# 1
# Introduction

This thesis is devoted to a number of extensions of a seminal result in the theory of semi-linear elliptic equations, obtained by Ambrosetti and Prodi in 1972 ([1]).

In order to give proper context for this work, we need to return to the "dawn" of the theory of nonlinear elliptic PDE, the late 1960's and early 1970's. Consider the basic equation in a smooth bounded domain

$$-Lu = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $L$ is a linear uniformly elliptic differential operator of order two, say $L = \Delta + \lambda$, where $\lambda$ is a parameter, with $\lambda \in (-\infty, \lambda_2)$ (we shall denote with $\lambda_i$ the $i$-th eigenvalue of the Laplacian). A classical result is that if the equation is linear, that is $f = f(x)$ is independent of $u$, then the solvability of (1.1) is a consequence of the Fredholm alternative, namely, if $\lambda \neq \lambda_1$, problem (1.1) has a solution for each $f$, while if $\lambda = \lambda_1$ (resonance) it has solutions if, and only if, $f$ is orthogonal to $\phi_1$, the first eigenfunction of the Laplacian.

The first fundamental results on nonlinear equations were obtained by that time by Krasnoselskii and Amann, among others, who showed that the existence result in the non-resonant case extends to nonlinearities $f(x,u)$ which grow *sub-linearly* in $u$ as $u \to \pm\infty$, thanks to Leray-Schauder degree and fixed point theory, see [2].

Another fundamental result, obtained by Landesman and Lazer [3] (see also [4]) in 1970, states that in the resonance case $\lambda = \lambda_1$ the problem is solvable provided $f$ is bounded and its limits as $u$ tends to plus or minus infinity are functions whose scalar product with $\phi_1$ have opposite signs. Not less importantly, it was observed that in this sublinear situation for values of $\lambda$ around $\lambda_1$, more than one solution may exist, and this started the prolific study of *multiplicity of solutions* of nonlinear elliptic equations.

Ambrosetti and Prodi obtained the first general existence and multiplicity result for nonlinear equations which are not sublinear. Specifically, $f$ has different linear asymptotic behaviour, with respect to the line $\lambda_1 u$ as $u$ tends to plus or minus infinity. This result induced huge interest in nonlinear elliptic PDE, and, together with the fundamental results on superlinear equations which started to appear after 1974 in the works of Amann, Ambrosetti, Crandall, Lions, Nirenberg, Rabinowitz, lead to the blossoming of this field which we have witnessed in the last 40 years.

The Laplacian is the most important example of the two main classes of elliptic operators — those in *divergence* form such as $Lu = \mathrm{div}(A(x)Du)$ and in *non-divergence* form such as $Lu = \mathrm{tr}(A(x)D^2u)$, where $A$ is a bounded uniformly positive matrix. Divergence form operators are typical for problems of the calculus of variations, while non-divergence form operators model non-homogeneous diffusions in biological, chemical or physical problems.

The theories of these operators have developed separately during the years, with the former preceding the latter. In the last 20 years there has been a lot of general effort of bringing the theory of non-divergence form elliptic PDE to the level of divergence form equations. We observe that this effort has been long hindered by the fact that $Lu = \mathrm{tr}(A(x)D^2u)$ is not self-adjoint and by the impossibility of representing the solutions of non-divergence form equations as critical points of functionals on Banach spaces. Only in the 1990's, after the founding works of Krylov-Safonov, Caffarelli, Berestycki-Nirenberg, methods based on the maximum principle for solving non-divergence form equations appeared. Our thesis is part of this general effort and concentrates specifically on the Ambrosetti-Prodi problem.

Let us now state precisely the result of Ambrosetti and Prodi. Consider the Laplacian with Dirichlet boundary conditions defined on a bounded $C^{2,\alpha}$-domain $\Omega \subset \mathbb{R}^n$. Define by $\lambda_1 < \lambda_2$ its two smallest eigenvalues.

