j}\left(\ell^{\mathrm{u}}_{*,j}\right)$ meets quasi-transversely $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_{i,j})$ in a point Z_j and $Z_j \to X$, when $j \to +\infty$, and

– $f^j_{i,j}\left(\ell^{\mathrm{u}}_{*,j}\right) \to \ell^{\mathrm{u}}$ as $j \to +\infty$ in the C^r-topology.

The next result state that small perturbations of diffeomorphisms in $\mathcal{M}^r$ generates a third quasi-transverse orbit between the one-dimensional invariant manifolds of the initial saddles.

Lemma 7.0.13 *Let f be a diffeomorphism in $\mathcal{M}^r$ as in Lemma 7.0.10. Then there is $\varepsilon_\ell \to 0^+$ and a ε_ℓ-C^r-perturbation f_ℓ of f such that*

– *the orbits $\mathcal{O}_f(X)$ and $\mathcal{O}_f(Z)$ are preserved by f_ℓ, i.e., in a neighbourhood of $\mathcal{O}_f(X) \cup \mathcal{O}_f(Z)$, the diffeomorphisms f and f_ℓ coincides, and*

– *the one-dimensional manifolds $W^{\mathrm{s}}(P, f_\ell)$ and $W^{\mathrm{u}}(Q, f_\ell)$ meet quasi-transversely at point Z^*_ℓ such that $Z^*_\ell \notin \mathcal{O}_f(X) \cup \mathcal{O}_f(Z)$.*

Proof. Fix $f \in \mathcal{M}^r$. Consider a segment ℓ^{s} of $W^{\mathrm{s}}(P, f)$, quasi-transverse to $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$ in X. From Lemma 7.0.7, the closure of $\left\{f^{-i}(\ell^{\mathrm{s}}) : i \in \mathbb{N}\right\}$ contain the local manifold $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f)$. In particular, this closure contains the non-trivial class $H(Q, f)$. Then, given $Z^* \in W^{\mathrm{s}}_{\mathrm{loc}}(Q, f) \pitchfork W^{\mathrm{u}}(Q, f)$ there is a sequence

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$\{Z_\ell^*\}_\ell$ in $W^{\mathrm{s}}(P,f)$ such that $Z_\ell^* \to Z^*$ (this sequence is associated a sequence of times n_ℓ such that $f^{n_\ell}(\ell^{\mathrm{s}})$ converge to the straight line in $W^{\mathrm{s}}_{\mathrm{loc}}(Q,f)$ that contains to Q and Z^*). Thus, for every ℓ large enough we can modify, by a local C^r-perturbation (like (6.6.12)), the local unstable manifold $W^{\mathrm{u}}_{\mathrm{loc}}(Z^*,f)$, so that for this perturbation f_ℓ of f it holds $Z_\ell^* \in W^{\mathrm{u}}_{\mathrm{loc}}(Z^*,f_\ell)$. We observe that the size of this C^r-perturbation is ε_ℓ, where ε_ℓ is the distance between $W^{\mathrm{u}}_{\mathrm{loc}}(Z^*,f)$ and Z_ℓ^{s}. Therefore, f_ℓ has a three quasi-transverse heteroclinic orbits. This completes the proof of the lemma. ■

Remark 7.0.14 In view of Lemmas 7.0.10 and 7.0.13 if $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ then there are a diffeomorphism $f_{i,j,\ell}$ arbitrarily C^r-close to f such that:

(i) $H(Q,f_{i,j,\ell})$ is not trivial,

(ii) $f_{i,j,\ell}$ preserves the cycle of f associated to heteroclinic points $X \in W^{\mathrm{s}}(P,f)\cap W^{\mathrm{u}}(Q,f)$ and $Y \in W^{\mathrm{u}}(P,f)\cap W^{\mathrm{s}}(Q,f)$.

(iii) $f_{i,j,\ell}$ has two quasi-transverse heteroclinic point Z_j and Z_ℓ^{s} such that $Z_j, Z_\ell^{\mathrm{s}} \to X$ when $j,\ell \to +\infty$.

We denote $\mathcal{L}^r$ to such C^r-dense set in $\mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$.

7.0.4
Transition map associated to the new heteroclinic orbit

Let $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^h_{\mathrm{quad}} \cup \mathcal{T}^{p,+,-}_{\mathrm{quad}})$ and $f_{i,j,\ell}$ the C^r-perturbation of f belong to $\mathcal{L}^r$, see Remark 7.0.14. Recall the $f_{i,j,\ell}$ has three quasi-transverse intersection points X, Z_j and Z_ℓ^{s} such that $Z_j, Z_\ell^{\mathrm{s}} \to X$, as $j,\ell \to +\infty$, where X is the heteroclinic point associated to the initial cycle of f. By construction, the diffeomorphisms in $f_{i,j,\ell}$ preserves the initial cycle of f (i.e., f and $f_{i,j,\ell}$ coincides in a neighbourhood of cycle of f). In particular, preserves its corresponding transitions maps (see (5.3.7) and (5.3.9)), that is,

$$f^{N_1}_{i,j,\ell}|_{U_X} = f^{N_1}|_{U_X} \quad \text{and} \quad f^{N_2}_{i,j,\ell}|_{U_Y} = f^{N_2}|_{U_Y}.$$

Note that for every j large enough, we have that $Z_j \in U_X$. Let us now describes the transition map associated to the points Z_j and $f^{N_1}(Z_j)$. For notational simplicity, in what follows, we will omit the subscripts i,ℓ in $f_{i,j,\ell}$. By construction, $Z_j = (x_j, y_j+1, z_j)$, where $x_j, y_j, z_j \to 0$ as $j \to +\infty$. Hence Z_j belongs to the domain U_X of the transition map f^{N_1} for every j large we can take a pair of small and disjoint neighbourhoods U_X and U_{Z_j} of X and Z_j (respectively) and consider the transition maps $f_j^{N_1} : U_X \to U_{\widetilde{X}}$ and

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$f_j^{N_1} : U_{Z_j} \to U_{\widetilde{Z}_j}$, where $f^{N_1}(Z_j) = \widetilde{Z}_j$ is close to $\widetilde{X}$. Note that $\tilde{f}_j^{N_1}|_{U_X} = f^{N_1}|_{U_X}$ and $\tilde{f}_j^{N_1}|_{U_{Z_j}} = f^{N_1}|_{U_{Z_j}}$.

Let $\widetilde{Z}_j := (1 + \tilde{x}_j, 0, 0) \in U_P$, with $\tilde{x}_j \to 0$ and $\tilde{x}_j \neq 0$. We note that

$$\widetilde{Z}_j = f^{N_1}(Z_j) = \widetilde{X} + Z_j - X + \widetilde{H}(Z_j).$$

We now explicit the transition maps $f^{N_1}|_{U_{Z_j}}$ around of Z_j. Considering the point Z_j as the center of U_{Z_j} and performing by an affine linear change of coordinates around of X we can write

$$f^{N_1}|_{U_{Z_j}} : \begin{pmatrix} x_j + x \\ 1 + y_j + y \\ z_j + z \end{pmatrix} \to \begin{pmatrix} 1 + \tilde{x}_j + x + \widetilde{H}_1^j(x, y, z) \\ y + \widetilde{H}_2^j(x, y, z) \\ z + \widetilde{H}_3^j(x, y, z) \end{pmatrix}, \tag{7.0.16}$$

where

$$\begin{aligned} \widetilde{H}_k^j(x, y, z) := {} & \widetilde{H}_k(x_j + x, y_j + y, z_j + z) - \\ & - \widetilde{H}_k(x_j, y_j, z_j), \quad k = 1, 2, 3. \end{aligned} \tag{7.0.17}$$

The maps $\widetilde{H}_k$ are defined in (5.3.7). Note that

$$\widetilde{H}_1^j(\mathbf{0}) = \widetilde{H}_2^j(\mathbf{0}) = \widetilde{H}_3^j(\mathbf{0}) = 0.$$

Remark 7.0.15 The higher order terms $\widetilde{H}_k^j(x, y, z)$ of the transition $f^{N_1}|_{U_{Z_j}}$ do not satisfy the "flat conditions" at the origin (i.e., for $X = \mathbf{0}$) satisfied by the maps $\widetilde{H}_k(x, y, z)$ of the transition $f^{N_1}|_{U_X}$, see (5.3.8). However these terms satisfy the following convergence property:

$$\frac{\partial}{\partial x}\widetilde{H}_k^j(\mathbf{0}) = \frac{\partial}{\partial y}\widetilde{H}_k^j(\mathbf{0}) = \frac{\partial}{\partial z}\widetilde{H}_k^j(\mathbf{0}) \to 0, \quad k = 1, 2, 3. \tag{7.0.18}$$

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8
Proof of Theorem 7.0.4: construction of g_k

We recall the properties in Remark 7.0.3 satisfied by a diffeomorphism f in $\mathcal{N}^r_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$. The sequence $\{g_k\}_k$ in Theorem 7.0.1 is constructed by arbitrarily small local C^r-perturbations of f. We start recalling the following type of perturbations: From Lemmas 7.0.10 and 7.0.13 (see also Remark 7.0.14). f can be C^r-approximated by diffeomorphisms $f_{i,j,\ell}$ such that:

- $f_{i,j,\ell}$ and f coincides in a neighbourhood of the initial cycle whose transitions maps are associated to a quasi-transverse intersection point $X \in W^{\mathrm{s}}(P,f) \cap W^{\mathrm{u}}_{\mathrm{loc}}(Q,f)$ and to a heterodimensional tangency $Y \in W^{\mathrm{u}}_{\mathrm{loc}}(P,f) \cap W^{\mathrm{s}}(Q,f)$.
- $f_{i,j,\ell}$ has two quasi-transverse intersection heteroclinic points Z_j, Z^*_ℓ converging to X as $j,\ell \to +\infty$.

  The next assert not involve the subscript i in $f_{i,j,\ell}$. For notational simplicity, their will be omitted.
- Consider $f_{j,\ell,\bar{\upsilon}_k}$ the renormalisation of $f_{j,\ell}$ given by Theorem 2 (see equation (5.3.19)). We will see that there is $k(j,\ell)$, with $k(j,\ell) \to +\infty$ as $j,\ell \to +\infty$, such that for every $k \geq k(j,\ell)$, Z_j and Z^*_ℓ are quasi-transverse intersection heteroclinic points of $f_{j,\ell,\bar{\upsilon}_k}$. Moreover we can unfold these heteroclinic points unmodified the orbit of blender $\Lambda_{j,\ell,k}$ of associated to $f_{j,\ell,\bar{\upsilon}_k}$.

We now explain how arbitrarily small local C^r-perturbations of $f_{j,\ell,\bar{\upsilon}_k}$ generated a C^r-robust cycle between $\Lambda_{j,\ell,k}$ and Q. This define our $g_k = g_{j,\ell,k}$.

★ ***Robust intersection between*** $W^{\mathrm{s}}(\Lambda_{j,\ell,k}, f_{j,\ell,\bar{\upsilon}_k})$ ***and*** $W^{\mathrm{u}}(Q, f_{j,\ell,\bar{\upsilon}_k})$*: unfolding of the heteroclinic quasi-transverse orbits.* Unfolding suitably the quasi-transverse Z_j we generates a uu-disc simultaneously contained in $W^{\mathrm{u}}(Q, f_{j,\ell,\bar{\upsilon}_k})$ and in the superposition region of the blender $\Lambda_{j,\ell,k}$ (see Proposition 8.1.4). From Lemma 3.4.5, the manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_{j,\ell,k}, f_{j,\ell,\bar{\upsilon}_k})$ and $W^{\mathrm{u}}(Q, f_{j,\ell,\bar{\upsilon}_k})$ is meet C^r-robustly. We continue to denote by $f_{j,\ell,\bar{\upsilon}_k}$ this last diffeomorphism. Z_j (see Section 8.1).

★ ***Transverse intersection between*** $W^{\mathrm{u}}(\Lambda_{j,\ell,k}, f_{j,\ell,\bar{\upsilon}_k})$ ***and*** $W^{\mathrm{s}}(Q, f_{j,\ell,\bar{\upsilon}_k})$*: growth of the size of the strong unstable leaves of the*

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blender. We begin observing that the definition of $G_{(\xi,\mu,0,0)}|_\Delta$ imply that the unstable manifolds of its left reference $P^+_{\xi,\mu}$, growth along of the uu-direction quadratically in relation to the height of the box Δ. This gives estimations on the growth of the unstable manifold of the reference $P^+_{j,\ell,k}$ of $\Lambda_{j,\ell,k}$ in the chart Φ_k^{-1} of the renomalisations scheme of f. Since $\Phi_k(\Delta) \to \{\widetilde{Y}\}$, then for every small neighbourhood $W^{\mathrm{s}} \subset W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{j,\ell,\bar{v}_k})$ of $\widetilde{Y}$ it holds that $\Psi_k^{-1}(W^{\mathrm{s}})$ moves away from Δ. A careful choices of increasing domains $\widetilde{\Delta}_k \subset \Delta$ we guarantees that $W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}|_{\widetilde{\Delta}_k})$ meet transversely to $\Psi_k^{-1}(W^{\mathrm{s}})$ (see Lemma 8.2.8) and that $\|(\Phi_k^{-1} \circ \mathcal{R}_k(f_{j,\ell,k}) \circ \Phi_k - G_{(\xi,\mu,0,0)})|_{\widetilde{\Delta}_k}\|_{C^r} \to 0$ (see Lemma 8.2.11). This imply that $W^{\mathrm{u}}(P^+_{j,\ell,k}, \Phi_k^{-1} \circ \mathcal{R}_k(f_{j,\ell,k}|_{\widetilde{\Delta}_k})$ and $\Psi_k^{-1}(W^{\mathrm{s}})$ it meets transversely (see Section 8.2).

★ ***Homoclinic relations between $\Lambda_{j,\ell,k}$ and P.*** These relations are obtained from the heteroclinic relation in the cycle above. Our bifurcation setting imply that $W^{\mathrm{u}}(P, f_{j,\ell,\bar{v}_k}) \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{j,\ell,\bar{v}_k}) \neq \emptyset$. On the other hand, the stable manifold of $P^+_{j,\ell,k}$ is dense in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{j,\ell,\bar{v}_k})$ and thus

$$W^{\mathrm{u}}(P, f_{j,\ell,\bar{v}_k}) \pitchfork W^{\mathrm{s}}(P^+_{j,\ell,k}, f_{j,\ell,\bar{v}_k}) \neq \emptyset.$$

The irrationality of angular argument and the heteroclinic connections associated to point Z_ℓ^* implies that the manifold $W^{\mathrm{s}}(P, f_{j,\ell,\bar{v}_k})$ is dense in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{j,\ell,\bar{v}_k})$. Since $W^{\mathrm{u}}(P^*_{j,\ell,k}, f_{j,\ell,\bar{v}_k})$ meet transversely the last manifolds obtaining

$$W^{\mathrm{s}}(P, f_{j,\ell,\bar{v}_k}) \pitchfork W^{\mathrm{u}}(P^*_{j,\ell,k}, f_{j,\ell,\bar{v}_k}) \neq \emptyset.$$

Therefore, P and $P^*_{j,\ell,k}$ are homclinically related (see Section 8.3.1).

We now go into the details of these constructions.

