2
Lipschitz nonlinearities

In this chapter, we always suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$a \leq \frac{f(x) - f(y)}{x - y} \leq b \ , \quad \text{for} \ \ x \neq y \quad \text{and} \ \ a < \lambda_1. \tag{2.1}$$

In section 2.1, we obtain the homeomorphism $\Phi : Y \to X$ mentioned in theorem 1.5. In section 2.2 we impose an extra condition on f and we obtain results about the behaviour at infinity of the map $t \mapsto \Phi(z + t\phi_1)$ for fixed $z \in H_Y$. In section 2.3 we obtain some results concerning the count of solutions of the equation $F(u) = g$ for $u \in X$ and $g \in Y$. Afterwards, in section 2.4 we obtain a characterization of properness for F which depends on the behaviour of the image of fibers at infinity. Lastly, in section 2.5, we provide the proof of a coercive property of the homeomorphism Φ used in the previous sections.

2.1
Lyapunov-Schmidt reduction

Take an elliptic operator $L : X \to Y$.

Definition 2.1 *Let L be elliptic. Define $\lambda_1(L, \Omega)$ by*

$$\inf \left\{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,n}_{loc}(\Omega) \ \textit{such that} \ \begin{cases} (L + \lambda)\phi \geq 0 & , \quad \Omega \\ \phi \leq 0 & , \quad \partial\Omega \\ \exists x_0 \in \Omega & : \quad \phi(x_0) > 0 \end{cases} \right\}.$$

Theorem 2.2 ([13]) *Let $L : X \to Y$ be an elliptic operator. Then, $\lambda_1(L, \Omega)$ is an isolated eigenvalue of $-L$ with smallest real part in $\sigma(-L)$. Also, $Ker(-L - \lambda_1(L, \Omega)I)$ is unidimensional ($\lambda_1(L, \Omega)$ is simple) and is generated by a positive function $\phi_1(L, \Omega)$.*

Theorem 2.3 ([23, proposition 1.1]) *If $L : X \to Y$ is an elliptic operator, then the operator $L^* : Y^* \to X^*$ has a simple isolated eigenvalue $\lambda_1(L^*, \Omega) =$*

$\lambda_1(L,\Omega)$ *with smallest real part in* $\sigma(-L^*)$ *to which one can associate a strictly positive eigenfunction* $\phi_1(L^*,\Omega) \in Y^*$.

For brevity, let $\lambda_1 := \lambda_1(L,\Omega)$, $\phi_1 := \phi_1(L,\Omega) > 0$, $\phi_1^* := \phi_1(L^*,\Omega) > 0$.

Decompose X and Y in direct sums. Define $H_Y = \langle \phi_1^* \rangle^\perp$ and $V_Y = \langle \phi_1 \rangle$. As $\phi_1 > 0$, $H_Y \cap V_Y = \{0\}$. Also, define $V_X = V_Y \cap X$ and $H_X = H_Y \cap X$. Then,

$$X = H_X \oplus V_X \ , \quad Y = H_Y \oplus V_Y .$$

Our main goal in this section is to prove the following theorem.

Theorem 2.4 *There exists* $B = B(L,\Omega,a) > \lambda_1$ *such that, if* $b \leq B$ *then, there exists a Lipschitz homeomorphism with Lipschitz inverse*

$$\Phi : H_Y \oplus V_Y = Y \to H_X \oplus V_X = X$$

such that, for every $z \in H_Y$ *and* $t \in \mathbb{R}$, $(F \circ \Phi)(z + t\phi_1) \in \{z + s\phi_1 : s \in \mathbb{R}\} \subset Y$. *In other words, for*

$$h(z,t) := \frac{\langle (F \circ \Phi)(z,t), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle},$$

we have $(F \circ \Phi)(z + t\phi_1) = z + h(z,t)\phi_1$.

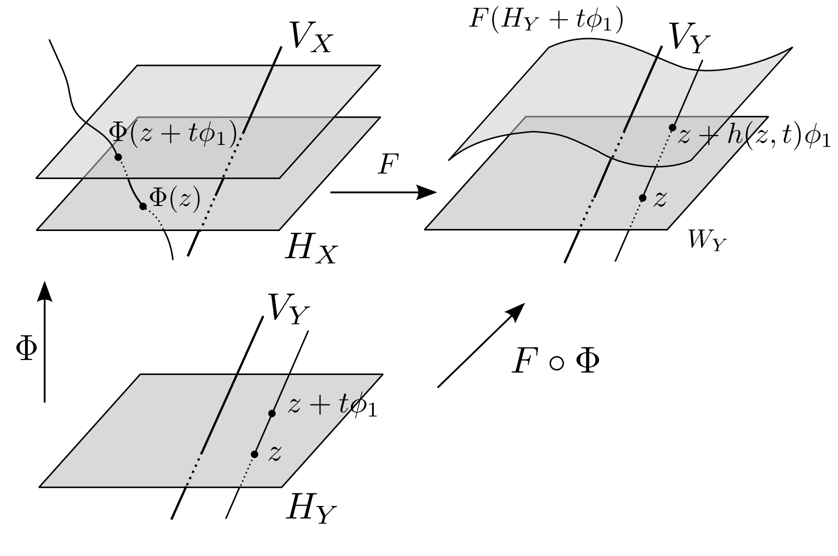

Figure 2.1: The function $F \circ \Phi$ trivializes the first coordinate.

Without loss, we can suppose that L is an isomorphism, i.e. its first eigenvalue is positive (see [13]). We can, for instance, translate both L and f by $aI : X \to Y$ as follows:

$$-Lu - f(u) = -(L + aI)u - (f(u) - au) = -(L + a)u - (f - a)(u).$$

The Fredholm alternative and the fact that λ_1 is a simple eigenvalue of L give that $u \in X$ is a solution of $Lu + \lambda_1 u = f$ if, and only if, $\langle \phi_1^*, f \rangle = 0$. As $L : X \to Y$ is an isomorphism, the aforementioned Fredholm alternative assures that $L : H_X \to H_Y$ is a well defined isomorphism.