**Theorem 1.1** *Let $f \in C^2(\mathbb{R})$ be such that $f'' > 0$, $f'(\mathbb{R}) = (a,b)$ and $0 < a < \lambda_1 < b < \lambda_2$. Then, there exists a closed connected $C^1$-manifold $\mathcal{M}$ of codimension 1 which splits the space $C^{0,\alpha}(\Omega)$ into three disjoint subsets, $C^{0,\alpha}(\Omega) = S_0 \cup \mathcal{M} \cup S_2$, such that the equation*

$$-\Delta u - f(u) = g \in C^{0,\alpha}(\Omega) \ , \quad u \in C^{2,\alpha}(\Omega) \cap C_0(\overline{\Omega})$$

*has no solution if $g \in S_0$, exactly one solution if $g \in \mathcal{M}$, and exactly two solutions if $g \in S_2$.*

In order to obtain that result Ambrosetti and Prodi used *topological methods* to prove the following remarkable feature of the map $F = -\Delta - f : C^{2,\alpha}(\Omega) \cap C_0(\overline{\Omega}) \to C^{0,\alpha}(\Omega)$. It turns out that the domain of this operator is also decomposed in disjoint components $R_0 \cup \mathcal{C} \cup R_2$ where $\mathcal{C}$ is the critical set of $F$ and a closed connected $C^1$-manifold of codimension 1, in such a way that each one of $R_0$ and $R_2$ is taken by $F$ diffeomorphically to $S_2$, while $\mathcal{C}$ is taken diffeomorphically (in the manifold sense) to $\mathcal{M}$. Consequently, $Im(-\Delta - f) = \mathcal{M} \cup S_2$. Soon after, Manes and Micheletti [5] weakened the

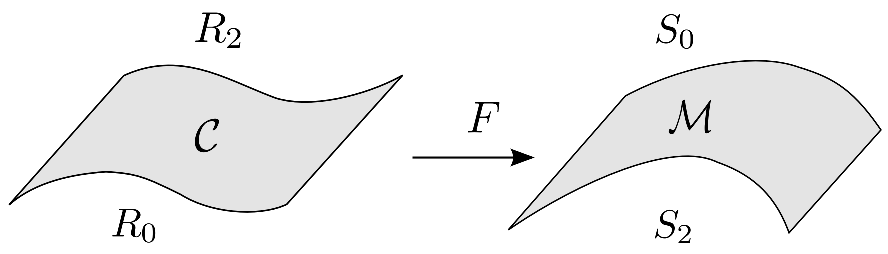


Figure 1.1: The components $R_0$ and $R_2$ are taken diffeomorphically to $S_2$.

hypothesis to just $a < \lambda_1$, thus allowing $a$ to be negative.

The use of topological methods (index theory) was furthered by Dancer ([6]), who obtained extensions of theorem 1.1, in particular for any self-adjoint operator in divergence form defined on the Sobolev space $H^1$, instead of the Laplacian. Furthermore, Dancer described the sets $R_0, \mathcal{C}$, and $R_2$ as containing precisely the functions $u$ such that the *linearized operator* $DF(u) = -\Delta - f'(u)$ (here $f'(u)$ is the operator which multiplies functions of the domain of $-\Delta$ by $f'(u)$) has a first eigenvalue of constant sign.

Another approach to the Ambrosetti-Prodi problem was provided by De Figueiredo and Solimini, [7], [8]. They make use of the variational structure of the equation $-\Delta u - f(u) = g$, that is, of the possibility to represent the weak Sobolev solutions of this equation as critical points of the associated energy functional. Then the set $R_0$ contains local minima of this functional, while the set $R_2$ contains points of higher Morse index, i.e., of mountain pass type.

Observe that the aforementioned topological and variational methods give a description of the domain of the operator, but not of its range. In other words, given a function $g$, it is not clear how to determine whether the equation is solvable at $g$ or how the solvability is affected by small variations of $g$.

A method for describing the counterdomain of $F$ was first developed by Berger and Podolak in [9]. They provided a description of the regions $R_0, R_2$ and $S_0, S_2$. To that purpose they took advantage of the self-adjoint structure of $-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$, which makes spectral estimates available by means of the Rayleigh quotient (quadratic forms), in order to introduce a *Lyapunov-Schmidt* type reduction approach to the problem.

