3
Differentiable Lipschitz nonlinearities

In this chapter we always suppose that $f \in C^k(\mathbb{R})$ for $k \geq 1$. We take advantage of the continuous inclusion $X \hookrightarrow C_0(\overline{\Omega})$ which provides, by corollary 6.2, that if $f \in C^k(\mathbb{R})$, then $F \in C^k(X, Y)$.

Then, we prove that $\Phi : Y \to X$ is a C^k-diffeomorphism. As a consequence, the map $(z, t) \to h(z, t) \in \mathbb{R}$ is C^k.

After that, we obtain a relationship between the principal eigenvalue $\lambda_1(u_z(t))$ of the linear elliptic operator given by the Jacobian of F at the point $u_z(t)$,

$$DF(u_z(t)) = -L - f'(u_z(t)) : X \to Y$$

and the first derivative of the height $t \mapsto h_z(t)$. Above, $f'(u_z(t))$ is the multiplicative operator taking functions $v \in X$ to $f'(u_z(t))v \in Y$. This relationship provides a criteria for deciding how deep a singularity of $t \mapsto h_z(t)$ is, in other words, if t_0 is a critical point of h_z, we provide a criteria for evaluating the number

$$M := \max\big\{m \in \mathbb{N} : \text{ if } j \leq m, \text{ then } \frac{\partial^j h_z}{\partial t^j}(t_0) = 0\big\}.$$

3.1
Regularity of fibers and heights

Since $f \in C^k(\mathbb{R})$ for $k \geq 1$, suppose additionally that $f' \leq B$ where B is obtained from theorem 2.4. Recall formula (2.6): $F(u(z, t)) = z + h(z, t)\phi_1$. By corollary 6.2, we already know that $F \in C^k(X, Y)$. By the chain rule of differentiation, if we knew that $(z, t) \mapsto u(z, t)$ was C^k, then $(z, t) \mapsto h(z, t)$ would be C^k. So, our goal is to show that, if $f \in C^k(\mathbb{R})$ then $(z, t) \mapsto u(z, t)$ is C^k.

We remind the reader that, by theorem 2.4

$$\Phi : Y = H_Y \oplus V_Y \to X = H_X \oplus V_X \ , \quad z + t\phi_1 \mapsto (PF_t)^{-1}(z) + t\phi_1 = u(z, t).$$

is a Lipschitz homeomorphism with Lipschitz inverse.

Lemma 3.1 *If $f \in C^k(\mathbb{R})$ then $\Phi : Y \to X$ is a C^k-diffeomorphism.*

Proof: Recall that

$$\Phi^{-1} : X = H_X \oplus V_X \to Y = H_Y \oplus V_Y \ , \quad w + t\phi_1 \mapsto PF_t(w) + t\phi_1.$$

By corollary 6.2, F is C^k. It follows that Φ^{-1} is C^k.

The Jacobian of Φ at a point $u = w + t\phi_1 \in X$ can be interpreted as the matrix

$$\begin{bmatrix} \partial_w PF_t(w) & \partial_t PF_t(w) \\ 0 & I \end{bmatrix} : H_X \times V_X \to H_Y \times V_Y.$$

Observe that it is invertible since $\partial_w PF_t(w) = D(PF_t)(w)$ which is invertible as proved in lemma 2.7. That said, by the inverse function theorem, we have that $\Phi^{-1} : X \to Y$ is a C^k diffeomorphism.

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Two immediate corollaries are the following.

Corollary 3.2 *If $f \in C^k(\mathbb{R})$, then the map $(z,t) \in H_Y \times \mathbb{R} \mapsto h(z,t)$ is C^k.*

Corollary 3.3 *Fix $z \in H_Y$. If $f \in C^k(\mathbb{R})$ then, given $z \in H_Y$, the fiber $t \mapsto u_z(t) \in X$ and the height $t \mapsto h_z(t) \in \mathbb{R}$ are C^k.*

Now, for fixed $z \in H_Y$, we want to differentiate $F(u_z(t))$ on t, that is, we want to differentiate F along a fiber $u_z(t)$.

For simplicity, given $t_0 \in \mathbb{R}$, set

$$\frac{\partial w_z}{\partial t}(t_0) := w_z'(t_0) \ , \quad \frac{\partial u_z}{\partial t}(t_0) := w_z'(t_0) + \phi_1 = u_z'(t_0) \ , \quad \frac{\partial h_z}{\partial t}(t_0) := h_z'(t_0).$$

In the spirit of equation (2.6), we use the chain rule to provide a formula for $t \mapsto h_z'(t)$

$$h_z'(t) = \lambda_1 - \frac{\int_\Omega f'(u_z(t))u_z'(t)\phi_1^*}{\int_\Omega \phi_1\phi_1^*}. \tag{3.1}$$

First recall that, for all $z \in H_Y$ and $t \in \mathbb{R}$ we have $F(u_z(t)) = z + h_z(t)\phi_1$. Differentiating on t we have, by the chain rule,

$$DF(u_z(t))u_z'(t) = -Lu_z'(t) - f'(u_z(t))u_z'(t) = h_z'(t)\phi_1. \tag{3.2}$$

3.2 Singularities

We begin with the following definition.