8.0.0.1 Unfolding of the heteroclinic quasi-transverse orbits

Let $f \in \mathcal{N}_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$. Consider the sequence of perturbations $f_{i,j,\ell}$ of f in Remark 7.0.14. The next results, not involves the subscript i, for this reason this will be omitted along of this section. Consider the renormalisation $f_{j,\ell,\bar{v}_k}$ of $f_{j,\ell}$ given in Theorem 2. Note that for any large and fixed k, ℓ, the points Z_j and Z_ℓ^* are heteroclinic points of $f_{j,\ell,\bar{v}_k}$ for every k large enough. Let us now to unfold the point Z_j of $f_{j,\ell,\bar{v}_k}$. More precisely, we consider perturbations of the form

$$\theta_{j,k} \circ f^{N_1}_{j,\ell,\bar{v}_k}|_{U_{Z_j}} : U_{Z_j} \to U_{\widetilde{Z}_j}, \tag{8.0.1}$$

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where $\theta_{j,k}$ is a C^r-perturbation of identity supported in a small neighbourhood of

$$\widetilde{Z}_j := f^{N_1}_{j,\ell,\bar{\upsilon}_k}(Z_j) \in W^{\mathrm{s}}_{\mathrm{loc}}(P, f_{j,\ell,\bar{\upsilon}_k}).$$

Note that by the comments above we have

$$\theta_{j,k} \circ f^{N_1}_{j,\ell,\bar{\upsilon}_k}|_{U_{Z_j}} = \theta_{j,k} \circ f^{N_1}_{j,\ell}|_{U_{Z_j}}, = \theta_{j,k} \circ f^{N_1}|_{U_{Z_j}},$$

for every k large enough. To build such perturbations of identity, we consider the following preliminary ingredients:

– A C^r-bump function

$$b : \mathbb{R} \to [0,1], \quad \begin{cases} b(x) = 0 \quad \text{for } |x| \geq \frac{1}{2}, \\ 0 < b(x) < 1 \quad \text{for } \frac{1}{3} < |x| < \frac{1}{2}, \\ b(x) = 1 \quad \text{for } |x| \leq \frac{1}{3}. \end{cases} \tag{8.0.2}$$

Let $C = C(r) > 0$ be the C^r-norm of b, i.e.,

$$C := \max_{1 \leq i \leq r} \sup_{x \in [-1/2, 1/2]} |b^{(i)}(x)|, \tag{8.0.3}$$

where $b^{(i)}(x)$ denote the i^{th}-derivative of b in x.

– Fix $0 < \lambda < 1$ such that $\lambda_Q < \lambda^r$. Recall the sojourn time n_k (see Subsection (5.3.5)) associated to the renormalisation $f_{j,\ell,\bar{\upsilon}_k}$ of $f_{j,\ell}$. Consider the sequence of neighbourhoods $U_{j,k} \subset U_P$ centered in $\widetilde{Z}_j$ given by

$$U_{j,k} := \left\{ (x,y,z) + \widetilde{Z}_j : |x|, |y|, |z| < \frac{\lambda^{n_k}}{2} \right\}. \tag{8.0.4}$$

– The sequence of C^r-bump functions $B_k : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$B_k(x,y,z) := \lambda_P^{n_k} \, B\left(\frac{x}{\lambda^{n_k}}, \frac{y}{\lambda^{n_k}}, \frac{z}{\lambda^{n_k}} \right), \tag{8.0.5}$$

where

$$B(x,y,z) = b(x)b(y)b(z).$$

Remark 8.0.1 Note that the support of B_k is $\overline{U}_{j,k} - \{\widetilde{Z}_j\}$ and

$$B_k(x,y,z) = \lambda_P^{n_k} \quad \text{if and only if} \quad |x|, |y|, |z| < \frac{\lambda^{n_k}}{3}.$$

.

Recall the convergence of the sequence $\sigma_P^{m_k} \lambda_Q^{n_k}$ in (5.3.18). We are now ready to define $\theta_{j,k}$.

Definition 8.0.2 The map $\theta_{j,k} : M \to M$ is defined by

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– if $(x,y,z)+\widetilde{Z}_j \in U_{j,k}$, then

$$\theta_{j,k}\Big(\widetilde{Z}_j+(x,y,z)\Big)=\widetilde{Z}_j+(x,y,z)+B_k(x,y,z)\left(0,\frac{\sqrt{2}}{\sigma_P^{m_k}\lambda_Q^{n_k}},0\right). \quad (8.0.6)$$

– otherwise the map is the identity.

Recalling that $\lambda_Q<\lambda^r$, the constant $C>0$ in (8.0.3) and taking $K>0$ such that $|\frac{\sqrt{2}}{\sigma_P^{m_k}\lambda_Q^{n_k}}|<K$ we have that the inequality it is easy to check that

$$\|\theta_{j,k}-\mathrm{id}\|_r \le K\,C^3\left(\frac{\lambda_Q}{\lambda^r}\right)^{n_k}\to 0. \quad (8.0.7)$$

Let $g_{j,\ell,k}$ the local C^r-perturbation of $f_{j,\ell,\bar{\upsilon}_k}$ defined by the condition

$$g^{N_1}_{j,\ell,k} := \theta_{j,k}\circ f^{N_1}_{j,\ell,\bar{\upsilon}_k}=\theta_{j,k}\circ f^{N_1}_{j,\ell,\bar{\upsilon}_k}|_{U_{Z_j}}. \quad (8.0.8)$$

Next lemma assert that for every large and fixed j,ℓ, we can take k large enough such that the perturbation $\theta_{j,k}$ does not modify the orbit of blender of of $f_{j,\ell,\bar{\upsilon}_k}$.

Lemma 8.0.3 *For every large and fixed j and ℓ there exists $k(j,\ell)\in\mathbb{N}$ such that for every $k\ge k(j,\ell)$ it holds:*

(i) the restrictions of $f_{j,\ell,\bar{\upsilon}_k}$ and $g_{j,\ell,k}$ to the domain of definition of the blender $\Lambda_{j,\ell,k}$ coincide and

(ii) Z^*_ℓ is a heteroclinic orbit of $g_{j,\ell,k}$.

Proof. We note that since $g_{j,\ell,k}$ is obtained from $f_{j,\ell,\bar{\upsilon}_k}$ by the bifurcation of the point $\widetilde{Z}_j=(1+\tilde{x}_j,0,0)$ it sufficient to verify that the renomalisations scheme not does modified in the neighbourhood of $\widetilde{X}=(1,0,0)$, that is, the coordinates of $(f_{i,j,\bar{\upsilon}_k})^{N_1+n_k}\circ\Psi_{m_k,n_k}(\Delta)$ is far of support of $\theta_{j,k}$ for every big sufficient k. To this, we will exhibit a $k(j,\ell)$ and a local C^r-perturbation of the identity $\widetilde{\theta}_k$, satisfying $\lim\|\widetilde{\theta}_k-\mathrm{id}\|_r=0$, and such that for every $k\ge k(j,\ell)$ it holds:

– $f^{N_1}_{j,\ell,\bar{\upsilon}_k}|_{f^{n_k}_{i,j,\bar{\upsilon}_k}\circ\Psi_{m_k,n_k}(\Delta)}=\widetilde{\theta}_k\circ f^{N_1}_{j,\ell}|_{f^{n_k}_{i,j,\bar{\upsilon}_k}\circ\Psi_{m_k,n_k}(\Delta)}=\widetilde{\theta}_k\circ f^{N_1}|_{f^{n_k}_{i,j,\bar{\upsilon}_k}\circ\Psi_{m_k,n_k}(\Delta)}$, and

– the sets $\mathrm{supp}(\widetilde{\theta}_k)$ and $\overline{U}_{j,k}$ are disjoint, and $f_{j,\ell,\bar{\upsilon}_k}(Z^*_\ell)\notin\mathrm{supp}(\widetilde{\theta}_k)\cup\overline{U}_{j,k}$.

These two points above imply the items (i) and (ii) in the lemma. We now proceed to construction of $\widetilde{\theta}_k$.

Recalling that $f^{N_1}_{j,\ell}(X)=f^{N_1}(X)=\widetilde{X}$ and the sojourn time n_k involved in $\bar{\upsilon}_k=\bar{\upsilon}_{m_k,n_k}$, consider the open sets

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$$U_{X,k} := \left\{(x,y,z) + \widetilde{X} : |x|, |y|, |z| < \frac{\lambda^{n_k}}{2}\right\}, \quad \lambda_Q < \lambda^r. \tag{8.0.9}$$

We define the local perturbations of the identity supported in $\overline{U}_{X,k}$ by

$$\widetilde{\theta}_k : U_{X,k} \to U_{X,k}, \qquad \widetilde{\theta}_k\Big((x,y,z) + \widetilde{X}\Big) = (x,y,z) + \widetilde{X} + B_k(x,y,z)\,\widetilde{\omega}_k,$$

where the map B_k is defined in (8.0.5) and the vector $\widetilde{\omega}_k$ is given by

$$\widetilde{\omega}_k := \left(\mathfrak{s}_k - \mathfrak{c}_k, \frac{\sqrt{2}}{\sigma_P{}^{m_k}\lambda_Q^{n_k}} - \lambda_Q^{n_k}\tilde{\rho}_{2,k}, -(\mathfrak{s}_k + \mathfrak{c}_k) - \lambda_Q^{n_k}\tilde{\rho}_{3,k}\right),$$

where $\mathfrak{s}_k, \mathfrak{c}_k, \sigma_P{}^{m_k}\lambda_Q^{n_k}$, and $\tilde{\rho}_{2,k}, \tilde{\rho}_{3,k}$ are the convergent sequences defined in Remark 5.3.10, (5.3.18) and (5.3.23), respectively. In particular, sequence of vectors $\widetilde{\omega}_k$ is convergent. This implies that $\widetilde{\theta}_k$ is a C^r-perturbation of the identity whose support is $\overline{U}_{X,k}$ for every k large enough. The lemma now follows from the next claim.

Claim 8.0.4 *For every big j,ℓ there exits $k(j,\ell) \geq 1$ such that for every large enough $k \geq k(j)$ it holds*

(i) $\overline{U}_{X,k} \cap \overline{U}_{j,k} = \emptyset$

(ii) $f_{j,\ell,\bar{v}_k}(Z_\ell^*) \notin \overline{U}_{X,k} \cup \overline{U}_{j,k} = \emptyset$ *and*

(iii) $\tilde{f}_{j,\ell,\bar{v}_k}^{N_1}|_{U_X} = \widetilde{\theta}_k \circ \tilde{f}_{j,\ell}^{N_1}|_{U_X} = \widetilde{\theta}_k \circ f^{N_1}|_{U_X}$.

Proof. For the first part recall that $\tilde{X} = (1,0,0)$ and $\tilde{Z}_j = (1+\tilde{x}_j, 0, 0)$, where $\tilde{x}_j \to 0$. We let $\widetilde{Z}_\ell^* := \tilde{f}_{j,\ell}(Z_\ell^*) := (1+\tilde{x}_\ell^*, 0, 0)$. We choose $k(j,\ell) \geq 1$ such that

$$\lambda^{n_k} < \min\Big\{|\tilde{x}_j|, |\tilde{x}_\ell^*|, |\tilde{x}_j - \tilde{x}_\ell^*|\Big\} \quad \text{for every } k \geq k(j,\ell).$$

This choice immediately implies that

$$\overline{U}_{X,k} \cap \overline{U}_{j,k} = \emptyset, \quad \widetilde{Z}_\ell^* \notin \overline{U}_{X,k} \cup \overline{U}_{j,k} \quad \text{for every } k \geq k(j,\ell).$$

For the second part, recall the coordinates in (5.3.25) and the equations (5.3.26) and (5.3.7). The neutral condition (5.3.18) imply that each coordinates from

$$f^{N_1}|_{U_X} \circ f^{n_k}_{Q,\varphi+\tilde{\alpha}_{m_k,n_k}(\varphi)} \circ \Psi_k(\Delta) - \widetilde{X},$$

has a symbol of Landau equal to $O(\lambda_Q^{n_k})$ (here $\tilde{\varphi}$ is the argument associated to rotation $f_{j,\ell}|_{U_P}$, φ is the argument associated to rotation $f_{j,\ell}|_{U_Q}$ and $\tilde{\alpha}_{m_k,n_k,\tilde{\varphi},\varphi}(\cdot) = \tilde{\alpha}_{m_k,n_k,\tilde{\varphi},\varphi}(\cdot)$). This imply that the map $B_k(x,y,z)$ restrict to set

$$f^{N_1}|_{U_X} \circ f^{n_k}_{Q,\varphi+\tilde{\alpha}_{m_k,n_k}(\varphi)} \circ \Psi_k(\Delta) - \widetilde{X},$$

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is equal to $\lambda_Q^{n_k}$ for every k big sufficient (see Remark 8.0.1). We conclude the proof observing that the vector ν_{m_k,n_k} in (5.3.24) satisfies $\lambda_Q{}^{n_k}\widetilde{\omega}_k = \nu_{m_k,n_k}$. ∎
The proof of the lemma is now complete. ∎

In the next remark we white explicitly the definition of the perturbation $g_{j,\ell,k}$ (including the omitted subscript i)

Remark 8.0.5 For $f \in \mathcal{N}_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$, we recall the sequence of C^r-perturbations $f_{i,j,\ell}$ of f in Remark 7.0.14. Recall that the local dynamics of $f_{i,j,\ell}$ in Remark 7.0.11 is given by

$$\mathrm{LocDyn}(f_{i,j,\ell}) = (\lambda_P, \sigma_P, \tilde{\varphi}_i, \lambda_Q, \sigma_Q, \hat{\varphi}_j),$$

such that $\tilde{\varphi}_i \to \varphi_P$, and $\hat{\varphi}_j \to \varphi_Q$. We observe that the subscript ℓ in $f_{i,j,\ell}$ is associated to a local perturbation along of the unstable manifold of Q that not modify the parameters above (see Lemma 7.0.13).

Finally, recalling the notation in (5.3.19) and taking $k \geq k(j,\ell)$ the diffeomorphism $g_{j,\ell,k} = g_{i,j,\ell,k}$ in (8.0.8) is defined by small C^r-perturbation of $f_{i,j,\ell}$ given by the following equations

$$\begin{aligned}
g_{j,\ell,k}|_{U_P} &= g_{i,j,\ell,k}|_{U_P} := f_{P,\varphi_i+\bar{\alpha}_{m_k,n_k}(\varphi_i)}, \quad \tilde{\varphi}_i \to \varphi_P;\\
g_{j,\ell,k}|_{U_Q} &= g_{i,j,\ell,k}|_{U_Q} = f_{Q,\hat{\varphi}_j+\bar{\alpha}_{m_k,n_k}(\hat{\varphi}_j)}, \quad \hat{\varphi}_j \to \varphi_Q;\\
g^{N_1}_{j,\ell,k}|_{U_X} &:= g^{N_1}_{i,j,\ell,k}|_{U_X} = (f_{i,j,\ell})^{N_1}_{X,\bar{\nu}_{m_k,n_k}},\\
g^{N_1}_{j,\ell,k}|_{U_{Z_j}} &:= g^{N_1}_{i,j,\ell,k}|_{U_{Z_j}} = f^{N_1}_{i,j,\ell,k}|_{U_{Z_j}} = \theta_{j,m_k} \circ f^{N_1}_{i,j,\ell}|_{U_{Z_j}},\\
g^{N_2}_{j,\ell,k}|_{U_Y} &:= g^{N_2}_{i,j,\ell,k}|_{U_Y} = (f_{i,j,\ell})^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu)}.
\end{aligned} \tag{8.0.10}$$

Remark 8.0.6 Recall the definition of $\bar{\alpha}_{m_k,n_k}(\cdot)$ in the renomalisation scheme, see Section 5.3.6. In the equation (8.0.10), it holds that

$$\bar{\alpha}_{m_k,n_k}(\cdot) = \bar{\alpha}_{m_k,n_k,\varphi_i,\hat{\varphi}_j}(\cdot).$$

We observe that, by definition, $\hat{\varphi}_j + \bar{\alpha}_{m_k,n_k}(\hat{\varphi}_j)$ is a irrational number for every j,k.