Adding the technical hypothesis $f'' \leq M$ to the Ambrosetti-Prodi theorem, define $F : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ by $F(u) = -\Delta u - f(u)$. Let $\phi_1 > 0$ be a normalized (in $L^2(\Omega)$) eigenfunction of $-\Delta$ associated to $\lambda_1$. Take $g$ and decompose it as the sum of a function orthogonal to $\phi_1$ and a multiple of $\phi_1$: $g = z + t\phi_1$. One of the results of Berger and Podolak says that for each $z \in \langle \phi_1 \rangle^\perp := \{u \in L^2(\Omega) : \langle u, \phi_1 \rangle = 0\}$, there exists a unique $t_z \in \mathbb{R}$ such that the equation

$$F(u) = -\Delta u - f(u) = z + t\phi_1 \ , \quad u \in H^2(\Omega) \cap H_0^1(\Omega)$$

has exactly two solutions if $t < t_z$, exactly one solution if $t = t_z$ and no solution if $t > t_z$. Another way of saying this is that they proved the Ambrosetti-Prodi theorem for the Laplacian in divergence form. A review of both the Ambrosetti-Prodi and Berger-Podolak results is provided in [10, pages 62-70].

Later on, in [11], Berger and Church weakened the hypothesis $f'' > 0$ to both $f'' \geq 0$ and $f''(0) > 0$, and, above all, gave a novel geometric description of the map $F$. They proved that there exist homeomorphisms $\Psi_1 : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$, $\Psi_2 : L^2(\Omega) \to L^2(\Omega)$, such that

$$(\Psi_2 \circ F \circ \Psi_1)(z + t\phi_1) = z - t^2\phi_1.$$

Another way of saying this is to call $F$ a *topological fold* (meaning that the domain of $F$ is *folded* by $F$ along its critical set, just like $p(x) = x^2$ folds $\mathbb{R}$ into $\mathbb{R}^+$ over its critical point 0). If the homeomorphisms $\Psi_1$ and $\Psi_2$ are $C^1$-diffeomorphisms, then $F$ is called a *differentiable fold*. We note that $F$ need not be differentiable to be a topological fold — it suffices that there exists some set that behaves like the critical set of a differentiable fold, this is the case of the extreme point 0 of the real function $p(x) = |x|$. These notions are Banach space generalizations of results concerning singularities of maps of the plane to the plane obtained by Whitney in 1955, in his classical paper [12].

The theorem below condenses the work of Berger, Podolak and Church.

**Theorem 1.2** *Let $f \in C^2(\mathbb{R})$ be convex such that $f''(0) > 0$ and $\overline{f'(\mathbb{R})} = [a, b]$ where $a < \lambda_1 < b < \lambda_2$. Then, $F$ is a topological fold.*

We now provide the context of our results. Let $\Omega$ be a bounded Lipschitz domain and $L$ be a second order uniformly elliptic operator *in non-divergence form* defined on $\Omega$,

$$L := a_{ij}\partial_i\partial_j + b_i\partial_i + c,$$

with coefficients satisfying the following assumptions: for some $\lambda > 0$,

$$a_{ij} = a_{ji} \in C(\overline{\Omega}) \ , \quad b_i \, , \, c \in L^\infty(\Omega) \ , \quad a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \ , \quad \xi \in \mathbb{R}^n.$$

Let $X := W^{2,n}(\Omega) \cap C_0(\overline{\Omega})$, $Y := L^n(\Omega)$ and define the operator $L : X \to Y$. Now, define the map $F : X \to Y$ such that $F(u) = -Lu - f(u)$.

According to Berestycki, Nirenberg and Varadhan [13], the operator $L$ has a simple eigenvalue $\lambda_1 = \lambda_1(L, \Omega)$ with positive eigenfunction $\phi_1 = \phi_1(L, \Omega)$,

$$L\,\phi_1 \ = \ -\,\lambda_1\,\phi_1 \ , \quad Ker(L + \lambda_1 I) = \langle \phi_1 \rangle \, .$$

Also, for any other eigenvalue $\lambda_i$, we have that $\lambda_1 < Re(\lambda_i)$.