In the following arguments, the letters w and z will be reserved, respectively, for members of the horizontal spaces H_X, H_Y. Consider the projection

$$P : H_Y \oplus V_Y = Y \to H_Y \oplus V_Y \ , \quad z + v \mapsto z.$$

Given $g = Pg + (I - P)g = z_g + t_g \phi_1 \in H_Y \oplus V_Y$, we want to solve the two equations and two variables nonlinear system

$$\begin{cases} PF(w + t\phi_1) = z_g \\ (I - P)F(w + t\phi_1) = t_g \phi_1 \end{cases} \quad \text{for the unknowns} \quad w \in H_X \ , \ t \in \mathbb{R}. \tag{2.2}$$

The system above motivates us to define, for $F_t(w) = F(w + t\phi_1)$, a family of maps $\{PF_t\}_{t \in \mathbb{R}}$

$$PF_t : H_X \to H_Y \ , \quad w \mapsto PF_t(w) = -Lw - Pf(w + t\phi_1),$$

and to prove the following, where a and b are as in equation (2.1),

Proposition 2.5 *Given $a < \lambda_1$, there exists $B = B(L, \Omega, a) > \lambda_1$ such that, if $b \leq B$, the mappings PF_t are Lipschitz homeomorphisms with Lipschitz inverses such that the Lipschitz constants of both PF_t and $(PF_t)^{-1}$ are independent of t.*

A consequence of this result is that, given $g = z_g + t_g \phi_1 \in H_Y \oplus V_Y$, for all $t \in \mathbb{R}$, there exists some $w(z_g, t)$ such that $PF(w(z_g, t) + t\phi_1) = z_g$. In other words, we can always solve the first equation of system (2.2), on $w \in H_X$, for every fixed t and z_g. Each z_g has exactly one pre-image in each horizontal affine subspace $t\phi_1 + H_X$ of X.

We use proposition 2.5 to define the map Φ

$$\Phi : H_Y \oplus V_Y \to H_X \oplus V_X \ , \quad z + t\phi_1 \mapsto (PF_t)^{-1}(z) + t\phi_1.$$

Note that Φ moves a point at height t to a point at the same height t. Turning back to system (2.2), we observe that, for all $z \in H_Y$ and $t \in \mathbb{R}$,

$$PF(\Phi(z + t\phi_1)) = PF((PF_t)^{-1}(z) + t\phi_1) = PF_t(PF_t^{-1}(z)) = z.$$

As a consequence,

$$\begin{aligned} F(\Phi(z+t\phi_1)) &= PF(\Phi(z+t\phi_1)) + (I-P)(F(\Phi(z+t\phi_1)) \\ &= z + (I-P)(F(\Phi(z+t\phi_1)) \in H_Y \oplus V_Y. \end{aligned} \tag{2.3}$$

Our goal now is to prove proposition 2.5 and theorem 2.4. The following theorem 2.6, which is proved at the end of this chapter (section 2.5), plays a central role in our arguments. It is a consequence of the maximum principle and results contained in [13] concerning the existence of lower bounds for the difference $\lambda_1(L,\Omega') - \lambda_1(L,\Omega)$, where $\Omega' \subset \Omega$ is a strict subset, in terms of the measure of the set $\Omega \setminus \Omega'$. Also, we use results on the stability of viscosity subsolutions and supersolutions under suitable hypotheses as in [25].

Theorem 2.6 *For all $a < \lambda_1$, there exist $B = B(L,\Omega,a) > \lambda_1$ and $C = C(L,\Omega,a) > 0$ such that, for all $f \in C(\mathbb{R})$ satisfying*

$$a \leq \frac{f(x)-f(y)}{x-y} \leq b \leq B\ , \quad x \neq y,$$

we have that, for every $t, \tilde{t} \in \mathbb{R}$ and $w, \tilde{w} \in H_X$,

$$\|PF_t(w) + t\phi_1 - PF_{\tilde{t}}(\tilde{w}) - \tilde{t}\phi_1\|_Y \geq C\|w-\tilde{w}\|_X.$$

This theorem is used not only to prove that PF_t and Φ are invertible, but also to obtain Lipschitz bounds for their inverses.

Now, we use theorem 2.6 to prove proposition 2.5 and theorem 2.4.

Proof: (proposition 2.5) We begin by proving the lemma below, which deals with the case where F is C^1.

Lemma 2.7 *The maps $PF_t : H_X \to H_Y$ have closed image and are injective. Also, if f is C^1, then the mappings PF_t are Lipschitz diffeomorphisms with Lipschitz inverse. Moreover, the constants of Lispchitz can be taken to be the same for every $t \in \mathbb{R}$.*

Proof: For fixed $t \in \mathbb{R}$, from theorem 2.6, we have that

$$\|PF_t(w) - PF_t(\tilde{w})\|_Y \geq C\|w-\tilde{w}\|_X,$$

so that PF_t is injective and has closed image.

Now, suppose that f is C^1. Note that $PF_t : H_X \to H_Y$ is C^1 — as a consequence of proposition 6.1, the Jacobian of PF_t at a point $w \in H_X$ is given by

$$D(PF_t)(w) : H_X \to H_Y \ , \quad v \mapsto -Lv - P\big(f'(w + t\phi_1)v\big).$$

We use theorem 2.6 to prove that it has invertible derivative at every point $w \in H_X$. Note that $v \mapsto P\big(f'(w+t\phi_1)v\big)$ is a compact map since $P : Y \to H_Y$ is continuous and H_X is compactly embbeded in H_Y. Also, $L : H_X \to H_Y$ is an isomorphism, so that $D(PF_t)(w)$ is the sum of the bijective map L with the compact map $Pf'(w + t\phi_1)$, hence, Fredholm of index zero. As it is injective (theorem 2.6), it follows that it is also surjective, thus it is an isomorphism. (There is another Fredholm index argument which does not depend on the compact inclusion mentioned above, it can be found in [17]).

From the inverse function theorem, PF_t is a local diffeomorphism, so that its image is open. As its image is open and closed (theorem 2.6), it equals H_Y, so that PF_t is surjective and, hence, bijective. Again, by theorem 2.6, we have that its inverse is Lipschitz with the same constant for every $t \in \mathbb{R}$, as we wanted.

□

The main difficulty to extend the result to the case where f is only Lipschitz is to prove that PF_t is surjective. In the previous argument, we used the fact that the image of PF_t is open, by means of the inverse function theorem, which is not readily available for Lipschitz maps.

We prove surjectivity for PF_t in the Lipschitz case by showing that it is the uniform limit of surjective functions and by using the fact that its image is closed. We use the next lemma 2.8 to aproximate PF_t uniformly by a sequence of C^1 surjective functions

$$PF_{k,t} : H_X \to H_Y \ , \quad w \mapsto PF_k(w + t\phi_1),$$

where $F_k(u) = -Lu - f_k(u)$, with $f_k \in C^1(\mathbb{R})$ obtained in the lemma below.