8.1
One-dimensional connections

The main result in this section is the next proposition that provides the one-dimensional connection between the stable manifold of blender $\Lambda_{i,j,\ell,k}$ and unstable manifold of the saddle Q of the diffeomorphism $g_{i,j,\ell,k}$ in (8.0.10). More precisely,

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Proposition 8.1.1 *For every i,j, ℓ and k large sufficient it holds*

$$W^{\mathrm{s}}\Big(\Lambda_{i,j,\ell,k}, g_{i,j,\ell,k}\Big) \pitchfork W^{\mathrm{u}}\Big(Q, g_{i,j,\ell,k}\Big) \neq \emptyset$$

C^r-robustly.

The proof of this result not involve the sub-scrips i and ℓ. For notational simplicity along of this section them will be omitted from notation above.

To prove this proposition we need to recall some properties and definitions. Recall the sequence of parameterisations $\Psi_{m_k,n_k} : \mathbb{R}^3 \to U_Q$ in (5.3.20), the coordinate changes $\Theta : \mathbb{R}^4 \to \mathbb{R}^4$ in (6.4.1) and $\widetilde{\Theta} : \mathbb{R}^3 \to \mathbb{R}^3$ in Remark 4.1.1, associated to these maps we consider

$$\Phi_{m_k,n_k} : \mathbb{R}^3 \to U_Q, \quad \Phi_{m_k,n_k}(X) = \Psi_{m_k,n_k} \circ \Theta \circ \tilde{\Theta}(X). \tag{8.1.1}$$

Remark 8.1.2 Notice that map Θ is a coordinate change in $\mathbb{R}^4$. The maps Φ_{m_k,n_k} is defined in precise form extending naturally the maps $\widetilde{\Theta}$ and Ψ_{m_k,n_k} to whole $\mathbb{R}^4$ as

$$(\mu, x, y, z) \to (\mu, \widetilde{\Theta}(x, y, z)), \quad (\mu, x, y, z) \to (\mu, \Psi_{m_k,n_k}(x, y, z)).$$

However, by notational simplicity, in what follows we will preserve the abuse of notation in the equation (8.1.1).

Recall also the renormalised sequences $\mathcal{R}_{m_k,n_k}(\cdot)$ in Theorem 2 and that the blender $\Lambda_{j,k}$ of $\mathcal{R}_{m_k,n_k}(g_{j,k})$ is the maximal invariant in

$$\Delta_k := \Phi_{m_k,n_k}(\Delta), \quad \Delta = [-4, 4]^2 \times [-40, 22] \subset \mathbb{R}^3, \tag{8.1.2}$$

that is,

$$\Lambda_{j,k} = \bigcap_{\ell \in \mathbb{Z}} \Big(\mathcal{R}_{m_k,n_k}(g_{j,k})\Big)^{\ell} (\Delta_k). \tag{8.1.3}$$

Let $G_{(\xi,\mu,\kappa,\eta)}|_{\Delta}$ be as in Theorem 2 and consider the curve $\bar{\ell} := \Big\{(0, y, 0) : |y| < 4\Big\}$. Then, $\bar{\ell}$ is a uu-disc in the region of superposition of the blender of $G_{(\xi,\mu,\kappa,\eta)}|_{\Delta}$. For large $k \geq 1$ we consider the curve

$$\bar{\ell}_k := \Phi_{m_k,n_k}(\bar{\ell}) \subset \Delta_k. \tag{8.1.4}$$

The C^r-convergence $\Phi^{-1}_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(g_{j,k}) \circ \Phi_{m_k,n_k} \to G_{(\xi,\mu,0,0)}$ imply that, for every sufficiently large k, the segment

$$\Phi^{-1}_{m_k,n_k} \circ (g_{j,k})^{N_2+m+N_1+n_k}(\bar{\ell}_k),$$

contains a uu-disc in the region of superposition of blender $\Lambda_{j,k}$.

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Following the construction in (21, Proposition 6.3) we now see that there is a disc contained in $W^{\mathrm{u}}(Q, g_{j,k})$ whose return to the heterodimensional tangency (considered in the chart $\Phi^{-1}_{m_k,n_k}$) is arbitrarily C^r-close to $\Phi^{-1}_{m_k,n_k} \circ (g_{j,k})^{N_2+m+N_1+n_k}(\bar{\ell}_k)$. We now go to the details of this construction.

Recall the choice of the heteroclinic point $Z_j = (x_j, 1 + y_j, z_j)$ in Remark 7.0.14 and the transition map $f_j^{N_1} : U_{Z_j} \to U_{\tilde{Z}_j}$ in (7.0.16), with $\widetilde{Z}_j = f_j(Z_j) = (1 + \tilde{x}_j, 0, 0)$,

$$f_j^{N_1} \begin{pmatrix} x + x_j \\ y + y_j + 1 \\ z + z_j \end{pmatrix} = \begin{pmatrix} 1 + \tilde{x}_j + x + \widetilde{H}_1^j(x,y,z) \\ y + \widetilde{H}_2^j(x,y,z) \\ z + \widetilde{H}_3^j(x,y,z) \end{pmatrix}, \tag{8.1.5}$$

where

$$\widetilde{H}_k^j(x,y,z) := \widetilde{H}_k(x_j + x, y_j + y, z_j + z) - \widetilde{H}_k(x_j, y_j, z_j), \quad k = 1, 2, 3.$$

Note that for $k = 1, 2, 3$, it holds $\widetilde{H}_k^j(\mathbf{0}) = 0$ and if $j \to +\infty$ then

$$\frac{\partial}{\partial x}\widetilde{H}_k^j(\mathbf{0}), \frac{\partial}{\partial y}\widetilde{H}_k^j(\mathbf{0}), \frac{\partial}{\partial z}\widetilde{H}_k^j(\mathbf{0}) \to 0, \quad k = 1, 2, 3. \tag{8.1.6}$$

Note that $f_j^{N_1}|_{U_{Z_j}} = f^{N_1}|_{U_{Z_j}}$. We write this transition map compactly as

$$f_j^{N_1}(Z_j + W) = \tilde{Z}_j + W + \widetilde{H}^j(W), \quad Z_j + W \in U_{Z_j}.$$

Choosing small $\delta > 0$ and an unitary vector V_j in $T_{Z_j}W^{\mathrm{u}}(Q, \tilde{f}_j)$ we can write (in local coordinates) the local unstable manifold of Z_j contained in U_{Z_j} as

$$\ell_j^{\mathrm{u}} = \Big\{ Z_j + t\, V_j + \bar{\rho}_j(t) : |t| < \delta \Big\}, \tag{8.1.7}$$

where

$$\begin{aligned} Z_j &= \Big(x_j, 1 + y_j, z_j\Big), \quad x_j, y_j, z_j \to 0; \\ V_j &= \Big(v_1^j, 1 + v_2^j, v_3^j\Big), \quad ||V_j|| = 1, \quad V_j \to (0, 1, 0); \\ \bar{\rho}_j(t) &= \Big(\rho_1^j(t), \rho_2^j(t), \rho_3^j(t)\Big), \quad \frac{d}{dt}\bar{\rho}_j(0) = \bar{\rho}_j(0) = \mathbf{0}. \end{aligned} \tag{8.1.8}$$

The λ-lemma (see (38, Theorem 2)) implies that $\|\bar{\rho}_j|_{[-\delta,\delta]}\|_r \to 0$, as $j \to +\infty$. Recall $g_{j,k}^{N_1}|_{U_{Z_j}}$ in (8.0.10). Consider the segment

$$\tilde{\ell}_{j,k}^{\mathrm{u}} := g_{j,k}^{N_1}(\ell_j^{\mathrm{u}}) = \theta_{j,k} \circ f_j^{\,N_1}(\ell_j^{\mathrm{u}}).$$

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From definition of $\theta_{j,k}$ in (8.0.6) we have that

$$\tilde{\ell}^{\mathrm{u}}_{j,k} = \Big\{ \widetilde{Z}_j + t\,V_j + \bar{\rho}_j(t) + \widetilde{H}^j(t\,V_j + \bar{\rho}_j(t)) + \\ + B_k\Big(t\,V_j + \bar{\rho}_j(t) + \widetilde{H}^j(t\,V_j + \bar{\rho}_j(t))\Big)\left(0, \frac{\sqrt{2}}{\sigma_P^{m_k}\,\lambda_Q^{n_k}}, 0\right) : |t| < \delta \Big\}.$$

We rewrite the segment $\tilde{\ell}^{\mathrm{u}}_{j,k}$ as follows

$$\tilde{\ell}^{\mathrm{u}}_{j,k} = \Big\{ \widetilde{Z}_j + t\,W_j + \tilde{\rho}_j(t) + \\ + B_k\Big(t\,W_j + \tilde{\rho}_j(t)\Big)\left(0, \frac{\sqrt{2}}{\sigma_P^{m_k}\,\lambda_Q^{n_k}}, 0\right) : |t| < \delta \Big\}, \tag{8.1.9}$$

where

$$W_j := \Big(\mathrm{id} + D\widetilde{H}^j(\mathbf{0})\Big)\,V_j, \quad \tilde{\rho}_j(t) := \bar{\rho}_j(t) + \widetilde{H}^j(t\,V_j + \bar{\rho}_j(t)) - t\,D\widetilde{H}^j(\mathbf{0})\,V_j.$$

Note that

$$\frac{d}{dt}(\tilde{\rho}_j)(0) = \tilde{\rho}_j(0) = \mathbf{0}. \tag{8.1.10}$$

We let $W_j = (w_1^j, w_2^j, w_3^j)$. From (8.1.8) and (8.1.6) it holds

$$\begin{pmatrix} w_1^j \\ w_2^j \\ w_3^j \end{pmatrix} = \begin{pmatrix} \Big(1 + \frac{\partial}{\partial x}\widetilde{H}_1^j(\mathbf{0})\Big)\,v_1^j + \frac{\partial}{\partial y}\widetilde{H}_1^j(\mathbf{0})\Big(1 + v_2^j\Big) + \frac{\partial}{\partial z}\widetilde{H}_1^j(\mathbf{0})\,v_3^j \\ \frac{\partial}{\partial x}\widetilde{H}_2^j(\mathbf{0})\,v_1^j + \Big(1 + \frac{\partial}{\partial y}\widetilde{H}_2^j(\mathbf{0})\Big)\Big(1 + v_2^j\Big) + \frac{\partial}{\partial z}\widetilde{H}_2^j(\mathbf{0})\,v_3^j \\ \frac{\partial}{\partial x}\widetilde{H}_3^j(\mathbf{0})\,v_1^j + \frac{\partial}{\partial y}\widetilde{H}_3^j(\mathbf{0})\Big(1 + v_2^j\Big) + (1 + \frac{\partial}{\partial z}\widetilde{H}_3^j(\mathbf{0}))\,v_3^j \end{pmatrix}, \tag{8.1.11}$$

where $w_1^j, w_3^j \to 0$ and $w_2^j \to 1$. For big $j \geq 1$, consider

$$\tilde{w}_1^j := \frac{w_1^j}{w_2^j} \quad \text{and} \quad \tilde{w}_3^j := \frac{w_3^j}{w_2^j} \quad \text{where} \quad \tilde{w}_1^j, \tilde{w}_3^j \to 0. \tag{8.1.12}$$

Thus, writing $\tilde{\rho}_j(t) = \Big(\tilde{\rho}_j^1(t), \tilde{\rho}_j^2(t), \tilde{\rho}_j^3(t)\Big)$, we can express the segment $\tilde{\ell}^{\mathrm{u}}_{j,k}$ as

$$\tilde{\ell}^{\mathrm{u}}_{j,k} := \Big\{ \Big(1 + \tilde{x}_j + \tilde{x}_{j,k}(t), \tilde{y}_{j,k}(t), \tilde{z}_{j,k}(t)\Big) : |t| < \delta \Big\},$$

where

$$\tilde{x}_{j,k}(t) = t\,\tilde{w}_1^j + \tilde{\rho}_j^1\left(\frac{t}{w_2^j}\right),$$
$$\tilde{y}_{j,k}(t) = t + \tilde{\rho}_j^2\left(\frac{t}{w_2^j}\right) + \frac{\sqrt{2}}{\sigma_P^{m_k}\,\lambda_Q^{n_k}}\,B_k\left(\frac{t}{w_2^j}\,W_j + \tilde{\rho}_j\left(\frac{t}{w_2^j}\right)\right),$$
$$\tilde{z}_{j,k}(t) = t\,\tilde{w}_3^j + \tilde{\rho}_j^3\left(\frac{t}{w_2^j}\right).$$

For sufficient large j, consider the sub-segment $\ell_{j,k}$ of $\tilde{\ell}^{\mathrm{u}}_{j,k}$ (obtained

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rescaling the parameter t by the factor $\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2$) given by

$$\ell_{j,k} := \Big\{(1+\tilde{x}_j + x_{j,k}(t), y_{j,k}(t), z_{j,k}(t)) : |t| < \delta\Big\},$$

where

$$\begin{aligned} x_{j,k}(t) &:= \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\tilde{w}_1^j\,\varsigma_2\,t + \widetilde{\rho}^1_{j,k}(t), \\ y_{j,k}(t) &:= \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2\,t + \widetilde{\rho}^2_{j,k}(t) + \\ &+ \frac{\sqrt{2}}{\sigma_P^{m_k}\lambda_Q^{n_k}}\,B_k\Big(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\,W_j + \\ &+ \widetilde{\rho}_j\Big(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\Big)\Big), \\ z_{j,k}(t) &:= \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\tilde{w}_3^j\,\varsigma_2\,t + \widetilde{\rho}^3_{j,k}(t), \end{aligned} \tag{8.1.13}$$

and

$$\widetilde{\rho}^k_{j,k}(t) = \widetilde{\rho}^k_j\left(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2\,t}{w_2^j}\right), \qquad k = 1,2,3.$$

Claim 8.1.3 *For every large sufficient k, it holds*

$$B_k\left(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\,W_j + \widetilde{\rho}_j\left(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\right)\right) = \lambda_Q^{n_k}.$$

Proof.

We recall the constant $0 < \lambda_Q < \lambda < 1$ involved in the definition of $B_k(\cdot)$ in (8.0.5). From Remark 8.0.1 it is sufficient verify that every coordinate of

$$\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\,W_j + \widetilde{\rho}_j\left(\frac{\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2}{w_2^j}t\right) \tag{8.1.14}$$

is less than $\frac{\lambda^{n_k}}{3}$. To see this, we note that as $|t| < \delta$, each coordinate of (8.1.14) has a Landau symbol equal to $\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}$. Since $\sigma_P^{m_k}\lambda_Q{}^{n_k}$ convergence to a number different of zero (see equation (5.3.18)) we have that

$$\sigma_P^{-2m_k}\sigma_Q{}^{-n_k} = \frac{\sigma_P^{-m_k}\sigma_Q{}^{-n_k}\lambda_Q^{n_k}}{\sigma_P^{m_k}\lambda_Q^{n_k}} \le \lambda_Q^{n_k},$$

for every big k. This imply that each coordinate in (8.1.14) is less than $\frac{\lambda^{m_k}}{3}$, for every k large enough. This completes the proof of claim. ■ Return to the coordinates in (8.1.13), we get that $y_{j,k}(t)$ is given by

$$y_{j,k}(t) = \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2\,t + \widetilde{\rho}^2_{j,k}(t) + \sqrt{2}\,\sigma_P^{-m_k}.$$

Recalling the definition of the curve $\bar{\ell}_k$ in (8.1.4) we state what following:

Proposition 8.1.4 *The C^r-distance between the segments $\Phi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k}(\ell_{j,k})$ and $\Phi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k+N_1+n_k}(\bar{\ell}_k)$ goes to zero as $j,k \to +\infty$.*

Remark 8.1.5 As an immediate consequence of Proposition 8.1.4 we have that the saddle Q activates the blender $\Lambda_{j,k}$, that is, the unstable manifold of Q contains a uu-disc in the region of the superposition of the blender, see Remark 3.4.3.