**Theorem 1.3 (section 4.3)** *Fix $a < \lambda_1$. There exists some $B(L, \Omega, a) > \lambda_1$ such that for all $f \in C^2(\mathbb{R})$ satisfying $\overline{f'(\mathbb{R})} = [a, b]$ and*

$$\lim_{s \to -\infty} f'(s) = a \leq f' \leq \lim_{s \to +\infty} f'(s) = b \leq B(L, \Omega, a),$$

*there exists $C^1$ diffeomorphisms $\Psi_1 : Y \to X$ ,$\Psi_2 : Y \to Y$ such that, for all $z \in \langle \phi_1^* \rangle^\perp$*

$$(\Psi_2 \circ F \circ \Psi_1)(z + t\phi_1) = z - t^2\phi_1$$

*(F is a differentiable fold) if, and only if, $f'' \geq 0$ and either $f''(0) > 0$ or both $f''(0) = 0$ and $f'(0) \neq \lambda_1$.*

The necessary conditions in the last theorem are new even for $L = \Delta$.

Some difficulties related to the new setting are the following. The operator $L$ is not self-adjoint, so that spectral estimates involving the quadratic form are lost, such as the possibility of relating spectrum and norms of functions of the operator. The operator might not even have a second smallest eigenvalue, that is, it is not clear what an explicit upper bound for $b$ as in the Ambrosetti-Prodi theorem could be.

In addition, we are interested in nonlinearities $f$ which are Lipschitz and not necessarily differentiable — local extrema are handled without differentiation. In theorem 5.1 of chapter 5, we obtain an extension of theorem 1.2, to the case where $L$ is an elliptic operator in non-divergence form as above and $f$ is both Lipschitz and convex. In section 5.2, we introduce *appropriate* functions $f$, which satisfy conditions generalizing those in the differentiable case (i.e. the necessary and sufficient condition in theorem 1.3).

**Theorem 1.4** *Given $a < \lambda_1$, there exists some $B(L, \Omega, a) > \lambda_1$ such that, if $f$ is convex, appropriate and $\lambda_1 < b \leq B(L, \Omega, a)$ then, $F$ is a topological fold.*

An example of an appropriate nonlinearity is the usual piecewise linear $f(x) = ax$ $(bx)$ for $x \leq 0$ $(x > 0)$ where $a < \lambda_1 < b \leq B(L, \Omega, a)$. We prove this theorem by approximating $F = -L - f$ by a sequence of maps $\{F_k = -L - f_k\}_{k \in \mathbb{N}}$ for convenient smooth $f_k$. The argument uses additional ingredients from [13] as well as the uniform Lipschitz bounds in theorem 1.5 depending only on $a, L$ and $\Omega$.

We explain in more detail the approach of Berger and Podolak ([9]) with the geometric vocabulary used in [14]. They obtained a diffeomorphism $\Phi : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ which may be used as a change of variables as indicated in figure 1.2. Vertical lines $\{z + t\phi_1 : t \in \mathbb{R}\}$, for $z \in \langle \phi_1 \rangle^\perp$, are taken to *fibers* $\{\Phi(z + t\phi_1) : t \in \mathbb{R}\}$, which in turn are taken by $F$ to vertical lines $\{z + h(z, t)\phi_1 : t \in \mathbb{R}\} \subset L^2(\Omega)$, for the *height* $h(z, t) = \langle F(\Phi(z + t\phi_1)), \phi_1 \rangle$ — formally, for all $z \in \langle \phi_1 \rangle^\perp$ and $t \in \mathbb{R}$, there exists some real number $h(z, t)$ such that $F(\Phi(z + t\phi_1)) = z + h(z, t)\phi_1$.