Lemma 2.8 *If $f : \mathbb{R} \to \mathbb{R}$ is such that*

$$a \le \frac{f(x) - f(y)}{x - y} \le b \le B \ , \quad x \ne y,$$

then there exists a sequence $\{f_k\} \in C^1(\mathbb{R})$ such that $f_k \to f$ uniformly and

$$a \le \frac{f_k(x) - f_k(y)}{x - y} \le b \ , \quad x \ne y.$$

Also, the sequence $\{F_k : X \to Y \ , \ u \mapsto -Lu - f_k(u)\}_k$ *converges uniformly to* F*.*

Proof: By propositon A.1, there exists a sequence of C^∞ real function $\{f_k\}$ which satisfies

$$a \leq \frac{f_k(x) - f_k(y)}{x - y} \leq b \leq B \ , \quad x \neq y.$$

and $f_k \to f$ uniformly.

Now we prove that $F_k \to F$ uniformly. Fix $\epsilon > 0$. Take N big enough so that, for all $k \geq N$ and $s \in \mathbb{R}$ we have $|f_k(s) - f(s)| < \epsilon / |\Omega|^{\frac{1}{n}}$. In particular, for all $u \in X$ and $x \in \Omega$ we have that $|f_k(u(x)) - f(u(x))| < \epsilon/|\Omega|^{\frac{1}{n}}$. It follows that, for all $u \in X$ and $k \geq N$ we have

$$\|F_k(u) - F(u)\|_Y = \|f_k(u) - f(u)\|_Y < \frac{\epsilon}{|\Omega|^{\frac{1}{n}}} |\Omega|^{\frac{1}{n}} = \epsilon.$$

□

Since $PF_{k,t}(w) = PF_k(w + t\phi_1)$, which is the result of a translation in the domain of F and the composition with a linear map, $F_k \to F$ uniformly implies that $PF_{k,t} \to PF_t$ uniformly.

By proposition 6.1, $PF_{k,t}$ is C^1 for all $k \in \mathbb{N}$. Then, lemma 2.7 implies that $PF_{k,t}$ are diffeomorphisms for every $k \in \mathbb{N}$, $t \in \mathbb{R}$ and, hence, surjective.

Now, fix $t \in \mathbb{R}$. We prove that, for every $z \in H_Y$, there exists $w \in H_X$ such that $PF_t(w) = z$, that is, that PF_t is surjective. Because $PF_{k,t}$ are surjective and $PF_{k,t} \to PF_t$ uniformly, there exists a sequence $\{w_k\}_k \in H_X$ such that $PF_{k,t}(w_k) = z$ and

$$\|PF_{k,t}(w_k) - PF_t(w_k)\|_Y = \|z - PF_t(w_k)\|_Y \to 0.$$

As the image of PF_t is closed, $z \in \mathrm{Ran}(PF_t)$.

Use theorem 2.6 to check that $(PF_t)^{-1} : H_Y \to H_X$ is Lipschitz for every $t \in \mathbb{R}$ with the same Lipschitz constant.

■

We proceed to the proof of theorem 2.4.

Proof: (theorem 2.4) Recall the definition of $\Phi : H_Y \oplus V_Y = Y \to X$,

$$\Phi(z + t\phi_1) = (PF_t)^{-1}(z) + t\phi_1 \in H_X \oplus V_X.$$

First we prove that Φ is injective. Take, in $H_Y \oplus V_Y$, $z + t\phi_1 \neq \tilde{z} + \tilde{t}\phi_1$. If $t \neq \tilde{t}$, clearly $\Phi(z + t\phi_1) \neq \Phi(\tilde{z} + \tilde{t}\phi_1)$. If $t = \tilde{t}$, then $z \neq \tilde{z}$. As PF_t is bijective, $(PF_t)^{-1}(z) \neq (PF_t)^{-1}(\tilde{z})$ so that $\Phi(z + t\phi_1) \neq \Phi(\tilde{z} + t\phi_1)$.

To see that Φ is surjective, take $w + t\phi_1 \in H_X \oplus V_X$. For such t, as $PF_t : H_X \to H_Y$ is bijective, there exists some $z \in H_Y$ such that $w = (PF_t)^{-1}(z)$. Then, for such z, we have $\Phi(z + t\phi_1) = (PF_t)^{-1}(z) + t\phi_1 = w + t\phi_1$.

It follows that Φ is invertible and

$$\Phi^{-1} : X = H_X \oplus V_X \to Y = H_Y \oplus V_Y \ , \quad w + t\phi_1 \mapsto PF_t(w) + t\phi_1$$

is well defined and Lipschitz, since $PF_t : H_X \to H_Y$ is uniformly Lipschitz on $t \in \mathbb{R}$ from proposition 2.5.

We prove that Φ is Lipschitz. Fix $t, \tilde{t} \in \mathbb{R}$ and $z, \tilde{z} \in H_Y$. Consider $(PF_t)^{-1}(z) = w$ and $(PF_{\tilde{t}})^{-1}(\tilde{z}) = \tilde{w}$. Substituting in theorem 2.6 we obtain

$$\|z + t\phi_1 - \tilde{z} - \tilde{t}\phi_1\|_Y \geq C\|(PF_t)^{-1}(z) - (PF_t)^{-1}(\tilde{z})\|_X.$$

As a consequence

$$\begin{aligned}
\|\Phi(z + t\phi_1) - \Phi(\tilde{z} + \tilde{t}\phi_1)\|_X &\leq \|(PF_t)^{-1}(z) - (PF_{\tilde{t}})^{-1}(\tilde{z})\|_X + \|(t - \tilde{t})\phi_1\|_X \\
&\leq \frac{1}{C}\|z - \tilde{z} + (t - \tilde{t})\phi_1\|_Y + \frac{\|\phi_1\|_X}{\|\phi_1\|_Y}\|(t - \tilde{t})\phi_1\|_Y \\
&\leq \left(\frac{1}{C} + 1 + \frac{\|\phi_1\|_X}{\|\phi_1\|_Y}\right)(\|z - \tilde{z}\|_Y + \|(t - \tilde{t})\phi_1\|_Y)
\end{aligned}$$

By an application of the closed graph theorem, for $z + t\phi_1 \in H_Y \oplus V_Y$, the norm given by $\|z + t\phi_1\|_{\Gamma(Y)} := \|z\|_Y + \|t\phi_1\|_Y$ is equivalent to the norm of Y, so that Φ is Lipschitz.