Proof.[Proof of Proposition 8.1.4] The proof follows considering calculations similar to the ones in the proof of Theorem 2. Recall the definition of $\Phi_{m_k,n_k} = \Psi_{m_k,n_k} \circ \Theta \circ \widetilde{\Theta}$ in (8.1.1). We will consider parameterizations $\bar{\gamma}_k(t)$ and $\gamma_{j,k}(t)$ (with common domains) of the curves $\bar{\ell}_k$ and $\ell_{j,k}$ and estimate the C^r-distance:

$$\begin{aligned}&\|\Phi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k+N_1+n_k}(\bar{\gamma}_k(t)) - \Phi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k}(\gamma_{j,k}(t))\|_r \leq \\ &\|\widetilde{\Theta}^{-1} \circ \Theta^{-1}\|_r \, \|\Psi^{-1}_{m_k,n_k} \circ \Big(g_{j,k}^{N_2+m_k+N_1+n_k}(\bar{\gamma}_k(t)) - g_{j,k}^{N_2+m_k}(\gamma_{j,k}(t)\Big)\|_r.\end{aligned} \tag{8.1.15}$$

Since the coordinate change $\Theta \circ \widetilde{\Theta}$ is bounded and independent of m_k and n_k, it is sufficient to check that the last C^r-norm goes to zero as $j,k \to +\infty$.

We recall that our bifurcation setting (see Remark 7.0.3) imply that

$$b_2 - b_3 = b_2 + b_3 - b_4 = 0.$$

• *The segment $\Psi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k+N_1+n_k}(\bar{\ell}_k)$.* Consider any point $(0,t,0) \in \Phi^{-1}_{m_k,n_k}(\bar{\ell}_k)$ where $|t| < 4$, and we let

$$\bar{\gamma}_k(t) = (\bar{x}_k(t), \bar{y}_k(t), \bar{z}_k(t)) := \Psi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k+N_1+n_k} \circ \Phi_{m_k,n_k}(0,t,0).$$

Recalling coordinates (5.3.37) we have that

$$\begin{aligned}\bar{x}_k(t) &= \varsigma_1\, \varsigma_2\, t + a_1\, {\lambda_P}^{m_k}\, \sigma_P^{m_k}\, {\sigma_Q}^{n_k}\, \tilde{H}_1\Big(\mathbf{x}_k(t)\Big) + \sigma_P^{m_k}\, {\sigma_Q}^{n_k} H_1\Big(\hat{\mathbf{x}}_k(t)\Big),\\ \bar{y}_k(t) &= \mu + \varsigma_2^3\, t^2 + b_1\, {\lambda_P}^{m_k}\, \sigma_P^{2m_k}\, {\sigma_Q}^{2n_k} \tilde{H}_1\Big(\mathbf{x}_k(t)\Big) + \sigma_P^{2m_k}\, {\sigma_Q}^{2n_k}\, H_2\Big(\hat{\mathbf{x}}_k(t)\Big),\\ \bar{z}_k(t) &= \varsigma_5\, \varsigma_2\, t + c_1\, {\lambda_P}^{m_k}\, \sigma_P^{m_k}\, {\sigma_Q}^{n_k} \tilde{H}_1\Big(\mathbf{x}_k(t)\Big) + \sigma_P^{m_k}\, {\sigma_Q}^{n_k}\, H_3\Big(\hat{\mathbf{x}}_k(t)\Big),\end{aligned}$$

where $\mathbf{x}_k(t)$ and $\hat{\mathbf{x}}_k(t)$ in the higher order terms are given by

$$\begin{aligned}\mathbf{x}_k(t) &= g_{j,k}^{n_k} \circ \Phi_{m_k,n_k}(0,t,0) - (0,1,0),\\ \hat{\mathbf{x}}_k(t) &= g_{j,k}^{m_k+N_1+n_k} \circ \Phi_{m_k,n_k}(0,t,0) - (0,1,1).\end{aligned} \tag{8.1.16}$$

• *The segment $\Psi^{-1}_{m_k,n_k} \circ g_{j,k}^{N_2+m_k}(\ell_{j,k})$.* Recall that

$$\ell_{j,k} = \Big\{\Big(1 + \tilde{x}_j + x_{j,k}(t), y_{j,k}(t), z_{j,k}(t)\Big) : |t| < \delta\Big\} \tag{8.1.17}$$

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where the coordinates $x_{j,k}(t), y_{j,k}(t)$, and $z_{j,k}(t)$ are given in (8.1.13). Write

$$\begin{aligned}\gamma_{j,k}(t) &= (\widetilde{x}_{j,k}(t), \widetilde{y}_{j,k}(t), \widetilde{z}_{j,k}(t)) \\ &:= \Psi^{-1}_{m_k,n_k} \circ g^{N_2+m_k}_{j,k}\Big(\tilde{x}_j + x_{j,k}(t), y_{j,k}(t), z_{j,k}(t)\Big).\end{aligned} \tag{8.1.18}$$

Applying $f^{m_k}_{j,k} = g^{m_k}_{j,k}$ (recall (8.0.10)) to $\Big(1+\tilde{x}_j + x_{j,k}(t), y_{j,k}(t), z_{j,k}(t)\Big)$ we get

$$(\widehat{x}_{j,k}(t), 1+\widehat{y}_{j,k}(t), 1+\widehat{z}_{j,k}(t)) := g^{m_k}_{j,k}\Big(1+\tilde{x}_j + x_{j,k}(t), y_{j,k}(t), z_{j,k}(t)\Big). \tag{8.1.19}$$

where

$$\begin{aligned}\widehat{x}_{j,k}(t) &= {\lambda_P}^{m_k}\,(1+\tilde{x}_j) + {\lambda_P}^{m_k}\sigma_P^{-2m_k}{\sigma_Q}^{-n_k}\,\tilde{w}^j_1\,\varsigma_2\,t + {\lambda_P}^{m_k}\widetilde{\rho}^1_{j,k}(t), \\ \widehat{y}_{j,k}(t) &= \left(\frac{1-\tilde{w}^j_3}{\sqrt{2}}\right)\sigma_P^{-m_k}\,{\sigma_Q}^{-n_k}\,\varsigma_2\,t + {\sigma_P}^{m_k}\,\left(\frac{\widetilde{\rho}^2_{j,k}(t) - \widetilde{\rho}^3_{j,k}(t)}{\sqrt{2}}\right), \\ \widehat{z}_{j,k}(t) &= \left(\frac{1+\tilde{w}^j_3}{\sqrt{2}}\right)\sigma_P^{-m_k}\,{\sigma_Q}^{-n_k}\,\varsigma_2\,t + {\sigma_P}^{m_k}\,\left(\frac{\widetilde{\rho}^2_{j,k}(t) + \widetilde{\rho}^3_{j,k}(t)}{\sqrt{2}}\right).\end{aligned} \tag{8.1.20}$$

The definitions of $\widetilde{\rho}^k_{j,k}(t)$ in (8.1.13) and $\widetilde{\rho}_j(t)$ in (8.1.9) imply that the symbols

$$O\Big(\widetilde{\rho}^2_{j,k}(t)\Big),\ O\Big(\widetilde{\rho}^3_{j,k}(t)\Big),\ O\Big(\widetilde{\rho}^2_{j,k}(t) \pm \widetilde{\rho}^3_{j,k}(t)\Big) \simeq O(\sigma_P^{-4m}{\sigma_Q}^{-2n_k}). \tag{8.1.21}$$

Thus the symbols of Landau of $\widehat{x}_{j,k}(t), \widehat{y}_{j,k}(t)$ and $\widehat{z}_{j,k}(t)$ are given by

$$\begin{aligned}\widehat{x}_{j,k}(t) &= O({\lambda_P}^{m_k}), \\ \widehat{y}_{j,k}(t) &= \widehat{z}_{j,k}(t) = O(\sigma_P^{-m_k}{\sigma_Q}^{-n_k}) + O(\sigma_P^{-3m_k}{\sigma_Q}^{-2n_k}).\end{aligned} \tag{8.1.22}$$

Finally, we apply $\Psi^{-1}_{m_k,n_k} \circ g^{N_2}_{j,k}$ to the point $(\widehat{x}_{j,k}(t), 1+\widehat{y}_{j,k}(t), 1+\widehat{z}_{j,k}(t))$ in (8.1.20) to get the coordinates $(\widetilde{x}_{j,k}(t), \widetilde{y}_{j,k}(t), \widetilde{z}_{j,k}(t))$ in (8.1.18). For notational simplicity, let us introduce the following terms

$$u_{j,k}(t) := \widetilde{\rho}^2_{j,k}(t) - \widetilde{\rho}^3_{j,k}(t); \quad v_{j,k}(t) := \widetilde{\rho}^2_{j,k}(t) + \widetilde{\rho}^3_{j,k}(t). \tag{8.1.23}$$

Recalling the definitions of $\Psi^{-1}_{m_k,n_k}$ in (5.3.36) and of $g^{N_2}_{j,k}$ in (8.0.10) and that $g^{N_2}_{j,k}$ coincides with $f^{N_2}_{Y,\bar{\mu}_{m_k,n_k}(\mu)}$ (see (5.3.19), Remark (5.3.4), and (5.3.9)), and the definition of $\bar{\mu}_{m_k,n_k}(\mu)$ in (5.3.21), we get

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$$\begin{aligned}
\widetilde{x}_{j,k}(t) &= a_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k}\, (1+\tilde{x}_j) + \\
&+ \left(a_1\, {\lambda_P}^{m_k} {\sigma_P}^{-m_k} \tilde{w}_1^j + a_2 \left(\frac{1-\tilde{w}_3^j}{\sqrt{2}} \right) + a_3 \left(\frac{1+\tilde{w}_3^j}{\sqrt{2}} \right) \right) \varsigma_2\, t + \\
&+ a_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k} \tilde{\rho}^1_{j,k}(t) + a_2\, \sigma_P^{2m_k} {\sigma_Q}^{n_k} u_{j,k}(t) + a_3\, {\sigma_P}^{2m_k} {\sigma_Q}^{n_k} v_{j,k}(t) + \\
&+ \sigma_P^{m_k} {\sigma_Q}^{n_k} H_1\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big); \\
\widetilde{y}_{j,k}(t) &= \mu + b_1\, {\lambda_P}^{m_k} \sigma_P^{2m_k} {\sigma_Q}^{2n_k}\, (1+\tilde{x}_j) + \\
&+ b_1\, {\lambda_P}^{m_k} \sigma_P^{2m_k} {\sigma_Q}^{2n_k} \tilde{\rho}^1_{j,k}(t) + b_1\, {\lambda_P}^{m_k} \sigma_Q^{n_k}\, \tilde{w}_1^j\, \varsigma_2\, t + \\
&+ \left(b_2 \left(\frac{(1-\tilde{w}_3^j)^2}{2} \right) + b_3 \left(\frac{(1+\tilde{w}_3^j)^2}{2} \right) + b_4 \left(\frac{1-(\tilde{w}_3^j)^2}{2} \right) \right) \varsigma_2^2\, t^2 + \\
&+ \sigma_P^{2m_k} {\sigma_Q}^{2n_k}\, u_{j,k}(t) \left(2\, b_2\, {\sigma_Q}^{-n_k} \left(\frac{1-\tilde{w}_3^j}{\sqrt{2}} \right) \varsigma_2\, t + b_2\, {\sigma_P}^{2m_k} u_{j,k}(t) \right) + \\
&+ \sigma_P^{2m_k} {\sigma_Q}^{2n_k}\, v_{j,k}(t) \left(2\, b_3\, {\sigma_Q}^{-n_k} \left(\frac{1+\tilde{w}_3^j}{\sqrt{2}} \right) \varsigma_2\, t + b_3\, {\sigma_P}^{2m_k} v_{j,k}(t) \right) + \\
&+ b_4\, \sigma_P^{2m_k} {\sigma_Q}^{n_k}\, \varsigma_2\, t \left(\left(\frac{1+\tilde{w}_3^j}{\sqrt{2}} \right) u_{j,k}(t) + \left(\frac{1-\tilde{w}_3^j}{\sqrt{2}} \right) v_{j,k}(t) \right) + \\
&+ b_4\, \sigma_P^{4m_k} {\sigma_Q}^{2n_k} u_{j,k}(t) v_{j,k}(t) + \sigma_P^{2m_k} {\sigma_Q}^{2n_k} H_2\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big); \\
\widetilde{z}_{j,k}(t) &= c_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k}\, (1+\tilde{x}_j) + \\
&+ \left(c_1\, {\lambda_P}^{m_k} {\sigma_P}^{-m_k} \tilde{w}_1^j + c_2 \left(\frac{1-\tilde{w}_3^j}{\sqrt{2}} \right) + c_3 \left(\frac{1+\tilde{w}_3^j}{\sqrt{2}} \right) \right) \varsigma_2\, t + \\
&+ c_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k} \tilde{\rho}^1_{j,k}(t) + c_2\, \sigma_P^{2m_k} {\sigma_Q}^{n_k} u_{j,k}(t) + c_3\, \sigma_P^{2m_k} {\sigma_Q}^{n_k} v_{j,k}(t) + \\
&+ \sigma_P^{m_k} {\sigma_Q}^{n_k} H_3\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big);
\end{aligned}$$

where

$$\hat{\mathbf{x}}_{j,k}(t) := \big(\widehat{x}_{j,k}(t), \widehat{y}_{j,k}(t), \widehat{z}_{j,k}(t)\big) = g_{j,k}^{m_k}\big(x_{i,k}(t), y_{j,k}(t), z_{j,k}(t)\big) - (0,1,1).$$

Recall this last terms in equation (8.1.20).