The change of variables $\Phi$ provides a *global Lyapunov-Schmidt reduction* for $F$: it trivializes $F$ on a subspace of codimension 1, turning the problem of finding solutions for $F(u) = g$ into a scalar problem, the so called *bifurcation equation*, as done below. Let $P$ be the orthogonal projection of $Y$ onto $\langle \phi_1 \rangle^\perp$. For some fixed $g = z + \langle g, \phi_1 \rangle \phi_1$, we want to solve the system below for $u \in Y$:

$$\begin{cases} P\, F(\Phi(u)) = z \\ (I - P)\, F(\Phi(u)) = \langle g, \phi_1 \rangle \phi_1. \end{cases}$$

Note that the problem above is essentially one dimensional. The vertical line through $g = z + \langle g, \phi_1 \rangle \phi_1$ has its pre-images in the vertical line $\ell = \{z + t\phi_1 : t \in \mathbb{R}\}$, that is, if a solution $u \in Y$ exists, it must be in $\ell$. Now, we *walk* along $\ell$ and check how many times we encounter $t$'s such that $h(z, t) = \langle g, \phi_1 \rangle$. Indeed, to prove the Ambrosetti-Prodi theorem, Berger and Podolak simply showed that, for all $z \in \langle \phi_1 \rangle^\perp$

1. $\lim_{|t| \to \infty} h(z, t) = -\infty$,
2. the real function $t \mapsto h(z, t)$ is $C^1$ and has a unique critical point.

The geometry of the Lyapunov-Schmidt reduction invites to consider more general scenarios. Thus, for example, the Laplacian may be replaced by

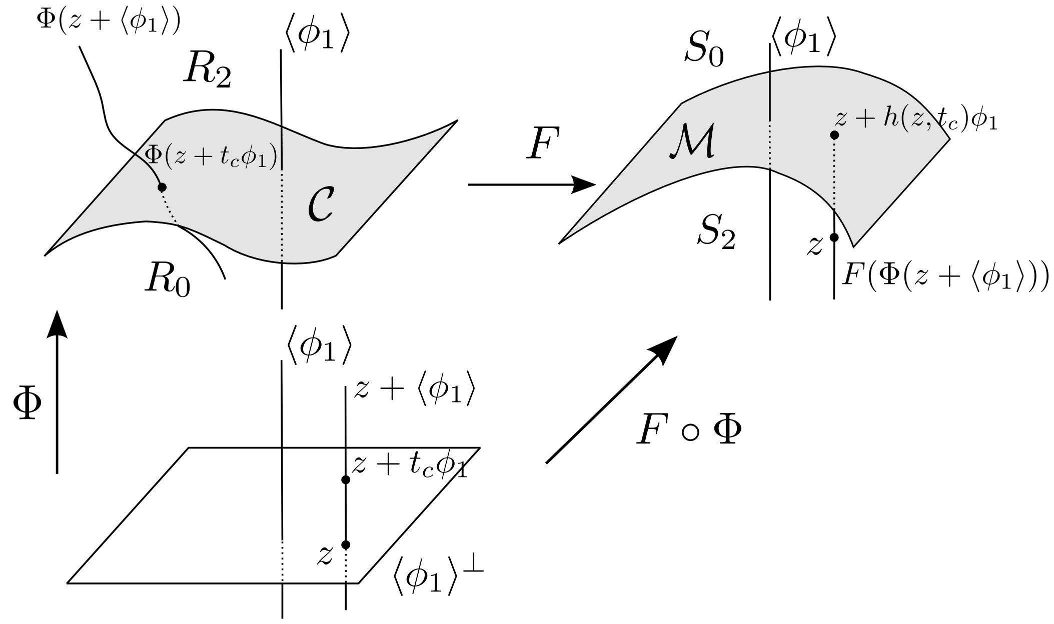


Figure 1.2: The point $z + t_c\phi_1$ is a critical point of $F \circ \Phi$.



Schrödinger operators in bounded and unbounded domains, so as to include the hydrogen atom and the quantum harmonic oscillator, together with integral operators in bounded domains with positive kernel. These generalizations were considered, among other results, in our works [15], [16]. The increased flexibility gained through the methods introduced in these papers led us to consider non self-adjoint problems, such as uniformly elliptic operators in non-divergence form, in the spirit of this thesis.

The reduction also allows for robust numerical analysis for standard Ambrosetti-Prodi equations ([17], [18], [19]). From our results, mainly theorem 1.5, the expansion of numerical techniques towards non self-adjoint problems is natural, although we do not treat the issue here.