Now we verify that $(F \circ \Phi)(z + t\phi_1) = z + h(z, t)\phi_1$ where $h(z, t)$ is defined by the multiple of ϕ_1 such that $(I - P)(F \circ \Phi)(z + t\phi_1) = h(z, t)\phi_1$. From equation (2.3),

$$(F \circ \Phi)(z + t\phi_1) = z + (I - P)(F \circ \Phi)(z + t\phi_1) \in H_Y \oplus V_Y$$

That is, $(F \circ \Phi)(z + t\phi_1) = z + h(z, t)\phi_1$. Recalling that $H_Y \perp \langle \phi_1^* \rangle$ and $z \in H_Y$, apply the functional ϕ_1^* to both sides of the equation above to obtain

$$h(z, t) = \frac{\langle (F \circ \Phi)(z + t\phi_1), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle}. \tag{2.4}$$

■

Define $w(z,t) := (PF_t)^{-1}(z) \in H_X$ and

$$u(z,t) := w(z,t) + t\phi_1 = \Phi(z + t\phi_1).$$

Applying the new notation to equation (2.3), we have

$$F(u(z,t)) = -Lu(z,t) - f(u(z,t)) = z + h(z,t)\phi_1. \tag{2.5}$$

Now we obtain a formula for $h(z,t)$ by expanding the one in (2.4)

$$\begin{aligned} h(z,t) &= \frac{\langle F(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} \\ &= \frac{\langle -Lw(z,t) - tL\phi_1, \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} - \frac{\langle f(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} \\ &= \lambda_1 t - \frac{\langle f(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} = \lambda_1 t - \frac{\int_\Omega f(u(z,t))\phi_1^*}{\int_\Omega \phi_1 \phi_1^*}. \end{aligned} \tag{2.6}$$

Definition 2.9 (fiber) *For fixed $z \in H_Y$, the fiber related to z is the function*

$$u_z : \mathbb{R} \to X \ , \quad t \mapsto u_z(t) = w_z(t) + t\phi_1 := u(z,t),$$

where $w_z(t) := w(z,t) = (PF_t)^{-1}(z)$.

Definition 2.10 (height) *For fixed $z \in H_Y$, the height related to z is the function*

$$h_z : \mathbb{R} \to \mathbb{R} \ , \quad t \mapsto h_z(t) := h(z,t).$$

The propositions below are immediate consequences of theorem 2.4 and the definition of fibers and heights.

Proposition 2.11 *For every $z \in H_Y$, the map $t \mapsto u_z(t) = u(z,t) = \Phi(z + t\phi_1)$ is Lipschitz with Lipschitz constant independent of z.*

Proposition 2.12 *For each $z \in H_Y$, the set $\{u_z(t) : t \in \mathbb{R}\} \subset X$ is the pre-image of $\{z + t\phi_1 : t \in \mathbb{R}\} \subset Y$ by F. Moreover, $F(u_z(t)) = z + h_z(t)\phi_1$, where $h_z(t)$ is Lipschitz on both z and t.*

Turning back to the system of equations (2.2), note that, given $g = z_g + t_g\phi_1 \in H_Y \oplus V_Y$, to solve $F(w + t\phi_1) = g = z_g + t_g\phi_1$ for $w \in H_X$ and $t \in \mathbb{R}$, take the fiber $t \mapsto u_{z_g}(t)$ and observe that

$$\begin{cases} PF(u_{z_g}(t)) = PF_t(w_{z_g}(t)) = z_g \\ (I - P)F(u_{z_g}(t)) = h_{z_g}(t)\phi_1 \end{cases}$$

so that the equation $F(u) = g$ has as many solutions as the equation $h_{z_g}(t) = t_g$, for $t \in \mathbb{R}$.

2.2
Behaviour of fibers at infinity

Now we discuss the behaviour of a fiber at infinity supposing that

$$a \leq \frac{f(x) - f(y)}{x - y} \leq b \leq B \ , \quad x \neq y \tag{2.7}$$

$$a \leq \lim_{s \to -\infty} \frac{f(s)}{s} = \tilde{a} \leq \lim_{s \to +\infty} \frac{f(s)}{s} = \tilde{b} \leq b. \tag{2.8}$$

The lemma below proves that $u_z(t) = o(t) + t\phi_1$, that is, $w_z(t) = o(t)$.

Lemma 2.13 *For every $z \in H_Y$ we have $\lim_{|t| \to \infty} \|w_z(t)/t\|_X = 0$, that is,*

$$\lim_{|t| \to \infty} \left\| \frac{u_z(t)}{t} - \phi_1 \right\|_X = 0.$$

Proof: Fix $z \in H_Y$. By theorem 2.6, for some $C > 0$,

$$\left\| \frac{z}{t} - \frac{PF_t(0)}{t} \right\|_Y = \frac{1}{|t|} \|PF_t(w_z(t)) - PF_t(0)\|_Y \geq C \left\| \frac{w_z(t)}{t} \right\|_X,$$

so that we need to prove that the left hand side converges to 0. It suffices to prove that $\|PF_t(0)/t\|_Y \to 0$ as $t \to \pm\infty$. We use the dominated convergence theorem to prove this.

Note that $PF_t(0) = -Pf(t\phi_1)$. Suppose that $t \to +\infty$. Then, for every $x \in \Omega$, we have $f(t\phi_1(x))/t \to \tilde{b}\phi_1(x)$. Moreover, for $t \geq 1$,

$$\left| \frac{f(t\phi_1(x))}{t} \right| \leq \left| \frac{f(t\phi_1(x)) - f(0)}{t} \right| + \left| \frac{f(0)}{t} \right| \leq \max\{|a|, |b|\} \, \phi_1(x) + |f(0)|$$

with the right hand side belonging to $Y = L^n(\Omega)$, so that the dominated convergence theorem assures that $f(t\phi_1)/t \to \tilde{b}\phi_1$ in Y. Finally, $PF_t(0)/t = -Pf(t\phi_1)/t \to -\tilde{b}P\phi_1 = 0$.

The case $t \to -\infty$ is proved similarly using $\lim_{s \to -\infty} f(s)/s = \tilde{a}$.