We are now ready to compare the coordinates of $\gamma_{j,k}(t)$ and $\bar{\gamma}_k(t)$. For that first recall that

$$\varsigma_1 = \frac{a_2+a_3}{\sqrt{2}}, \quad \varsigma_2 = \frac{b_2+b_3+b_4}{2}, \quad \varsigma_5 = \frac{c_2+c_3}{\sqrt{2}}$$

we have that

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$$\begin{aligned}
\widetilde{x}_{j,k}(t)-\bar{x}_k(t) &= a_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\,\tilde{x}_j+\left(a_1\,{\lambda_P}^{m_k}{\sigma_P}^{-m_k}\tilde{w}_1^j+\frac{(a_3-a_2)}{\sqrt{2}}\,\tilde{w}_3^j\right)\varsigma_2\,t+\\
&+a_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{\rho}^1_{j,k}(t)+a_2\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}u_{j,k}(t)+a_3\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}v_{j,k}(t)+\\
&+\sigma_P^{m_k}{\sigma_Q}^{m_k}H_1\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big)-a_1{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{H}_1\Big(\mathbf{x}_k(t)\Big)-\sigma_P^{m_k}{\sigma_Q}^{n_k}H_1\Big(\widehat{\mathbf{x}}_k(t)\Big);\\
\widetilde{y}_{j,k}(t)-\bar{y}_k(t) &= b_1{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\,\tilde{x}_j+b_1\,{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\tilde{\rho}^1_{j,k}(t)+b_1\,{\lambda_P}^{m_k}\sigma_Q^{n_k}\,\tilde{w}_1^j\,t+\\
&+\left(b_2\left(-\tilde{w}_3^j+\frac{(\tilde{w}_3^j)^2}{2}\right)+b_3\left(-\tilde{w}_3^j+\frac{(\tilde{w}_3^j)^2}{2}\right)-b_4\frac{(\tilde{w}_3^j)^2}{2}\right)\varsigma_2^2\,t^2+\\
&+\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\,u_{j,k}(t)\left(2\,b_2\,\sigma_Q^{-n_k}\left(\frac{1-\tilde{w}_3^j}{\sqrt{2}}\right)\varsigma_2\,t+b_2\,{\sigma_P}^{2m_k}u_{j,k}(t)\right)+\\
&+\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\,v_{j,k}(t)\left(2\,b_3\,\sigma_Q^{-n_k}\left(\frac{1+\tilde{w}_3^j}{\sqrt{2}}\right)\varsigma_2\,t+b_3{\sigma_P}^{2m_k}v_{j,k}(t)\right)+\\
&+b_4\sigma_P^{2m_k}{\sigma_Q}^{n_k}\,\varsigma_2\,t\left(\left(\frac{1+\tilde{w}_3^j}{\sqrt{2}}\right)u_{j,k}(t)+\left(\frac{1-\tilde{w}_3^j}{\sqrt{2}}\right)v_{j,k}(t)\right)+\\
&+b_4\sigma_P^{4m_k}{\sigma_Q}^{2n_k}u_{j,k}(t)\,v_{j,k}(t)+\sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big)-\\
&-b_1{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\tilde{H}_1\Big(\mathbf{x}_k(t)\Big)-\sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\Big(\widehat{\mathbf{x}}_k(t)\Big);\\
\widetilde{z}_{j,k}(t)-\bar{z}_k(t) &= c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\,\tilde{x}_j+\left(c_1\,{\lambda_P}^{m_k}{\sigma_P}^{-m_k}\tilde{w}_1^j+\frac{(c_3-c_2)}{\sqrt{2}}\tilde{w}_3^j\right)\varsigma_2\,t+\\
&+c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{\rho}^1_{j,k}(t)+c_2\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}u_{j,k}(t)+c_3\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}v_{j,k}(t)+\\
&+\sigma_P^{m_k}{\sigma_Q}^{n_k}H_3\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big)-c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{H}_1\Big(\mathbf{x}_k(t)\Big)-\sigma_P^{m_k}{\sigma_Q}^{n_k}H_3\Big(\widehat{\mathbf{x}}_k(t)\Big).
\end{aligned}$$

Landau symbols of the coordinates of $\widehat{\mathbf{x}}_{j,k}(t)$ in (8.1.22), imply the convergence to 0 of the following higher order terms

$$\sigma_P^{m_k}{\sigma_Q}^{n_k}H_l\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big),\quad {\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{H}_l\Big(\mathbf{x}_k(t)\Big),\quad \sigma_P^{m_k}{\sigma_Q}^{n_k}H_l\Big(\widehat{\mathbf{x}}_k(t)\Big),\quad l=1,3,$$

in the expresions $\widetilde{x}_{j,k}(t)-\bar{x}_k(t)$ and $\widetilde{z}_{j,k}(t)-\bar{z}_k(t)$.

On the other hand, the C^r-convergence to zero of the higher order terms

$$\sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\Big(\widehat{\mathbf{x}}_{j,k}(t)\Big),\quad {\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\tilde{H}_1\Big(\mathbf{x}_k(t)\Big),\quad \sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\Big(\widehat{\mathbf{x}}_k(t)\Big),$$

in $\widetilde{y}_{j,k}(t)-\bar{y}_k(t)$ were already obtained in the proof of Theorem 2 (see (5.3.8)).

Now we analyze the convergence of the associated terms $\tilde{\rho}^1_{j,k}(t)$, $u_{j,k}(t)$ and $v_{j,k}(t)$ contained in $\Big(\widetilde{x}_{j,k}(t)-\bar{x}_k(t),\widetilde{y}_{j,k}(t)-\bar{y}_k(t),\widetilde{z}_{j,k}(t)-\bar{z}_k(t)\Big)$.

The Landau symbols of $\tilde{\rho}^1_{j,k}(t)$, $u_{j,k}(t)$ and $v_{j,k}(t)$ in (8.1.21) imply that

$${\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{\rho}^1_{j,k}(t)=O({\lambda_P}^{m_k}\sigma_P^{-3m_k}{\sigma_Q}^{-n_k}),$$

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and

$$\sigma_P^{2m_k}{\sigma_Q}^{2n_k}u_{j,k}(t) = \sigma_P^{2m_k}{\sigma_Q}^{2n_k}v_{j,k}(t) = O({\sigma_P}^{-2m_k}).$$

Thus, the terms ${\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\tilde{\rho}^1_{j,k}(t)$, $\sigma_P^{2m_k}{\sigma_Q}^{2n_k}u_{j,k}(t)$ and $\sigma_P^{2m_k}{\sigma_Q}^{2n_k}v_{j,k}(t)$ are convergent to zero in the C^r-topology as $k \to +\infty$.

Therefore, it remains to study the convergence of the linear parts in $\widetilde{x}_{j,k}(t) - \bar{x}_k(t)$ and $\widetilde{z}_{j,k}(t) - \bar{z}_k(t)$, and of the quadratic part of $\widetilde{y}_{j,k}(t) - \bar{y}_k(t)$.

The linear part in $\widetilde{x}_{j,k}(t) - \bar{x}_k(t)$ is given by

$$a_1{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\,\tilde{x}_j + \left(a_1\,{\lambda_P}^{m_k}{\sigma_P}^{-m_k}\tilde{w}_1^j + \frac{(a_3-a_2)}{\sqrt{2}}\tilde{w}_3^j\right)\varsigma_2\,t. \tag{8.1.24}$$

For that recall the spectral condition in (5.3.4), $0 < ({\lambda_P}^{\frac{1}{2}}\sigma_P)^\eta\sigma_Q < 1$, with $\eta = \frac{\log\lambda_Q^{-1}}{\log\sigma_P}$. By Lemma 5.3.6 there is a constant $C > 0$ such that

$$\left({\lambda_P}^{\frac{1}{2}}\sigma_P\right)^{m_k}\sigma_Q^{n_k} < C\Big(({\lambda_P}^{\frac{1}{2}}\sigma_P)^\eta\sigma_Q)^{n_k}.$$

Thus, when $k \to +\infty$ we have that

$${\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k} = \left({\lambda_P}^{\frac{m_k}{2}}{\sigma_P}^{m_k}\sigma_Q^{n_k}\right)^2 < \left(C({\lambda_P}^{\frac{1}{2}}\sigma_P)^\eta\sigma_Q)^{n_k}\right)^2 \to 0. \tag{8.1.25}$$

This implies that the constant term in (8.1.24) goes to 0 as $k \to +\infty$.

Recalling that $\tilde{w}_1^j, \tilde{w}_3^j \to 0$ in (8.1.12), we get that factor that multiply t in (8.1.24)

$$a_1\,{\lambda_P}^{m_k}{\sigma_P}^{-m_k}\tilde{w}_1^j + \frac{(a_3-a_2)}{\sqrt{2}}\tilde{w}_3^j$$

tends to zero, when $j, k \to +\infty$. This last assertion jointly with the convergence in (8.1.25), imply that the C^1-norm (and therefore the C^r-norm) of linear part (8.1.24) tend to zero when j and k tends to infinity. Therefore, $|\widetilde{x}_{j,k}(t) - \bar{x}_k(t)|_r$ tend to zero when j and k tend to infinity. The same arguments apply to the convergence of $|\widetilde{z}_{j,k}(t) - \bar{z}_k(t)|_r$.

On the other hand, the quadratic part of $\widetilde{y}_{j,k}(t) - \bar{y}_k(t)$ is given by

$$\begin{aligned} &b_1\,{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}(\tilde{x}_j - 1) + b_1\,{\lambda_P}^{m_k}\sigma_Q^{n_k}\,\tilde{w}_1^j\,t + \\ +\quad &\left(b_2\left(-\tilde{w}_3^j + \frac{(\tilde{w}_3^j)^2}{2}\right) + b_3\left(-\tilde{w}_3^j + \frac{(\tilde{w}_3^j)^2}{2}\right) - b_4\,\frac{(\tilde{w}_3^j)^2}{2}\right)\varsigma_2^2\,t^2 \end{aligned} \tag{8.1.26}$$

The convergence $\tilde{w}_3^j \to 0$, imply that the C^2-norm (and therefore the C^r-norm) of the expression in (8.1.26) tend to zero when j and k goes to infinity. This implies that $|\widetilde{y}(t) - y(t)|_r$ tends to zero when j and k tend to infinity. This completes the proof of Proposition 8.1.4. ■

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8.2
Two-dimensional connections

The main result in this section is the next proposition that provides the two-dimensional connection between the unstable manifold of blender $\Lambda_{i,j,\ell,k}$ and stable manifold of the saddle Q to the diffeomorphism $g_{i,j,\ell,k}$ in (8.0.10). More precisely, we have the following proposition.

Proposition 8.2.1 *For every i, j, ℓ and k we have the following transverse intersection*

$$W^{\mathrm{u}}\Big(\Lambda_{i,j,\ell,k}, g_{i,j,\ell,k}\Big) \pitchfork W^{\mathrm{s}}\Big(Q, g_{i,j,\ell,k}\Big) \neq \emptyset.$$

The results of this section do not depend on the subscript i, j, ℓ thus, by simplicity, the will be omitted and we will write g_k and Λ_k in the places of $g_{i,j,\ell,k}$ and $\Lambda_{i,j,\ell,k}$.

Before going to the details of the proof of the proposition let us explain briefly its main steps. Recall the renormalised sequence $\mathcal{R}_{m_k,n_k}(g_k)$ in Section 5.3.6. Recall that

$$G_k : \mathbb{R}^3 \to \mathbb{R}^3, \quad G_k(X) := \Phi_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(g_k) \circ \Phi^{-1}_{m_k,n_k}(X)$$

is a sequence of diffeomorphisms defined in whole $\mathbb{R}^3$ that converges in the C^r-topology to the endomorphism $G_{(\xi,\mu,0,0)} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ on compact sets. For k large sufficient we denote by $\widetilde{\Lambda}_k := \Phi^{-1}_{m_k,n_k}(\Lambda_k)$ the respective blender of G_k.

Consider the left reference $P^+_{\xi,\mu} = (x^+_{\xi,\mu}, y^+_{\xi,\mu}, z^+_{\xi,\mu})$ of blender of $G_{(\xi,\mu,0,0)}|_\Delta$. For large k we denote by $P^+_{k,\xi,\mu}$ the continuation of $P^+_{\xi,\mu}$ for G_k.

8.2.1
Strategy of the proof of proposition

Note first that the domain of definition $\Delta_k := \Phi_{m_k,n_k}(\Delta)$, see (8.1.3), of the blender of g_k converges exponentially to $\widetilde{Y} \in W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_k)$ as $k \to +\infty$. Thus, any (fixed) small neighbourhood $W^{\mathrm{s}} \subset W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_k)$ of $\widetilde{Y}$ measured in the $\Phi^{-1}_{m_k,n_k}$-charts, it moves exponentially fast away from $\Delta \subset \mathbb{R}^3$ as $k \to +\infty$. This is illustrated in Figure 8.2. With this in mind, we define a suitable increasing sequence of domains $\widehat{\Delta}_k \subset \mathbb{R}^3$, $\Delta \subset \widehat{\Delta}_k \subset \widehat{\Delta}_{k+1}$, such that $W^{\mathrm{u}}\Big(P^+_{k,\xi,\mu}, G_k|_{\widehat{\Delta}_k}\Big)$ meets transversely $\Phi^{-1}_{m_k,n_k}(W^{\mathrm{s}})$. This intersection is obtained if we guarantee the follwing two facts:

(1) $\|(G_{(\xi,\mu,0,0)} - G_k)|_{\widehat{\Delta}_k}\|_r \to 0$, when $k \to +\infty$, and

(2) For every $k \geq 1$ large enough it holds

$$W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}|_{\widehat{\Delta}_k}) \pitchfork \Phi^{-1}_{m_k,n_k}(W^{\mathrm{s}}) \neq \emptyset,$$

The first condition above implies that for every k large enough, the manifolds $W^{\mathrm{u}}(P^+_{k,\xi,\mu}, G_k|_{\widehat{\Delta}_k})$ and $W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}|_{\widehat{\Delta}_k})$ are C^r-close. The second condition imply that $W^{\mathrm{u}}(P^+_{k,\xi,\mu}, G_k|_{\widehat{\Delta}_k})$ and $\Phi^{-1}_{m_k,n_k}(W^{\mathrm{s}})$ meet transversely. As consequence we get that

$$W^{\mathrm{u}}(\widetilde{P}^+_{k,\xi,\mu}, g_k) \pitchfork W^{\mathrm{s}} \neq \emptyset,$$

where $\widetilde{P}^+_{k,\xi,\mu} = \Phi_{m_k,n_k}(P^+_{k,\xi,\mu})$ denotes the reference of the blender of $\mathcal{R}_{m_k,n_k}(g_k)$ in Δ_k.

We now we provide the precise proofs of the steps above.

8.2.1.1
The unstable manifold of the fixed saddles of the Hénon-like family

In this sections we study the growth (along of the unstable direction) of the unstable manifold of the saddle $P^+_{\xi,\mu,\kappa}$ of $G_{(\xi,\mu,\kappa,0)}$ for parameters $(\xi,\mu,\eta,0) \in (1.18, 1.19) \times (-10,-9) \times (-\epsilon,\epsilon)$ as in Theorem 1. Figure 8.2 illustrates this growth.

Recall that from Lemma 5.1.1, for every $(\xi,\mu) \in \mathcal{P} = (1.18, 1.19) \times (-10,-9)$ the left reference $P^+_{\xi,\mu} = (x^+_\mu, y^+_\mu, z^+_{\xi,\mu})$ of the blender-horseshoe of $G_{(\xi,\mu,0,0)}|_\Delta$ is a partially hyperbolic point with two unstable directions (a *strong* one and a *weak* one) satisfying the relations

$$x^+_\mu = y^+_\mu = \mu + (y^+_\mu)^2 = (1-\xi)\, z^-_{\xi,\mu}, \quad y^+_\mu > 0. \tag{8.2.1}$$

Lemma 8.2.2 *Let $\Delta^+ = [-4,4] \times [0,4] \times [-40,22] \subset \mathbb{R}^3$. Then, $G_{(\xi,\mu,0,0)}(\Delta^+) \cap \Delta \subset W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)})$.*

Proof. Consider the projection $\Pi_{12}(x,y,z) = (x,y)$. Recall the nested sequence of discs in (6.6.3):

$$\ell^+_{\mu,0} = \Pi_{12}\Big(G_{(\xi,\mu,0,0)}(\Delta^+) \cap \Delta\Big), \quad (\ell^+_{\mu,n})_n \subset \Pi_{12}\Big(G_{(\xi,\mu,0,0)}(\Delta^+) \cap \Delta\Big),$$

satisfying for every $n \in \mathbb{N}$:

$$(x^+_\mu, y^+_\mu) \in \ell^+_{\mu,n}, \quad g_\mu(\ell^+_{\mu,n+1}) = \ell^+_{\mu,n}, \quad g_\mu(x,y) = (y, \mu + y^2).$$

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For every n consider the two-disc :

$$\widetilde{\Gamma}_{\mu,n} := \left\{(x, y, z^+_\mu + t) : (x, y) \in \ell^+_{\mu,n},\ \xi^{-n}(-40 - z^+_\mu) \leq t \leq \xi^{-n}(22 - z^+_\mu)\right\}.$$

Note that $\widetilde{\Gamma}_{\mu,0} = \ell^+_{\mu,0} \times [-40, 22] = G_{(\xi,\mu,0,0)}(\Delta^+) \cap \Delta$.