In the self-adjoint scenario, eigenvalue estimates are used in the construction of a global Lyapunov-Schmidt reduction (all the above quoted works concern equations in divergence form). In the non-divergence setting these methods cannot be used, nor a Hilbert structure is available. To circumvent these difficulties, topological (index) methods have been applied to non-divergence form equations, see [20], [21], [22], with the drawback that no exact count of solutions is obtained and, of course, no description of the counterdomain (i.e. solvability of the equation for a given $g$) is possible.

In this thesis we obtain the global Lyapunov-Schmidt reduction for equations in non-divergence form, by using the maximum principle and its consequences, together with elliptic regularity. This seems to be the first time an *exact* multiplicity result appears for a semilinear elliptic equation driven by an operator in a general non-divergence form. Along the way we establish a number of results which are also new for equations in divergence form.

We now present the Lyapunov-Schmidt reduction obtained in the non self-adjoint case, along with an overview of the proofs of our theorems.

By standard functional analysis, $L$ has a dual operator $L^* : Y^* \to X^*$ with $Ker(L^* + \lambda_1 I)$ being one dimensional and $Y^* = L^{\frac{n}{n-1}}(\Omega)$. As proved by Birindelli in [23], $L^*$ has an everywhere positive eigenfunction $\phi_1^* = \phi_1(L^*, \Omega)$ associated to $\lambda_1$ (in particular, $\langle \phi_1, \phi_1^* \rangle \neq 0$).

Decompose $Y$ into a direct sum of *horizontal* and *inclined* complementary subspaces $\langle \phi_1^* \rangle^\perp \oplus \langle \phi_1 \rangle = Y$. Although in that scenario we lack a precise meaning for orthogonality, we refer to the subspace $\langle \phi_1 \rangle$ as the *vertical* subspace. In the Berger-Podolak spirit, we obtain a global Lyapunov-Schmidt reduction for $F$. Note that, by the hypotheses below, $f$ is just Lipschitz.

**Theorem 1.5 (section 2.1)** *Given $F : X \to Y$ as above and $a < \lambda_1$, there exists $B := B(L, \Omega, a) > \lambda_1$ such that, if*

$$a \leq \frac{f(x) - f(y)}{x - y} \leq B \ , \quad x \neq y,$$

*then there exists a Lipschitz homeomorphism $\Phi : Y \to X$ with Lipschitz inverse such that, for all $z \in \langle \phi_1^* \rangle^\perp$, we have $(F(\Phi(z + t\phi_1)) = z + h(z,t)\phi_1$ where*

$$h(z,t) := \frac{\langle F(\Phi(z + t\phi_1)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle}.$$

Thus, each *fiber*, i.e., the pre-image of each *vertical* line $\{z + t\phi_1 \ : \ t \in \mathbb{R}\}$ ($z \in \langle \phi_1^* \rangle^\perp$) by $F$, is parametrized as the graph of the Lipschitz function $t \mapsto \Phi(z + t\phi_1)$. Also, $F(\Phi(z + t\phi_1)) \in Y$ has *height* $h(z,t)$.

The proof of theorem 1.5 is based on theorem 2.6 (chapter 2), which is fundamental to our work. Its proof uses elliptic estimates and a result contained in [13] which provides lower bounds for the increase of the eigenvalue $\lambda_1$ when restricting $L$ to strict subdomains $\Omega' \subset \Omega$ in terms of the measure of the difference $\Omega \setminus \Omega'$. The positivity of $\phi_1^*$ is also essential to our arguments. We point out that the constant $B(L, \Omega, a)$ is actually quantified while we only prove the existence of the Lipschitz constant of $\Phi$.

As far as we know, similar Lipschitz estimates appeared first in a paper by Podolak ([24]). She describes some interesting ideas on how to construct a Lyapunov-Schmidt reduction in a geometric setting including both the self-adjoint and non self-adjoint cases. However, in the case of interest, her techniques would require the somewhat unnatural hypothesis that the difference $b - a$ be very small. The special case of theorem 1.5 for general *self-adjoint* operators $T$ perturbed by a Lipschitz nonlinearity with constant $\ell$ for which $[-\ell, \ell] \cap \sigma(T)$ is finite is treated in our works [15] and [16].