■

2.3 Behaviour of heights at infinity

We have already seen in proposition 2.12 that, for a fiber $u_z(t)$ we have

$$F(u_z(t)) = z + h_z(t)\phi_1.$$

Suppose, again, that conditions (2.7) and (2.8) are valid. We claim that, under these hypotheses,

$$\begin{aligned} t \to +\infty &\implies \|f(u_z(t))/t - \tilde{b}\phi_1\|_Y \to 0 \\ t \to -\infty &\implies \|f(u_z(t))/t - \tilde{a}\phi_1\|_Y \to 0. \end{aligned}$$

Observe that

$$\left|\frac{f(u_z(t))}{t} - \frac{f(t\phi_1)}{t}\right| \leq \max\{|a|,|b|\}\left|\frac{w_z(t)}{t}\right|.$$

From lemma 2.13, $\|w_z(t)/t\|_Y \to 0$ for $|t| \to \infty$. Moreover, we have already seen in the proof of the same lemma that $\|f(t\phi_1)/t - \tilde{b}\phi_1\|_Y \to 0$ if $t \to +\infty$ and $\|f(t\phi_1)/t - \tilde{a}\phi_1\|_Y \to 0$ if $t \to -\infty$, so that the claim is proved.

Proposition 2.14 *Suppose* (2.7) *and* (2.8). *If* $\tilde{a} < \lambda_1 < \tilde{b}$ *then, for every bounded set* $B \subset H_Y$ *we have that* $|t| \to \infty$ *implies that* $h_z(t) \to -\infty$ *uniformly on* $z \in B$.

Proof: Fix any $z \in H_Y$. By equation (2.5),

$$\frac{F(u_z(t))}{t} = -\frac{Lu_z(t) + f(u_z(t))}{t} = \frac{z}{t} + \frac{h_z(t)}{t}\phi_1.$$

Take $t \to +\infty$.

By lemma 2.13, $\|u_z(t)/t - \phi_1\|_X \to 0$, so that $\|-Lu_z(t)/t - \lambda_1\phi_1\|_Y \to 0$. On the other hand, as we have proved above, $\|f(u_z(t))/t - \tilde{b}\phi_1\|_Y \to 0$. Then,

$$\frac{h_z(t)}{t}\phi_1 = \frac{F(u_z(t)) - z}{t} \to (\lambda_1 - \tilde{b})\phi_1\ , \quad \text{in } Y,$$

so that $t \to +\infty$ implies that $h_z(t)/t \to \lambda_1 - \tilde{b} < 0$.

For $t \to -\infty$, a similar argument provides $h_z(t)/t \to \lambda_1 - \tilde{a} > 0$, and, again, $h_z(t) \to -\infty$.

Now, take $z, z_0 \in B \subset H_Y$ where B is bounded. As $F(u_z(t)) = F(\Phi(z + t\phi_1)) = z + h(z,t)\phi_1$ is Lipschitz, it follows that $h(z,t) = h_z(t)$ is

Lipschitz. Then,

$$|h_z(t) - h_{z_0}(t)| \, \|\phi_1\|_Y \leq C\|z - z_0\|_Y,$$

which assures that the convergence is uniform on B for $|t| \to \pm\infty$.

∎

We observe that in this proof, the Lipschitz property of Φ plays a fundamental role as we used it to prove that $\|u_z(t)/t - \phi_1\|_X \to 0$ and to obtain the uniform convergence. Also, this approach has the advantage of proving what the limits of $h_z(t)/t$ are when $t \to \pm\infty$.

On the other hand, the techniques used by Berger and Podolak in [9], if applied to that case, would use only the compact inclusion $X \hookrightarrow Y$ instead of lemma 2.13, but do not provide uniform convergence on bounded sets nor the limits $\lim_{t\to\pm\infty} h_z(t)/t$.

We state similar results below but we omit their proofs as they are analogous to the one in proposition 2.14.

Proposition 2.15 *Under hypotheses* (2.7) *and* (2.8) *we have*

1. $\tilde{a}, \tilde{b} < \lambda_1 \implies \lim_{t\to-\infty} h_z(t) = -\infty$ *and* $\lim_{t\to+\infty} h_z(t) = +\infty$.
2. $\tilde{a}, \tilde{b} > \lambda_1 \implies \lim_{t\to-\infty} h_z(t) = +\infty$ *and* $\lim_{t\to+\infty} h_z(t) = -\infty$.
3. $\tilde{a} > \lambda_1 > \tilde{b} \implies \lim_{t\to-\infty} h_z(t) = +\infty$ *and* $\lim_{t\to+\infty} h_z(t) = +\infty$.

For slightly different hypotheses on f there is a similar approach which relies on the positivity of ϕ_1^* and does not depend on the existence of a limit for $u_z(t)/t$. It provides uniform convergence on z of the height functions $h_z(t)$ to $-\infty$ regardless of the Lipschitz bounds for Φ, but we do not know what the limits of $h_z(t)/t$ are when $t \to \pm\infty$.

Proposition 2.16 *Suppose that there exist lines* $a_0 + \tilde{a}s$ *and* $b_0 + \tilde{b}s$, $s \in \mathbb{R}$, $\epsilon > 0$ *with* $\tilde{a} < \lambda_1 < \tilde{b}$ *such that*

$$a_0 + \tilde{a}s \leq f(s) \quad \text{if} \quad s < 0\ , \qquad b_0 + \tilde{b}s \leq f(s) \quad \text{if} \quad s > 0. \tag{2.9}$$

Then, $|t| \to \infty$ *implies that* $h_z(t) \to -\infty$ *uniformly on* $z \in H_Y$.

Proof: Fix $z \in H_Y$. Equation (2.6) provides

$$h_z(t) = \lambda_1 t - \frac{\int_\Omega f(u_z(t))\phi_1^*}{\int_\Omega \phi_1 \, \phi_1^*}.$$

By hypothesis, for all $s \in \mathbb{R}$, $b_0 + \tilde{b}s \leq f(s)$. Since $\phi_1^* > 0$,

$$-f(u_z(t))\phi_1^* \leq -b_0\phi_1^* - \tilde{b}u_z(t)\phi_1^*.$$

Substituting in the equation above, we have

$$\begin{aligned} h_z(t) &= \lambda_1 t - \frac{\int_\Omega f(u_z(t))\phi_1^*}{\int_\Omega \phi_1\, \phi_1^*} \\ &\leq \lambda_1 t - \frac{b_0 \int_\Omega \phi_1^* + \tilde{b} \int_\Omega (w_z(t) + t\phi_1)\phi_1^*}{\int_\Omega \phi_1\, \phi_1^*} \\ &\leq \lambda_1 t - \tilde{b}t - b_0 \frac{\int_\Omega \phi_1^*}{\int_\Omega \phi_1\, \phi_1^*} \end{aligned}$$

Since $\lambda_1 < \tilde{b}$, taking $t \to +\infty$ we have that, $h_z(t) \to -\infty$. As the right hand side does not depend on $z \in H_Y$, the uniform convergence is clear.