By construction, the sequence $(\widetilde{\Gamma}_{\mu,n})_n$ satisfy

$$P^+_{\xi,\mu} \in \widetilde{\Gamma}_{\mu,n}, \quad \widetilde{\Gamma}_{\mu,n+1} \subset \widetilde{\Gamma}_{\mu,n}, \quad \text{and} \quad G_{(\xi,\mu,0,0)}(\widetilde{\Gamma}_{\mu,n+1}) = \widetilde{\Gamma}_{\mu,n} \quad (n \in \mathbb{N}).$$

This completes the proof. ■

We define the local unstable manifold of the saddle $P^+_{\xi,\mu}$ of $G_{(\xi,\mu,0,0)}$ as

$$W^{\mathrm{u}}_{\mathrm{loc}}\Big(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}\Big) = G_{(\xi,\mu,0,0)}(\Delta^+) \cap \Delta. \tag{8.2.2}$$

Remark 8.2.3 The continuity of the local unstable manifold implies that for every small $\epsilon > 0$ (fixed) and every $(\xi, \mu, \kappa, \eta) \in (1.18, 1.19) \times (-10, -9) \times (-\epsilon, \epsilon)^2$ it holds that

$$W^{\mathrm{u}}_{\mathrm{loc}}\Big(P^+_{\xi,\mu,\kappa,\eta}, G_{(\xi,\mu,\kappa,\eta)}\Big) = G_{(\xi,\mu,\eta,\eta)}(\Delta^+) \cap \Delta, \tag{8.2.3}$$

where $P^+_{\xi,\mu,\kappa,\eta} = (x^+_{\xi,\mu,\kappa,\eta}, y^+_{\xi,\mu,\kappa,\eta}, z^-_{\xi,\mu,\kappa,\eta})$ it is the continuation of $P^+_{\xi,\mu}$ satisfying

$$\begin{aligned} x^+_{\xi,\mu,\kappa,\eta} &= y^+_{\xi,\mu,\kappa,\eta}, \quad y^+_{\xi,\mu,\kappa,\eta} > 0; \\ y^+_{\xi,\mu,\kappa,\eta} &= \mu + (y^+_{\xi,\mu,\kappa,\eta})^2 + \kappa\,(z^+_{\xi,\mu,\kappa,\eta})^2 + \eta\,(y^+_{\xi,\mu,\kappa,\eta})\,(z^+_{\xi,\mu,\kappa,\eta}); \\ z^-_{\xi,\mu,\kappa,\eta} &= \xi\, z^-_{\xi,\mu,\kappa,\eta} + y^+_{\xi,\mu,\kappa,\eta}. \end{aligned} \tag{8.2.4}$$

Let us investigate the size and growth of the unstable manifold $W^{\mathrm{u}}_{\mathrm{loc}}\Big(P^+_{\xi,\mu,\kappa}, G_{(\xi,\mu,\kappa,0)}\Big)$. For this, we consider $W^{\mathrm{uu}}\Big(P^+_{\xi,\mu,\kappa}, G_{(\xi,\mu,\kappa,0)}\Big)$ the strong invariant manifold of $P^+_{\xi,\mu,\kappa}$ and the following subsets of this

$$W^{\mathrm{uu},+}\Big(P^+_{\xi,\mu,\kappa}, G_{(\xi,\mu,\kappa,0)}\Big) = W^{\mathrm{uu}}\Big(P^+_{\xi,\mu,\kappa}, G_{(\xi,\mu,\kappa,0)}\Big) \cap \Big\{y \geq y^+_{\xi,\mu,\kappa}\Big\}$$

and

$$W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu,\kappa} = W^{\mathrm{uu},+}(P^+_{\xi,\mu,\kappa}, G_{(\xi,\mu,\kappa,0)}) \cap \Delta.$$

We write

$$W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu,\kappa} := \Big\{P^+_{\xi,\mu,\kappa} + \varphi_{\xi,\mu,\kappa}(t) : y^+_{\xi,\mu,\kappa} \leq t \leq 4\Big\},$$

where $\varphi_{\xi,\mu,\kappa}(t) := \Big(\varphi_{1,\xi,\mu,\kappa}(t), \varphi_{2,\xi,\mu,\kappa}(t), \varphi_{3,\xi,\mu,\kappa}(t)\Big)$ is the (unique) invariant curve tangent to $E^{\mathrm{uu}}(P^+_{\xi,\mu,\kappa})$ such that

$$\varphi_{i,\xi,\mu,\kappa}(y^+_{\xi,\mu,\kappa}) = 0, \ (i = 1, 2, 3) \quad \text{and}$$

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Consider the (family of) sequences of maps,

$$\alpha_{n,\xi,\mu}(t) := \Pi_2\Big(G^n_{(\xi,\mu,0,0)}\Big(P^+_{\xi,\mu} + \varphi_{\xi,\mu}(t)\Big)\Big), \quad n \geq 0.$$

Lemma 8.2.4 *For every $(\xi,\mu) \in \mathcal{P}$ and every $0 < t < \delta$ the sequence $(\alpha_n(t))_n$ is strictly increasing and satisfies $\lim\limits_{n\to+\infty} \alpha_n(t) = +\infty$.*

Proof. Let $(\xi,\mu) \in \mathcal{P}$. We claim that:

(i) For $i = 1,2$ and every small $t > 0$, it holds $\varphi_{i,\xi,\mu}(t) > 0$; and

(ii) for every small $t > 0$, it holds $\varphi_{2,\xi,\mu}(t) = 2\,y^+_\mu\,\varphi_{1,\xi,\mu}(t) + \varphi_{1,\xi,\mu}(t)^2$.

Indeed, by definition of $W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}$, we have that

$$\Big(y^+_\mu + \varphi_{1,\xi,\mu}(t), y^+_\mu + \varphi_{2,\xi,\mu}(t)\Big) \in \Big\{(x,y) : y = \mu + x^2,\ x \geq y^+_\mu,\ y \geq y^+_\mu\Big\}. \quad (8.2.5)$$

Thus, $t > 0$ imply that $\varphi_1(t) > 0$ and $\varphi_2(t) > 0$. Moreover, recalling $\mu + (y^+_\mu)^2 =, y^+_\mu$ in (5.1.2), we get

$$\begin{aligned} y^+_\mu + \varphi_{2,\xi,\mu}(t) &= \mu + (y^+_\mu + \varphi_{1,\xi,\mu}(t))^2 \\ &= \mu + (y^+_\mu)^2 + 2\,y^+_\mu\,\varphi_{1,\xi,\mu}(t) + \varphi_1(t)^2 \\ &= y^+_\mu + 2\,y^+_\mu\,\varphi_{1,\xi,\mu}(t) + \varphi_{1,\xi,\mu}(t)^2. \end{aligned} \quad (8.2.6)$$

This completes the proof of our claim.

Note that $\alpha_{0,\xi,\mu}(t) = y^+_\mu + \varphi_{1,\xi,\mu}(t)$. Is easy to see that

$$\alpha_{n,\xi,\mu}(t) = \mu + \alpha_{n-1,\xi,\mu}(t)^2. \quad (8.2.7)$$

Claim 8.2.5 *For every $(\xi,\mu) \in \mathcal{P}$ and for every small $t > 0$, it holds $\alpha_{n-1,\xi,\mu}(t) < \alpha_{n,\xi,\mu}(t)$.*

Proof. Let y^-_μ be the root of $y^2 - y + \mu$ different from y^+_μ. Then $y^-_\mu < y^+_\mu$ and

$$y^2 - y + \mu > 0 \quad \text{if, and only if,} \quad y \notin [y^-_\mu, y^+_\mu].$$

We claim that for every $(\xi,\mu) \in \mathcal{P}$, every small $t > 0$, and every $n \geq 1$ we have that $\alpha_{n,\xi,\mu}(t) > y^+_{\xi,\mu}$. To see why this is so note that $\alpha_{0,\xi,\mu}(t) > y^+_{\xi,\mu}$ for every small $t > 0$. Thus, by (8.2.7)

$$\alpha_{1,\xi,\mu}(t) = \mu + \alpha_{0,\xi,\mu}(t)^2 > \mu + (y^+_\mu)^2 = y^+_\mu.$$

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Proceeding inductively we get that for every $n > 1$ and every small $t > 0$, it holds $\alpha_{n,\xi,\mu}(t) > y^+_\mu$. Therefore,

$$0 < \alpha_{n,\xi,\mu}(t)^2 - \alpha_{n,\xi,\mu}(t) + \mu = \alpha_{n+1,\xi,\mu}(t) - \alpha_{n,\xi,\mu}(t),$$

proving the claim. ■

Easily it follows that for every $(\xi,\mu) \in \mathcal{P}$ and every small $t > 0$, the sequence $(\alpha_{n,\xi,\mu}(t))_{n\geq 1}$ is unbounded. This completes the proof of Lemma 8.2.4. ■

Remark 8.2.6 From Lemma 8.2.4 we have that

$$W^{\mathrm{uu},+}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}) = \bigcup_{n\geq 1} G^n_{(\xi,\mu,0,0)}\Big(W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}\Big)$$

and that

$$\Pi_2\Big(W^{\mathrm{uu},+}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)})\Big) = [y^+_\mu, +\infty).$$

Lemma 8.2.4 also implies that the size of $G^n_{(\xi,\mu,0,0)}\Big(W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}\Big)$ along to the positive semi y-axis only depends on the y-coordinate of $G^{n-1}_{(\xi,\mu,0,0)}\Big(W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}\Big)$ and the size $|\Pi_2\Big(G^n_{(\xi,\mu,0,0)}\Big(W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}\Big)\Big)|$ is approximately $|\Pi_2\Big(G^{n-1}_{(\xi,\mu,0,0)}\Big(W^{\mathrm{uu},+}_{\mathrm{loc},\xi,\mu}\Big)\Big)|^2$. We observe also that the endomorphism $G_{(\xi,\mu,0,0)}$ collapses the x-directions (this direction is a eigenspace of $DG_{(\xi,\mu,0,0)}$ with eigenvalue equal to zero, see Lemma 5.1.1).

Let

$$W^{\mathrm{u},+}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}) := W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}) \cap \Big\{y \geq y^+_\mu\Big\} \tag{8.2.8}$$

and consider the projection

$$\Pi_{23}(x,y,z) = (y,z).$$

Claim 8.2.7 *For every $(\xi,\mu) \in \mathcal{P}$ it holds that*

$$\begin{aligned} W^{\mathrm{u},+}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}) &= G_{(\xi,\mu,0,0)}\Big(\mathbb{R} \times [y^+_\mu, +\infty) \times \mathbb{R}\Big) \\ &= G_{(\xi,\mu,0,0)}\Big(\{y^+_\mu\} \times [y^+_\mu, +\infty) \times \mathbb{R}\Big). \end{aligned}$$

In particular,

$$\Pi_{23}\Big(W^{\mathrm{u},+}(P^+_{\xi,\mu}, \widetilde{G}_{(\xi,\mu,0,0)})\Big) = [y^+_\mu, +\infty) \times \mathbb{R}.$$

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Proof. Consider the plane $\mathfrak{L}_{\xi,\mu}$ parallel to the xz-plane through of $P^+_{\xi,\mu}$:

$$\mathfrak{L}_{\xi,\mu} = \mathbb{R} \times \Big\{(y^+_\mu, z^+_{\xi,\mu} + t) : t \in \mathbb{R}\Big\}.$$

Then, the definition of $G_{(\xi,\mu,0,0)}$ imply that

$$G_{(\xi,\mu,0,0)}\Big(\mathfrak{L}_{\xi,\mu}\Big) = \Big\{(y^+_\mu, y^+_\mu, z^+_{\xi,\mu} + t) : t \in \mathbb{R}\Big\}.$$

The claim now follows from Remark 8.2.6. ■

8.2.2
Relative positions around of heterodimensional tangency.

We study the relative position of the blender-horseshoe associated to g_k as well the positions of the invariant manifolds (after of the unfolding) around of $\widetilde{Y}$ using the coordinates $\Phi_{m_k,n_k} : \mathbb{R}^3 \to U_Q$. Recall that

$$\Phi_{m_k,n_k}(x,y,z) = \Psi_{m_k,n_k} \circ \Theta \circ \widetilde{\Theta}(x,y,z),$$

where

$$\begin{aligned}
\Psi_{m_k,n_k}(x,y,z) &= \Big(1 + \sigma_Q^{-n_k} \sigma_P{}^{-m_k}\, x, \sigma_Q{}^{-n_k} + \sigma_P^{-2n_k} \sigma_P{}^{-2m_k}\, y, 1 + \sigma_Q^{-n_k}\, \sigma_P{}^{-m_k}\, z\Big);\\
\Theta(\mu,x,y,z) &= \varsigma_2^{-1}\,(\mu, \varsigma_1\, x, y, \varsigma_5\, z);\\
\widetilde{\Theta}(x,y,z) &= (z,y,x).
\end{aligned}$$

We observe that when the heterodimensional tangencies of the cycle of $f \in \mathcal{N}^r_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$ are of type elliptic, the condition $\varsigma_2 < 0$ (see Remark 7.0.3) implies, that the blender-horseshoe Λ_k associated to g_k is “encapsulated” by the two-dimensional invariant manifolds of the saddles P and Q. Thus, in this case we seek an intersection as illustrated in Figure 8.1.

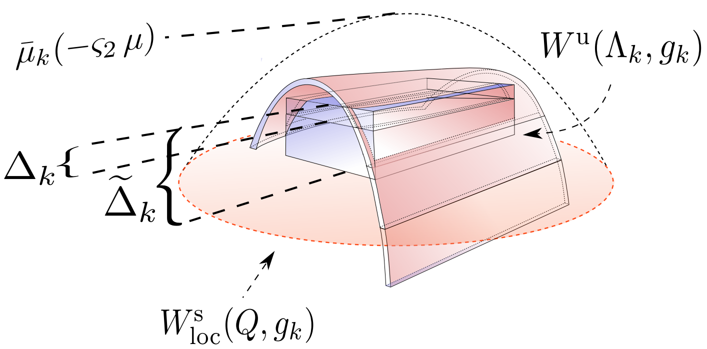

Figure 8.1: “Encapsulated” blender-horseshoes.

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Fix small $\delta > 0$ and consider the δ-ball $W^{\mathrm{s}}_{\delta}(\widetilde{Y})$ of $\widetilde{Y} = (1,0,1)$ in $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_k)$ given by

$$W^{\mathrm{s}}_{\delta}(\widetilde{Y}) = \Big\{(1+x, 0, 1+z) : |x|, |z| \leq \delta\Big\} \subset U_Q.$$

Thus

$$\Phi^{-1}_{m_k,n_k}\Big(W^{\mathrm{s}}_{\delta}(\widetilde{Y})\Big) = I^1_{k,\delta} \times \Big\{ -\sigma_Q^{n_k} {\sigma_P}^{2m_k} \varsigma_2\Big\} \times I^3_{k,\delta}, \tag{8.2.9}$$

where

$$I^1_{k,\delta} = \left[-\frac{|\varsigma_2|}{|\varsigma_1|}\,\delta\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}, \frac{|\varsigma_2|}{|\varsigma_1|}\,\delta\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}\right], \quad I^3_{k,\delta} = \left[-\frac{|\varsigma_2|}{|\varsigma_5|}\,\delta\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}, \frac{|\varsigma_2|}{|\varsigma_5|}\,\delta\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}\right].$$

Recall also the sequence of (global) diffeomorphisms $G_k : \mathbb{R}^3 \to \mathbb{R}^3$

$$G_k(X) := \Phi_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(g_k) \circ \Phi^{-1}_{m_k,n_k}(X), \quad X \in \mathbb{R}^3.$$

converging to $G_{(\xi,\mu,0,0)}$ in the C^r-topology on compact sets (see Remark 7.0.3).