Now that fibers and heights are available, we extend the construction by Berger and Podolak to equations in non-divergence form. As before, to get the representation of $F$ as a differentiable fold (theorem 1.3), we walk along a fiber and use the following result.

**Proposition 1.6 (section 4.1)** *Fix $a < \lambda_1$. There exists some $B(L, \Omega, a) > \lambda_1$ such that, if $f \in C^2(\mathbb{R})$, $f'' \geq 0$, $\overline{f'(\mathbb{R})} = [a, b]$ with $\lambda_1 < b \leq B(L, \Omega, a)$ and either $f''(0) > 0$ or both $f''(0) = 0$ and $f'(0) \neq \lambda_1$. Then, the homeomorphism $\Phi$ of theorem 1.5 is actually a $C^2$-diffeomorphism such that, for all $z \in \langle \phi_1^* \rangle^\perp$,*

1. $\lim_{|t| \to \infty} h(z, t) = -\infty$,

2. $\frac{\partial h}{\partial t}(z, t_0) = 0 \implies \frac{\partial^2 h}{\partial t^2}(z, t_0) < 0$,

The techniques employed to prove item 1 include, in particular, an use of the Lipschitz estimates on $\Phi$ in order to prove that $\Phi(z + t\phi_1)/t$ converges to $\phi_1$ as $|t| \to \infty$. Regarding item 2, we give two proofs, both of which use proposition 3.5, which states that, if 0 is an eigenvalue of $DF(u) = -L - f'(u)$, then it is its smallest eigenvalue. One of them provides a precise formula for the second derivative of $t \mapsto h(z, t)$. The other one involves the differentiability of the principal eigenfunction $\lambda_1(u)$ of the Jacobian of $F$ at $u \in X$ given by $DF(u)v = -Lv - f'(u)v$, $v \in X$. This last proof provides the basis to obtain new criteria for classifying critical points of the height function $t \mapsto h(z, t)$ — we show that, if $F$ is $C^k$ for $k \geq 1$ then there exists a *positive* $C^{k-1}$ function $p : \langle \phi_1^* \rangle^\perp \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{\partial h}{\partial t}(z, t) = p(z, t)\, \lambda_1(\Phi(z + t\phi_1)).$$

That result was first stated for the self-adjoint scenario in [16] and its proof is to be given in [15], both our works.

In particular, the derivative of $h$ along a vertical line has the same sign as the first eigenvalue of the linearized operator along the corresponding fiber. As a corollary, Dancer's characterization for the domain of $-\Delta - f$ is valid for our $F = -L - f$. We use this fact and the validity of the maximum principle relative to the sign of the principal eigenvalue to prove that, if $f''(0) > 0$, then $t \mapsto h(z,t)$ is concave to the left of its critical point. This result is the content of proposition 4.8. To our knowledge, this was also unknown for $L = \Delta$.

Finally, we state more precisely the result which proves the necessity of the convexity assumption in theorem 1.3.

**Theorem 1.7 (section 4.2)** *Take* $a < \lambda_1 < b \leq B(L, \Omega, a)$. *Suppose that* $f \in C^2(\mathbb{R})$, $\overline{f'(\mathbb{R})} = [a, b]$ *and* $\lim_{s \to -\infty} f'(s) = a \leq f' \leq \lim_{s \to +\infty} f'(s) = b$. *If there exists some* $r \in \mathbb{R}$ *such that* $f''(r) < 0$, *then there exists some* $g \in Y$ *such that* $F(u) = g$ *has at least* *__four__* *solutions in* $X$.

This result was stated by us for the self-adjoint case in [16] — its proof is due to appear in [15]. There, $B(L, \Omega, a)$ is any number in $(\lambda_1, \lambda_2)$. This theorem answers a long standing open problem proposed by Dancer in [6]: the necessity of convexity to the validity of the Ambrosetti-Prodi theorem.

This thesis contains six chapters including this introduction, and three appendices.