An analogous argument provides that $h_z(t) \to -\infty$ for $t \to -\infty$ uniformly on z using the inequality $a_0 + \tilde{a}s \leq f(s)$, where $\tilde{a} < \lambda_1$.

■

Also, with hypotheses (2.7) and (2.9) instead of (2.7) and (2.8), proposition 2.15 is valid. Clearly, the results in proposition 2.15 provide non-exact counts of solutions for the equation $F(u) = g$, for $g \in Y$ and $u \in X$. These results are well known for the case $L = \Delta$, as it can be seen, for example, in [1], [9], [24], jus to name a few.

2.4 Criteria for properness

In this section we obtain a criteria for deciding if F is proper, that is, if the pre-image of compact subsets of Y are compact in X. This is the same as saying that: for every sequence $\{y_k\} \in Im(F)$ that converges in Y, the set $\{u \in X : \text{ for some } k \in \mathbb{N}, F(u) = y_k\}$ has a convergent subsequence.

It is possible to prove that F is proper with the use of the maximum principle and the compact inclusion $X \hookrightarrow Y$. This is what Ambrosetti and Prodi did in [1], but we would like to give an alternate geometric approach by considering fibers, that is, the Lyapunov-Schmidt reduction of F.

All we need is the following hypothesis

$$a \leq \frac{f(x) - f(y)}{x - y} \leq b \leq B \;, \quad x \neq y$$

where B is given in theorem 2.4, and the easy lemma below, which we do not prove.

Lemma 2.17 *Let $F : X \to Y$ be continuous and let $G : Y \to X$ be an homeomorphism. Then $F \circ G$ is proper if, and only if, F is proper.*

Now we provide a criteria for deciding if F is proper.

Proposition 2.18 *The map $F : X \to Y$ given by $F(u) = -Lu - f(u)$ is proper if, and only if, for every $z \in H_Y$ we have $\lim_{|t|\to\infty} |h_z(t)| = \infty$.*

Proof: ($\Longleftarrow$) Suppose that, for all $z \in H_Y$, we have that $\lim_{|t|\to\infty} |h_z(t)| = \infty$. By lemma 2.17, we just need to prove that $F \circ \Phi$ is proper, where Φ is given by theorem 2.4. Take $z_k + s_k\phi_1 \in Im(F) \subset H_Y \oplus V_Y$ converging to $z_0 + s_0\phi_1$. Then, for some sequence $z_k + t_k\phi_1 \in H_Y \oplus V_Y$ we have

$$F(\Phi(z_k + t_k\phi_1)) = z_k + h(z_k, t_k)\phi_1 = z_k + s_k\phi_1.$$

First we prove that $h(z_0, t_k) \to s_0$. By theorem 2.4, $F \circ \Phi$ is Lipschitz, so that h is Lipschitz. Then, for some constant $C > 0$,

$$|h(z_k, t_k) - h(z_0, t_k)| \leq C\|z_k - z_0\|_Y \to 0.$$

Since $h(z_k, t_k) \to s_0$, we have that $h(z_0, t_k) \to s_0$. It follows, by hypothesis, that $\{t_k\}_k$ is bounded, otherwise $|h(z_0, t_k)|$ would have a subsequence going to infinity. Then, it has a convergent subsequence $t_{k_i} \to t_0$. Clearly, $z_{k_i} + t_{k_i}\phi_1 \to z_0 + t_0\phi_1$, that is, $z_k + t_k\phi_1$ has a convergent subsequence and thus $F \circ \Phi$ is proper.

($\Longrightarrow$) Suppose that F is proper. Then, $F \circ \Phi$ is proper, by lemma 2.17. Fix some $z \in H_Y$. Take $|t_k| \to \infty$. Suppose, by contradiction, that $h_z(t_k)$ does not go to infinity, that is, it has bounded subsequence — $\{h_z(t_k)\}$ has a convergent subsequence $h_z(t_{k_i}) \to s_0$. It follows that

$$F(\Phi(z + t_{k_i}\phi_1)) = z + h(z, t_{k_i})\phi_1 \to z + s_0\phi_1.$$

Since $F \circ \Phi$ is proper, then $\{t_{k_i}\}_i$ must have a convergent subsenquence, which is a contradiction with the fact that $|t_k| \to \infty$. ■

An immediate corollary of propositions 2.15 and 2.18 is the following.

Corollary 2.19 *Under hypotheses (2.7) and (2.8), if $\tilde{a}, \tilde{b} \neq \lambda_1$, then F is proper.*

2.5
Proof of theorem 2.6

The reader should now be convinced that the estimate contained in theorem 2.6 is fundamental to our construction of the Lyapunov-Schmidt reduction.

The theorem below, which was first proved in [13, theorem 2.4], is the cornerstone of our argument.

Theorem 2.20 *Let $\Omega' \subset \Omega$ be a closed subset in Ω such that, for some $\delta \in \mathbb{R}$ satisfying $0 < \delta < |\Omega|$, we have $|\Omega'| \leq |\Omega| - \delta$. Then, there exists some $\eta = \eta(L, \Omega, \delta)$ such that*

$$\lambda_1(L, \Omega') - \lambda_1(L, \Omega) \geq \eta.$$

The proof is made by contradiction in two steps. First we obtain upper bounds (which depend only on L, Ω and a) for the measure of the subsets of Ω where a certain function $u \in X$ is positive and negative. This provides a lower bound for the increase of the principal eigenvalue of L when restricted to such subdomains. Then, we choose B to be equal to $\lambda_1(L, \Omega)$ plus this lower bound and use this information to obtain a contradiction.

Proof: (theorem 2.6)

Step 1: Recall that $a < \lambda_1$. Suppose, by contradiction, that there exist sequences $w_k, \tilde{w}_k \in H_X$, $t_k, \tilde{t}_k \in \mathbb{R}$ and a sequence of real functions $\{f_k\}_k$ satisfying

$$a \leq \frac{f_k(x) - f_k(y)}{x - y} \leq \lambda_1 + \frac{1}{k}\ , \quad x \neq y$$

such that, for every $k \in \mathbb{N}$, and

$$\begin{aligned} \frac{1}{k} &> \|PF_k(w_k + t_k\phi_1) - PF_k(\tilde{w}_k + \tilde{t}_k\phi_1) + (t_k - \tilde{t}_k)\phi_1\|_Y \\ &= \Big\|L\frac{w_k - \tilde{w}_k}{\|w_k - \tilde{w}_k\|_X} + P\frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X} + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\Big\|_Y. \end{aligned}$$

Observe that,

$$0 \leq \frac{f_k(x) - f_k(y)}{x - y} - a = \frac{(f_k - a)(x) - (f_k - a)(y)}{x - y} \leq \lambda_1 + \frac{1}{k} - a\ , \quad x \neq y$$

so that, $-Lu - f_k(u) = -(L - aI)u - (f_k - aI)(u)$. It follows from $a < \lambda_1$ that, without loss, we can suppose that $a = 0$, $\lambda_1(L, \Omega) > 0$ and $\lambda_1 + 1/k$ is a

Lipschitz bound for f_k. That said, all the constants we take will depend on a as well.