We now construct the sequence of domains of blenders $\widehat{\Delta}_k \subset \mathbb{R}^3$ mentioned in Section 8.2.1. For that consider the sequence $a_k = \frac{1}{n_k}$. Note that

$$\lim_{k\to+\infty} a_k = 0, \quad \text{and} \quad \lim_{k\to+\infty} a_k^2\,\sigma_Q^{n_k} = +\infty. \tag{8.2.10}$$

Consider the sequence of domains $\widehat{\Delta}_k$ in $\mathbb{R}^3$ given by

$$\widehat{\Delta}_k := [-4,4] \times [-4, a_k\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}] \times [-40, 22]. \tag{8.2.11}$$

Recall that $\Delta = [-4,4]^2 \times [-40,22] \subset \mathbb{R}^3$, see (8.1.2) Hence $\Delta \subset \widehat{\Delta}_k$ for sufficiently large k. Let

$$\widehat{\Delta}_{k,\mu} := \widehat{\Delta}_k \cap \Big\{y \geq y_\mu^+\Big\}.$$

Then, from Claim 8.2.7 we have that

$$G_{(\xi,\mu,\kappa,0)}\Big(\widehat{\Delta}_{k,\mu}\Big) \subset W^{\mathrm{u}}\Big(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}\Big). \tag{8.2.12}$$

Lemma 8.2.8 $G_{(\xi,\mu,0,0)}\Big(\widehat{\Delta}_{k,\mu}\Big) \pitchfork \Phi^{-1}_{m_k,n_k}\Big(W^{\mathrm{s}}_{\delta}(\widetilde{Y})\Big) \neq \emptyset$, *for every large $k \geq 1$.*

Proof. Note that

$$\begin{aligned}
\Pi_1\Big(G_{(\xi,\mu,\kappa,0)}\Big(\widehat{\Delta}_{k,\mu}\Big)\Big) &= [y_\mu^+, a_k\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}],\\
\Pi_2\Big(G_{(\xi,\mu,\kappa,0)}\Big(\widehat{\Delta}_{k,\mu}\Big)\Big) &= [y_\mu^+, \mu + a_k^2\,\sigma_Q^{2n_k}\,{\sigma_P}^{2m_k}],\\
\Pi_3\Big(G_{(\xi,\mu,\kappa,0)}\Big(\widehat{\Delta}_{k,y_\mu^+}\Big)\Big) &= [-40\xi - 4, a_k\,\sigma_Q^{n_k}\,{\sigma_P}^{m_k}].
\end{aligned} \tag{8.2.13}$$

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The properties of $(a_k)_k$ in (8.2.10) imply that for every big sufficient $k \geq 1$ it holds

$$\begin{aligned} a_k\, \sigma_Q^{n_k}\, {\sigma_P}^{m_k} &< \frac{|\varsigma_2|}{|\varsigma_1|}\, \delta\, \sigma_Q^{n_k}\, {\sigma_P}^{m_k}, \\ -\varsigma_2 \sigma_Q^{n_k}\, {\sigma_P}^{2m_k} &< \mu + a_k^2\, \sigma_Q^{2n_k}\, {\sigma_P}^{2m_k}, \\ a_k\, \sigma_Q^{n_k}\, {\sigma_P}^{m_k} &< \frac{|\varsigma_2|}{|\varsigma_5|}\, \delta\, \sigma_Q^{n_k}\, {\sigma_P}^{m_k}. \end{aligned} \tag{8.2.14}$$

Bearing in mind (8.2.9), from (8.2.14) we get that for every k large enough it holds

$$G_{(\xi,\mu,0,0)}\Big(\widehat{\Delta}_{k,\mu}\Big) \pitchfork \Phi_{m_k,n_k}^{-1}\Big(W_\delta^{\mathrm{s}}(\widetilde{Y})\Big) \neq \emptyset.$$

This completes the proof of the lemma. ■

Remark 8.2.9 The transverse intersection $G_{(\xi,\mu,0,0)}\Big(\widehat{\Delta}_{k,\mu}\Big) \pitchfork \Phi_{m_k,n_k}^{-1}\Big(W_\delta^{\mathrm{s}}(\widetilde{Y})\Big)$ is given by the following straight line segment:

$$L_k := \Big\{\Big(y_k, -\varsigma_2 \sigma_Q^{n_k}\, {\sigma_P}^{2m_k}, \xi\, z + y_k\Big) : y_k := \sqrt{-\varsigma_2 \sigma_Q^{n_k}\, {\sigma_P}^{2m_k} - \mu},\ -40 \leq z \leq 22\Big\}.$$

Let

$$\widetilde{\Delta}_k := \Phi_{m_k,n_k}\Big(\widehat{\Delta}_k\Big) \subset U_Q. \tag{8.2.15}$$

We now to check that the renormalised sequence $\mathcal{R}_{m_k,n_k}(g_k)|_{\widetilde{\Delta}_k}$ is well defined and satisfies $G_k|_{\widehat{\Delta}_k} = \Phi_{m_k,n_k} \circ \mathcal{R}_{m_k,n_k}(g_k) \circ \Phi_{m_k,n_k}^{-1}|_{\widehat{\Delta}_k}$

Lemma 8.2.10 *The renormalised sequence $\mathcal{R}_{m_k,n_k}(g_k)|_{\widetilde{\Delta}_k}$ is well defined.*

Proof. We need to check that the following convergences hold:

$$\widetilde{\Delta}_k \to \widetilde{Y}, \quad g_k^{n_k}\Big(\widetilde{\Delta}_k\Big) \to X, \quad g_k^{m_k+N_1+n_k}\Big(\widetilde{\Delta}_k\Big) \to Y, \quad k \to +\infty.$$

Thus the return map $\mathcal{R}_{m_k,n_k}(g_k)|_{\widetilde{\Delta}_k} := g_k^{N_2+m_k+N_1+n_k}|_{\widetilde{\Delta}_k}$ is well defined.

We will estimate the Landau symbols of

$$\widetilde{\Delta}_k - \widetilde{Y}, \quad g_k^{n_k}\Big(\widetilde{\Delta}_k\Big) - X, \quad g_k^{m_k+N_1+n_k}\Big(\widetilde{\Delta}_k\Big) - Y,$$

as in Theorem 2. Recalling the definition of Φ_{m_k,n_k} in (8.1.1), we have that the coordinates (x_k, y_k, z_k) of the points in $\widehat{\Delta}_k - \widetilde{Y}$ satisfy

$$x_k = z_k = O(\sigma_Q^{-n_k}\ {\sigma_P}^{-m_k}) \quad \text{and} \quad y_k = O(a_k\, \sigma_Q^{-n_k}\ {\sigma_P}^{-m_k}).$$

Thus, $\widetilde{\Delta}_k \to \widetilde{Y}$ as $k \to +\infty$.

Similarly, the coordinates $(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$ of the points in $g^{n_k}\Big(\widetilde{\Delta}_k\Big) - X$ satisfy

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$$\tilde{x}_k = \tilde{z}_k = O(\lambda_Q^{n_k}) \quad \text{and} \quad \tilde{y}_k = O(a_k\, \sigma_P{}^{-m_k}). \tag{8.2.16}$$

therefore $g^{n_k}\big(\widetilde{\Delta}_k\big) \to X$.

Finally, recalling the definitions of $g_k^{N_1}$ and $g_k^{m_k}$ in (8.0.10) we get that the the coordinates of the points $(\hat{x}_k, \hat{y}_k, \hat{z}_k)$ of $g_k^{m_k+N_1+n_k}\big(\widetilde{\Delta}_k\big) - X$ satisfy

$$\hat{x}_k = O(\lambda_P{}^{m_k}) \quad \text{and} \quad \hat{y}_k = \hat{z}_k = O(a_k). \tag{8.2.17}$$

therefore $g_k^{m_k+N_1+n_k}\big(\widetilde{\Delta}_k\big) \to X$. This completes the proof of the lemma. ■

Recalling the coordinates in the equation (5.3.37) we have that the sequence of diffeomorphisms $G_k : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\begin{aligned} G_k(x,y,z) = \Big(& - c_1\, \lambda_P^{m_k}\, \lambda_Q^{n_k}\, x + y + \mathrm{h.o.t}_k^1(x,y,z), \\ & \mu - b_1\, \lambda_P^{m_k}\, \sigma_P^{m_k}\, \lambda_Q^{n_k}\, \sigma_Q^{n_k}\, \varsigma_5\, x + y^2 + \mathrm{h.o.t}_k^2(x,y,z), \\ & - a_1\, \lambda_P^{m_k}\, \lambda_Q^{n_k}\, \varsigma_5\, \varsigma_1^{-1}\, x + y + \sigma_P^{m_k}\, \lambda_Q^{n_k} \left(\frac{a_3 - a_2}{2}\right) z + \mathrm{h.o.t}_k^3(x,y,z)\Big) \end{aligned}$$

where

$$\begin{aligned} \mathrm{h.o.t}_k^1(x,y,z) &:= \varsigma_2\, \varsigma_5^{-1} c_1\, \lambda_P^{m_k}\, \sigma_P^{m_k}\, \sigma_Q^{n_k}\, \widetilde{H}_1(\mathbf{x}_k^*) + \varsigma_2\, \varsigma_5^{-1}\, \sigma_P^{m_k}\, \sigma_Q^{n_k}\, H_3(\hat{\mathbf{x}}_k^*), \\ \mathrm{h.o.t}_k^2(x,y,z) &:= \varsigma_2\, b_1\, \lambda_P^{m_k}\, \sigma_P^{2m_k}\, \sigma_Q{}^{2n_k}\, \widetilde{H}_1(\mathbf{x}_k^*) + \varsigma_2\, \sigma_P^{2m_k}\, \sigma_Q{}^{2n_k}\, H_2(\hat{\mathbf{x}}_k^*), \\ \mathrm{h.o.t}_k^3(x,y,z) &:= \varsigma_2\, \varsigma_1^{-1} a_1\, \lambda_P{}^{m_k}\, \sigma_P^{m_k}\, \sigma_Q{}^{n_k}\, \widetilde{H}_1(\mathbf{x}_k^*) + \varsigma_2\, \varsigma_1^{-1}\, \sigma_P^{m_k}\, \sigma_Q{}^{n_k}\, H_1(\hat{\mathbf{x}}_k^*), \end{aligned}$$

and

$$\mathbf{x}_k^*(x,y,z) := \mathbf{x}_k \circ \Theta \circ \widetilde{\Theta}(x,y,z), \quad \hat{\mathbf{x}}_k^*(x,y,z) := \hat{\mathbf{x}}_k \circ \Theta \circ \widetilde{\Theta}(x,y,z),$$

where $\mathbf{x}_k = \mathbf{x}_k(x,y,z)$ and $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k(x,y,z)$ are defined as in (5.3.28) and (5.3.35), respectively.

Finally we estimate the C^r-distance between $G_{(\xi,\mu,0,0)}$ and G_k in the set $\widehat{\Delta}_k$.

Lemma 8.2.11 $\lim_k \|\big(G_{(\xi,\mu,0,0)} - G_k\big)|_{\widehat{\Delta}_k}\|_r = 0$.

Proof. By the estimates of G_k above and the definition of $\widetilde{G}_{(\xi,\mu,0,0)}$ we have that

$$\begin{aligned} G_k(x,y,z) - G_{(\xi,\mu,0,0)}(x,y,z) = \Big(& - c_1\, \lambda_P^{m_k}\, \lambda_Q{}^{n_k}\, x + \mathrm{h.o.t}_k^1(x,y,z), \\ & - b_1\, \lambda_P^{m_k}\, \sigma_P{}^{m_k}\, \lambda_Q^{n_k}\, \sigma_Q{}^{n_k}\, \varsigma_5\; x + \mathrm{h.o.t}_k^2(x,y,z), \\ & - a_1\, \lambda_P^{m_k}\, \lambda_Q{}^{n_k}\, \varsigma_5\, \varsigma_1^{-1}\, x + \Big(\sigma_P{}^{m_k}\, \lambda_Q^{n_k} \left(\tfrac{a_3 - a_2}{2}\right) - \xi\Big)\, z + \\ & + \mathrm{h.o.t}_k^3(x,y,z)\Big). \end{aligned}$$

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Recall that by equation (5.3.18)

$$\sigma_P{}^{m_k}\,\lambda_Q^{n_k}\left(\tfrac{a_3-a_2}{2}\right)\to\xi.$$

As the first and third coordinates of the points in the $\widehat{\Delta}_k$ are bounded, we only we need to study the convergence to zero of the higher order terms $\mathrm{h.o.t}_k^i|_{\widehat{\Delta}_k}$, for $i=1,2,3$. This covergence is covered in the next claim.

Claim 8.2.12

(1) $\lim_k \widetilde{H}_1(\boldsymbol{x}_k^*(x,y,z))|_{\widehat{\Delta}_k}=0.$

(2) $\lim_k \sigma_P^{m_k}\,\sigma_Q^{n_k}\,H_i(\hat{\boldsymbol{x}}_k^*(x,y,z))|_{\widehat{\Delta}_k}=0,\quad i=1,3$

(3) $\lim_k \sigma_P^{2m_k}\,\sigma_Q^{2n_k}\,H_2(\hat{\boldsymbol{x}}_k^*(x,y,z))|_{\widehat{\Delta}_k}=0.$

Proof. We observe that the Landau symbols of the coordinates of the vector $\mathbf{x}_k^*(x,y,z)|_{\widehat{\Delta}_k}$ are

$$O(\lambda_Q^{n_k}),\quad O(a_k\,\sigma_P{}^{-m_k}),\quad \text{and}\quad O(\lambda_Q^{n_k}),$$

respectively. This imply that $\widetilde{H}_1(\mathbf{x}_k^*(x,y,z))|_{\widehat{\Delta}_k}\to 0$.