Note that the terms

$$L\frac{w_k-\tilde{w}_k}{\|w_k-\tilde{w}_k\|_X}+P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}\in H_Y\ ,\ \frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\phi_1\in V_Y,$$

are linearly independent, so that both of them converge to 0. It follows that

$$\frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\to 0.$$

For simplicity, let $(w_k-\tilde{w}_k)/\|w_k-\tilde{w}_k\|_X=u_k\in H_X$. By the compact inclusion $X\hookrightarrow C^{0,\alpha}(\overline{\Omega})$, there exists a subsequence, already relabelled, such that $u_k\overset{C^{0,\alpha}(\overline{\Omega})}{\longrightarrow}u\in H_X\cap C^{0,\alpha}(\overline{\Omega})$ for, say, $\alpha=1/2\in(0,1)$. Note that,

$$Lu_k=\psi_k-P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}-\frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\phi_1,$$

with $\psi_k\overset{Y}{\longrightarrow}0$, so that

$$\psi_k=Lu_k+P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}+\frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\phi_1\overset{Y}{\longrightarrow}0.$$

We use elliptic and α-Hölder estimates on u_k to prove that $\|u\|_\infty:=\|u_k\|_{L^\infty(\Omega)}\neq 0$. For some $C_1=C(\Omega)$ and $C_2=C(L,\Omega)$, we have

$$\|u_k\|_{C^{0,\alpha}(\overline{\Omega})}\le C_1\|u_k\|_X=C_1\le$$
$$C_1C_2\Big(\|u_k\|_\infty+\Big\|\psi_k-P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}-\frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\phi_1\Big\|_Y\Big)$$

As $k\to\infty$ we have

$$\|\psi_k\|_Y+\Big\|\frac{\tilde{t}_k-t_k}{\|w_k-\tilde{w}_k\|_X}\phi_1\Big\|_Y\to 0,$$

so we just have to deal with the term

$$P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}.$$

It is easy to see that,

$$\begin{aligned}
&\|P\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}\|_Y \\
\leq&\|P\|\,\|\frac{f_k(w_k+t_k\phi_1)-f_k(\tilde{w}_k+\tilde{t}_k\phi_1)}{\|w_k-\tilde{w}_k\|_X}\|_Y \\
\leq&(\lambda_1+\frac{1}{k})\,\|P\|\,\|\frac{w_k-\tilde{w}_k}{\|w_k-\tilde{w}_k\|_X}+\frac{t_k-\tilde{t}_k}{\|w_k-\tilde{w}_k\|_X}\phi_1\|_Y \to \lambda_1\|P\|\,\|u\|_Y.
\end{aligned}$$

Note that $\|u\|_Y = \|u\|_{L^n(\Omega)} \leq |\Omega|^{1/n}\|u\|_\infty$, and we have, for $k \to \infty$

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1 \leq C_1C_2(\|u\|_\infty + \lambda_1|\Omega|^{1/n}\|P\|\,\|u\|_\infty) = C(L,\Omega)\|u\|_\infty,$$

and it is proved that $\|u\|_\infty > 0$. As a consequence, $\|u\|_{C^{0,\alpha}(\overline{\Omega})} > 0$.

Set $w = u/\|u\|_{L^\infty(\Omega)} \in H_X$, which changes sign, as $\int_\Omega w\,\phi_1^* = 0$. For every $x, y \in \Omega$,

$$|w(x) - w(y)| \leq C(L,\Omega)|x-y|^\alpha = C|x-y|^\alpha.$$

In particular for $x_1 = arg\,max\{|w(x)|\}$ and $x_0 = arg\,min\{|x_1 - x_0| : w(x_0) = 0, x_0 \in \overline{\Omega}\}$ we have $|w(x_1)| = 1$ and $w(x_0) = 0$, so that

$$\frac{1}{C^{1/\alpha}} \leq |x_1 - x_0|.$$

Clearly, in $B_{C^{-1/\alpha}}(x_1) \subset \Omega$, $w(x)$ does not change sign. For every $z \in C(\overline{\Omega})$ define

$$\Omega_z^+ := \{x \in \Omega : z(x) > 0\}\ ,\quad \Omega_z^- := \{x \in \Omega : z(x) > 0\}.$$

If $w(x_1) = 1$, then $|\Omega_w^+| \geq |B_{C^{-1/\alpha}}|$, so half of our work is done.

Now we need to find a positive lower bound for $\sup w^-$ depending only on L and Ω. Note that the ball around x_1 where $w(x) \geq 1/2$ has radius $(2C)^{-1/\alpha} = r = r(L,\Omega)$.

As $w \in H_X$, $\int_\Omega w^+\phi_1^* = \int_\Omega w^-\phi_1^*$ and $\phi_1^* > 0$, we have

$$\begin{aligned}
\sup_\Omega w^- \int_\Omega \phi_1^* \geq \int_\Omega w^-\phi_1^* &= \int_\Omega w^+\phi_1^* \\
&\geq \frac{1}{2}\int_{B_r(x_1)} \phi_1^* \\
&\geq \frac{1}{2}\inf\Big\{\int_{B_r(x)} \phi_1^* : B_r(x) \subset \Omega\Big\}
\end{aligned}$$

which implies that $\sup_\Omega w^- > \epsilon = \epsilon(L,\Omega)$.

For $w(x_2) = -\sup_\Omega w^- = \inf_\Omega w < -\epsilon$ and $x_0 = arg\,min\{|x_2 - x_0| : w(x_0) = 0, x_0 \in \overline{\Omega}\}$,

$$\epsilon \leq \sup_\Omega w^- = |w(x_2) - w(x_0)| \leq C|x_2 - x_0|^\alpha \implies (\epsilon/C)^{1/\alpha} \leq |x_2 - x_0|.$$

In the ball $B_{(\epsilon/C)^{1/\alpha}}(x_2)$, $w(x) < 0$. Now take $\delta = |B_{(\epsilon/C)^{1/\alpha}}|$. Note that $\delta = \delta(L, \Omega)$, that is, δ does not depend on w, only on L and Ω.