Since $\lambda_P{}^{m_k}\,\sigma_P^{m_k}\,\sigma_Q{}^{n_k}\to 0$, as $k\to+\infty$ (see (5.3.43)) it holds

$$\lambda_P{}^{m_k}\,\sigma_P^{m_k}\,\sigma_Q{}^{n_k}\,\widetilde{H}_1(\mathbf{x}_k^*)|_{\widehat{\Delta}_k}\to 0.$$

In similar way, we observe that the Landau symbols of the coordinates of the vector $\hat{\mathbf{x}}_k^*(x,y,z)|_{\widehat{\Delta}_k}$ are

$$O(\lambda_P^{m_k}),\quad O(a_k),\quad \text{and}\qquad O(a_k),$$

respectively. Since $\lim_k \lambda_P^{m_k}\,\sigma_P^{2m_k}\sigma_Q{}^{2n_k}=0$ (see (8.1.25)) the Taylor's expansion of $H_3(\cdot)$ (around of $\mathbf{0}$) and the conditions (7.0.6) implies that

$$\sigma_P^{2m_k}\,\sigma_Q{}^{2n_k}\,H_i(\hat{\mathbf{x}}_k^*)|_{\widehat{\Delta}_k}\to 0,\quad (i=1,2,3).$$

Thus, $\mathrm{h.o.t}_k^1|_{\widehat{\Delta}_k}, \mathrm{h.o.t}_k^2|_{\widehat{\Delta}_k}, \mathrm{h.o.t}_k^3|_{\widehat{\Delta}_k}\to 0$ as $k\to+\infty$ in the C^r-topology. This completes the claim. ■

Thus we get that $\|\big(G_{(\xi,\mu,0,0)}-G_k\big)|_{\widehat{\Delta}_k}\|_r\to 0$ completing the proof of the lemma. ■

Lemma 8.2.13 *For each $(\xi,\mu)\in\mathcal{P}$ consider the left reference saddle $P^+_{k,\xi,\mu}$ associated to blender of G_k (the continuation of $P^+_{\xi,\mu}$). Then, for every k large enough the unstable manifold $W^{\mathrm{u}}(P^+_{k,\xi,\mu},G_k|_{\widehat{\Delta}_k})$ meets transversely to $\Phi^{-1}_{m_k,n_k}\Big(W^{\mathrm{s}}_\delta(\widetilde{Y})\Big)$.*

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Proof. From (8.2.12) and Lemma 8.2.8, for $\widehat{\Delta}_{k,\mu} \subset \widehat{\Delta}_k$ it holds and

$$G_{(\xi,\mu,0,0)}(\widehat{\Delta}_{k,\mu}) \subset W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}|_{\widehat{\Delta}_k})$$

and

$$G_{(\xi,\mu,0,0)}\left(\widehat{\Delta}_{k,\mu}\right) \pitchfork \Phi^{-1}_{m_k,n_k}\left(W^{\mathrm{s}}_{\delta}(\widetilde{Y})\right) \neq \emptyset.$$

Lemma 8.2.11 implies that $W^{\mathrm{u}}(P^+_{\xi,\mu}, G_{(\xi,\mu,0,0)}|_{\widehat{\Delta}_k})$ and $W^{\mathrm{u}}(P^+_{k,\xi,\mu}, G_k|_{\widehat{\Delta}_k})$ are C^r-close. Therefore, for every $k \geq 1$ large enough we have that

$$W^{\mathrm{u}}(P^+_{k,\xi,\mu}, G_k|_{\widehat{\Delta}_k}) \pitchfork \Phi^{-1}_{m_k,n_k}\left(W^{\mathrm{s}}_{\delta}(\widetilde{Y})\right) \neq \emptyset.$$

This ends the proof of the lemma. ■

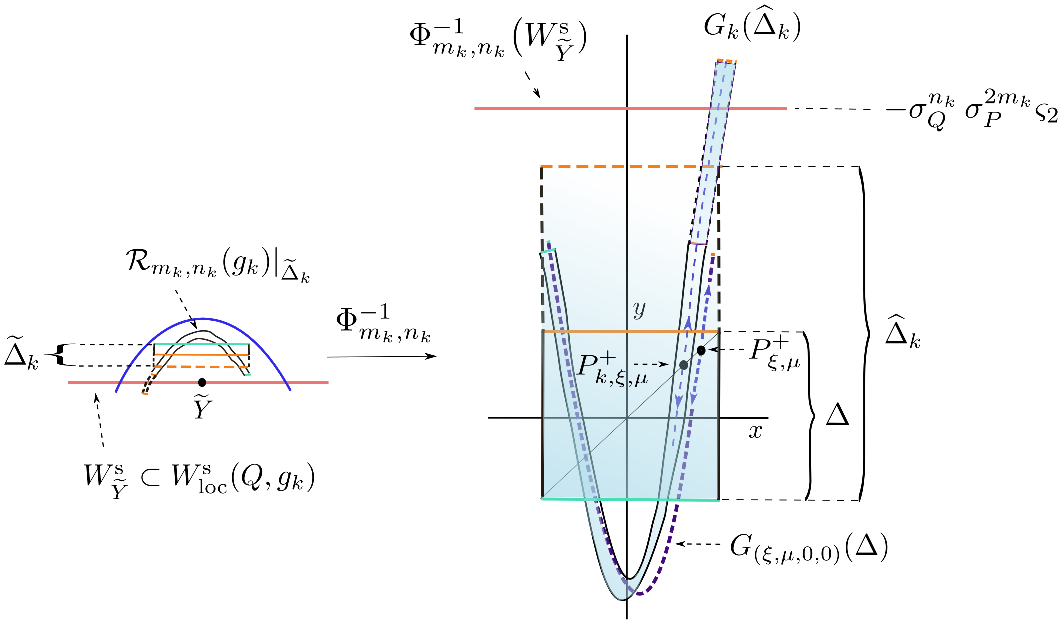

Figure 8.2: Growth of the unstable manifolds of blender along of the uu-direction.

8.3 Generation of non-dominated homoclinic classes

In this section we see that blender $\Lambda_{i,j,\ell,k}$ and the saddle P of the diffeomorphism $g_{i,j,\ell,k}$ in (8.0.10) are homoclinically related. We prove also that $Q \in H(P, g_{i,j,\ell,k})$ in a C^r-robust way. In particular, this implies that the set $H(P, g_{i,j,\ell,k})$ does not admit any dominated splitting C^r-robustly, see Remark 3.2.3. Let us now to claim in precise form our statement. The next

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results do not depend on the subscript i, j thus, by simplicity, the will be omitted and we will write $g_{\ell,k}$ and $\Lambda_{\ell,k}$ in the places of $g_{i,j,\ell,k}$ and $\Lambda_{i,j,\ell,k}$.

Recall the sets $\mathcal{P}^{\mathrm{s}}_f$ and $\mathcal{P}^{\mathrm{u}}_f$ in (7.0.3) and (7.0.4) (associated to a inicial diffeomorphism f in $\mathcal{N}^r_{P,Q}(\mathcal{T}^*_{\bar{b}_0})$) and consider the continuations $\mathcal{P}^{\mathrm{s}}_{\ell,k}$ and $\mathcal{P}^{\mathrm{u}}_{\ell,k}$ associated to the diffeomorphism $g_{\ell,k}$. Recall that our bifurcation setting implies that for every big k it holds

$$\tau_{\ell,k} := \mathcal{P}^{\mathrm{u}}_{\ell,k} \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k}) \neq \emptyset. \tag{8.3.1}$$

In the case in that the cycle of f has a heterodimensional tangency of type elliptic or hyperbolic, then the intersection (8.3.1) consist of a closed curve or two curves with boundary, respectively. More explicitly, for every large k and up to a small C^r-error, we have the following two possibilities for the curve $\tau_{\ell,k}$:

(1) $\tau_{\ell,k}$ *is a ellipse.* This is the case when the heterodimensional tangency $\widetilde{Y}$ is of elliptic type see Section 6.1.

(2) $\tau_{\ell,k}$ *consist of two curves with boundary.* Here we distinguish two cases.

- Two symmetrical curves contained in a hyperbola whose center is close to $\widetilde{Y}$. This is the case in that $\widetilde{Y}$ is of hyperbolic type, see Section 6.1.
- Two segments of parallel straight lines. This is the case in that $\widetilde{Y}$ is a degenerated tangency.

Lemma 8.3.1 *For every large ℓ and k, the radial projection of the curves $\tau_{\ell,k}$ on $\mathbb{S}^1_Q \subset W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ contains intervals.*

Proof. It is clear that if $\tau_{\ell,k}$ is a closed curve or consists of two segments contained in a hyperbola, the radial projection of $\tau_{\ell,k}$ contains intervals, independently of the relative position of $\tau_{\ell,k}$ around of $\widetilde{Y}$. If $\tau_{\ell,k}$ consists of two segments of parallel straight lines, we observe that if the radial projection of one of the segments is a point then the projection of the second segment is a interval. This completes the proof of claim. ∎

Lemma 8.3.2 *For every big ℓ and k, the homoclinic class $H(P, g_{\ell,k})$ is non-trivial.*

Proof.

From Lemma 8.3.1 the set $\mathcal{P}^{\mathrm{u}}_{\ell,k} \subset W^{\mathrm{u}}(P, g_{\ell,k})$ contains two-discs with positive radial projection on $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$. Recall that the argument of non-real eigenvalue of $Dg_{k,\ell}(Q)$ is a irrational number (see Remark (8.0.6)) and

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recall the quasi-transverse heteroclinic orbit

$$\mathcal{O}_{g_{\ell,k}}(Z_\ell^*) \subset W^{\mathrm{s}}(P, g_{\ell,k}) \cap W^{\mathrm{u}}(Q, g_{\ell,k}),$$

in Remark 7.0.14. Lemma 7.0.8 implies that $W^{\mathrm{s}}(P, g_{\ell,k}) \pitchfork \mathcal{P}^{\mathrm{u}}_{\ell,k} \neq \emptyset$. This completes the proof of the lemma. ■

8.3.1 Heteroclinic relations implies Homoclinic relations

In this section, we use the heteroclinic connections in Propositions 8.1.1 and 8.2.1 to prove that the reference saddle $\widetilde{P}^+_{\ell,k}$ of blender $\Lambda_{\ell,k}$ and the saddle P of $g_{\ell,k}$ are homoclinically related. In what follows we can assume that, after an arbitrarily small C^r-perturbation the following holds:

Theorem 8.3.3 *For every $k \geq 1$ large enough, the saddles $\widetilde{P}^+_{\ell,k}$ and P of $g_{\ell,k}$ are homoclinically related. Moreover, the saddle Q belongs to the homoclinic class $H(P, g_{\ell,k})$ C^r-robustly. In particular, $H(P, g_{\ell,k})$ is not dominated in a C^r-robust way.*

Note that the last assertion is an immediate consequence of Remark 3.2.3.
Proof. We need to check that for every large enough ℓ and k it holds that:

$$W^{\mathrm{u}}(P, g_{\ell,k}) \pitchfork W^{\mathrm{s}}(\widetilde{P}^+_{\ell,k}, g_{\ell,k}) \neq \emptyset, \quad W^{\mathrm{s}}(P, g_{\ell,k}) \pitchfork W^{\mathrm{u}}(\widetilde{P}^+_{\ell,k}, g_{\ell,k}) \neq \emptyset.$$

Claim 8.3.4 $W^{\mathrm{u}}(P, g_{\ell,k}) \pitchfork W^{\mathrm{s}}(\widetilde{P}^+_{\ell,k}, g_{\ell,k}) \neq \emptyset$.

Proof. Recall that from Proposition 8.1.1 the saddle Q activates the blender $\Lambda_{\ell,k}$ (Remark 8.1.5). Thus,

$$W^{\mathrm{u}}(Q, g_{\ell,k}) \cap W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_{\ell,k}, g_{\ell,k}) \neq \emptyset.$$

Let $X_{\ell,k} \in \Lambda_{\ell,k}$ such that $W^{\mathrm{u}}(Q, g_{\ell,k}) \cap W^{\mathrm{s}}_{\mathrm{loc}}(X_{\ell,k}, g_{\ell,k}) \neq \emptyset$. Thus, $W^{\mathrm{u}}(Q, g_{\ell,k})$ and $W^{\mathrm{s}}(X_{\ell,k}, g_{\ell,k})$ meet along of a quasi-transverse orbit. The irrationality of the argument of the contractive eigenvalue of $Dg_{\ell,k}(Q)$ and Lemma 7.0.8 imply that $W^{\mathrm{s}}(X_{\ell,k}, g_{\ell,k})$ meet tranversely any two-disc S tranverse to $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ with positive radial projection. Then, Lemma 8.3.1 imply that $W^{\mathrm{s}}(X_{\ell,k}, g_{\ell,k}) \pitchfork \mathcal{P}^{\mathrm{u}}_{\ell,k} \neq \emptyset$. Consider $N^* \in \mathbb{N}$ such that $g^{N^*}_{\ell,k}(\mathcal{P}^{\mathrm{u}}_{\ell,k}) \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(X_{\ell,k}, g_{\ell,k}) \neq \emptyset$ and consider $P^*_{\ell,k} \in \Lambda_{\ell,k}$ a saddle homoclinically related to $\widetilde{P}^+_{\ell,k}$ and sufficiently close to $X_{\ell,k}$ (recall that the set of saddles homoclinically related to $\widetilde{P}^+_{\ell,k}$ is dense in $\Lambda_{\ell,k}$) so that $g^{N^*}_{\ell,k}(\mathcal{P}^{\mathrm{u}}_{\ell,k}) \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(P^*_{\ell,k}, g_{\ell,k}) \neq \emptyset$. The λ-lemma guarantees that $g^{N^*}_{\ell,k}(\mathcal{P}^{\mathrm{u}}_{\ell,k}) \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(P^+_{\ell,k}, g_{\ell,k}) \neq \emptyset$. This completes the proof of claim. ■

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Claim 8.3.5 $W^{\mathrm{s}}(P, g_{\ell,k}) \pitchfork W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}) \neq \emptyset$.

Proof. By Proposition 8.2.1, the two-dimensional manifold $W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k})$ meet transversely to $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ in a curve C^r-close to the straight line segment $\widetilde{L}_k := \Phi_{m_k,n_k}(L_k)$ where L_k is as in Remark 8.2.9. Note that $\widetilde{L}_k$ is parallel to X_Q-coordinate in U_Q, see Remark 8.2.9. Therefore its radial projection in $\mathbb{S}^1_Q \subset W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ is a interval. Thus, every curve C^1-close to $\widetilde{L}_k$ project a interval in $\mathbb{S}^1_Q$. On the other hand, using the quasi-transverse heteroclinic orbit $\mathcal{O}_{g_{\ell,k}}(Z^*_\ell) \subset W^{\mathrm{s}}(P, g_{\ell,k}) \cap W^{\mathrm{u}}(Q, g_{\ell,k})$ (see Remark 7.0.14) and applying Lemma 7.0.8, we obtain a transverse intersection between $W^{\mathrm{s}}(P, g_{\ell,k})$ and $W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k})$, proving the claim. ■

This completes the first part of theorem. We now go to the second part of theorem. To see that $Q \in H(P, g_{\ell,k})$ in C^r-robust form, we consider the curve $\widetilde{\tau}_{\ell,k}$ in $W^{\mathrm{u}}(\widetilde{P}^{*}_{k}, g_{\ell,k}) \pitchfork W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ with positive radial projection obtained in Proposition 8.2.1.

Lemma 8.3.6 *For every large sufficient ℓ and k it holds $\widetilde{\tau}_{\ell,k} \subset H(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k})$ in C^r-robust way.*

Proof. Consider $X_{\ell,k}$ in $\widetilde{\tau}_{\ell,k}$. We will see that given $\epsilon > 0$ it holds that

$$B(X_{\ell,k}, \epsilon) \cap W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}) \pitchfork W^{\mathrm{s}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}) \neq \emptyset.$$

Consider

$$S_\epsilon := B(X_{\ell,k}, \epsilon) \cap W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}).$$

Note that S_ϵ is a two-disc transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g_{\ell,k})$ having positive radial projection. Thus, arguing as Claim 8.3.5 we have that

$$S_\epsilon \pitchfork W^{\mathrm{s}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}) \neq \emptyset.$$

Thus, we get that

$$\widetilde{\tau}_{\ell,k} \subset \overline{W^{\mathrm{u}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}) \pitchfork W^{\mathrm{s}}(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k})} = H(\widetilde{P}^{+}_{\ell,k}, g_{\ell,k}),$$

which proves the lemma. ■

The proof of the Theorem 7.0.1 is now complete. ■

Remark 8.3.7 The set $\Gamma_{\ell,k}$ given by

$$\Gamma_{\ell,k} := \left\{\widetilde{P}^{+}_{\ell,k}, Q\right\} \cup \left\{g^n_{\ell,k}(\widetilde{\tau}_{\ell,k}) : n \in \mathbb{Z}\right\},$$

is a closed invariant subset of $H(\widetilde{P}^{*}_{\ell,k}, g_{\ell,k})$ which is robustly non-dominated. See Definition 3.2.1 and Remark 3.2.3.

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