The case $\sup|w| = -w(x_1) = 1$ is handled in a similar fashion providing again $\delta = |B_{(\epsilon/C)^{1/\alpha}}|$.

By theorem 2.20, given $\delta = |B_{(\epsilon/C)^{1/\alpha}}|$, there exists $\eta = \eta(L, \Omega, \delta) = \eta(L, \Omega) > 0$ such that for every $\Omega' \subset \Omega$, satisfying $|\Omega'| \leq |\Omega| - \delta$ we have $\lambda_1(L, \Omega') \geq \lambda_1 + \eta$.

Step 2: Recall that

$$Lu_k + P\frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X} + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 = \psi_k \to 0,$$

so that, there exists a bounded sequence $\{s_k\} \in \mathbb{R}$ such that

$$Lu_k + \frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X} + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 = \psi_k + s_k\phi_1.$$

To see that $\{s_k\}$ is bounded, note that $\{Lu_k\}$ is bounded, $(t_k - \tilde{t}_k)/\|w_k - \tilde{w}_k\|_X \to 0$, $u_k \to u$ in $C^{0,\alpha}(\overline{\Omega})$ and

$$\left\|\frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X}\right\|_Y \leq (\lambda_1 + \frac{1}{k})\left\|u_k + \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\right\|_Y.$$

Take a subsequence of indices for which the (already relabelled) sequence $\{s_k\}$ converges to s_0.

First, suppose $s_0 \geq 0$. Define

$$\Omega_k^+ := \{x \in \Omega : w_k + t_k\phi_1 \geq \tilde{w}_k + \tilde{t}_k\phi_1\}.$$

Set, for $\Omega' \subset \Omega$, $\chi(\Omega')(x) = 1$ if $x \in \Omega'$ and 0 if not. For all $k \in \mathbb{N}$, we have

$$\begin{aligned} &Lu_k + (\lambda_1 + \frac{1}{k})\chi(\Omega_k^+)\left(u_k + \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\right) + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 \\ \geq &Lu_k + \frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X} + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 = \psi_k + s_k\phi_1. \end{aligned}$$

Recall that $L = a_{ij}\partial_i\partial_j + b_i\partial_i + c$, where $c \in L^\infty(\Omega)$. Now, we have the following inequality

$$(L-c)u_k \geq$$
$$\psi_k + \left(s_k - \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\right)\phi_1 - (\lambda_1 + \frac{1}{k})\chi(\Omega_k^+)\left(u_k - \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\right) - c\,u_k,$$

with the right hand side converging to $s_0\phi_1 - \lambda_1\chi(\Omega_u^+)u - c\,u$ in the norm of $Y = L^n(\Omega)$ and with $u_k \to u$ in the $C(\overline{\Omega})$ norm. We use the following result to be able to take limits on k in the inequality above. It is an easy corollary of the stability of viscosity subsolutions and supersolutions with respect to uniform convergence — see lemma 2.5, corollary 3.7 and theorem 3.8 of [25].

Theorem 2.21 *Let Ω have an exterior cone condition and T be an elliptic operator in Ω without terms of zero order. For $p \geq n$, let $g_k, g \in L^p(\Omega)$ be such that $g_k \to g$ in $L^p(\Omega)$. Moreover, let $u_k, u \in C(\Omega)$ be such that $u_k \to u$ in $C(\Omega)$ and, for all $k \in \mathbb{N}$, u_k is a strong subsolution (supersolution) of $Tu_k = g_k$ in Ω. Then u is a strong subsolution (supersolution) of $Tu = g$ in Ω.*

In theorem 2.21 take $p = n$ and let $T = L - c$. Also, set

$$\begin{aligned} g_k = {} & \psi_k + \left(s_k - \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\right)\phi_1 \\ & - (\lambda_1 + \frac{1}{k})\chi(\Omega_k^+)\left(u_k - \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\right) - c\,u_k \in L^n(\Omega) \\ g = {} & s_0\phi_1 - \lambda_1\chi(\Omega_u^+)u - c\,u \in L^n(\Omega) \end{aligned}$$

where $g_k \to g$ in $L^n(\Omega)$, since $u_k \to u$ in $C^{0,\alpha}(\Omega)$. Then, we have that

$$(L-c)u \geq s_0\phi_1 - \lambda_1\chi(\Omega_u^+)u - c\,u.$$

It follows that, in Ω_u^+

$$(L+\lambda_1)u \geq s_0\phi_1 \geq 0,$$

so that, by definition 2.1, $\lambda_1 = \lambda_1(L,\Omega) \geq \lambda_1(L,\Omega_u^+)$ with $|\Omega_u^+| \leq |\Omega| - \delta$, *i.e.*, $\lambda_1(L,\Omega) \geq \lambda_1(L,\Omega) + \eta$, a contradiction.

If $s_0 < 0$, define $\Omega_k^- := \{x \in \Omega : w_k + t_k\phi_1 < \tilde{w}_k + \tilde{t}_k\phi_1\}$. By a similar argument,

$$\begin{aligned} & Lu_k + (\lambda_1 + \frac{1}{k})\chi(\Omega_k^-)\left(u_k + \frac{t_k - \tilde{t}_k}{\|w_k - \tilde{w}_k\|_X}\phi_1\right) + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 \\ \leq & Lu_k + \frac{f_k(w_k + t_k\phi_1) - f_k(\tilde{w}_k + \tilde{t}_k\phi_1)}{\|w_k - \tilde{w}_k\|_X} + \frac{\tilde{t}_k - t_k}{\|w_k - \tilde{w}_k\|_X}\phi_1 = \psi_k + s_k\phi_1. \end{aligned}$$

As before, apply theorem 2.21 to obtain, in Ω_u^-, $(L+\lambda_1)u \leq s_0\phi_1 < 0$. That is,

$$(L+\lambda_1)(-u) \geq -s_0\phi_1 > 0.$$

so that $\lambda_1 = \lambda_1(L,\Omega) \geq \lambda_1(L,\Omega_u^-)$. Because $|\Omega_u^-| \leq |\Omega| - \delta$, we have $\lambda_1(L,\Omega) \geq \lambda_1(L,\Omega) + \eta$, and again, a contradiction.

■