Definition 3.4 *A principal eigenpair of* L *is a pair* $(\lambda_1(L,\Omega),\phi_1(L,\Omega))$ *where* $\phi_1(L,\Omega) > 0$. *Analogously, a principal eigenpair of* $L^* : Y^* \to X^*$ *is a pair* $(\lambda_1(L^*,\Omega),\phi_1(L^*,\Omega))$ *with* $\lambda_1(L^*,\Omega) = \lambda_1(L,\Omega)$ *and* $\phi_1(L^*,\Omega) > 0$.

Note that the Jacobian $DF(u) = -L - f'(u)$ is a Fredholm operator of index zero. It follows that it is not invertible if, and only if, 0 is an eigenvalue of $DF(u)$. The proposition below assures that if 0 is an eigenvalue of $DF(u)$ then its principal eigenvalue $\lambda_1(u)$ is zero — that result is crucial to the exact counting results we obtain in chapters 4 and 5.

Proposition 3.5 *Let* L *be an elliptic operator and suppose that* $d \in L^\infty(\Omega)$. *Then, there exists some* $B(L,\Omega) > \lambda_1(L,\Omega)$ *such that, if* $a \leq d(x) \leq B := B(L,\Omega)$ *and for some* $u \in X$ *such that* $u \neq 0$ *we have that* $Lu + d(x)u = 0$, *then* 0 *is the principal eigenvalue of* $L + d(x)$.

Proof: By theorem 2.20, for $\delta \in (0,|\Omega|)$, there exists $\eta = \eta(L,\Omega,\delta) > 0$ such that, for every $\Omega' \subset \Omega$ satisfying $|\Omega'| \leq |\Omega| - \delta$, we have that $\lambda_1(L,\Omega') > \lambda_1(L,\Omega) + \eta$. Set $\delta = |\Omega|/2$. Take $B = B(L,\Omega)$ such that $\lambda_1(L,\Omega) < B < \lambda_1(L,\Omega) + \eta$. Suppose that $d(x) \leq b \leq B(L,\Omega)$.

For $u \neq 0$ with $Lu + d(x)u = 0$, define

$$\Omega_u^+ := \{x \in \Omega : u(x) > 0\}\ , \quad \Omega_u^- := \{x \in \Omega : u(x) < 0\},$$

and $\chi(\Omega_u^+) : \Omega \to \mathbb{R}$ assuming the value 1 if $x \in \Omega_u^+$ and 0 otherwise. Suppose, by contradiction, that neither $\Omega_u^+ = \Omega$ nor $\Omega_u^- = \Omega$. Clearly, either $0 < |\Omega_u^+| \leq |\Omega| - \delta$ or $0 < |\Omega_u^-| \leq |\Omega| - \delta$. If $|\Omega_u^+| \leq |\Omega| - \delta$, then for $Lu + d(x)u = 0$ we have that

$$(L + aI)u + (B - a)\chi(\Omega_u^+)u \geq (L + aI)u + (d(x) - a)u = 0$$

Restricting the equation above to Ω_u^+, we have

$$(L + aI)u + (B - a)u = Lu + bu \geq 0.$$

By definition 2.1, this implies that $B \geq \lambda_1(L,\Omega_u^+) \geq \lambda_1(L,\Omega) + \eta > B$, an absurd. With that we conclude that u is an eigenfunction of $L + d(x)$ which has sign. By theorem [13, theorem 2.3] the only eigenfunction of $L + d(x)$

which does not change sign is its principal eigenfunction, thus 0 is the principal eigenvalue of $L + d(x)$.

The case $|\Omega_u^-| \leq |\Omega| - \delta$ is handled similarly.

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The following theorem is now easy to prove. It will serve as reference for the choice of B in the main theorems we prove in chapters 4 and 5.

Theorem 3.6 *Given $a < \lambda_1$, there exists some $B = B(L, \Omega, a) > \lambda_1(L, \Omega)$ such that both theorem 2.4 and proposition 3.5 are valid.*

Proof: Take $B(L, \Omega, a) > \lambda_1$ as the minimum between the bounds B obtained in both theorem 2.4 and proposition 3.5.

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From this point until the end of this chapter, we always take B as in theorem 3.6

We prove some interesting geometric consequences of the Lyapunov Schmidt reduction we obtained in chapter 2.

Lemma 3.7 *A point $u_0 = u(z, t_0) = u_z(t_0) \in X$ is a critical point of F if and only if $h_z'(t_0) = 0$.*

Proof: Consider the composition $(F \circ \Phi)(z + t\phi_1) = F(u_z(t)) = z + h_z(t)\phi_1$. We can represent the derivative of $F \circ \Phi$ at $z + t\phi_1$ by the matrix

$$DF(\Phi(z + t\phi_1))D\Phi(z + t\phi_1) = \begin{bmatrix} I & 0 \\ \partial_z h(z, t) & h_z'(t) \end{bmatrix} : H_Y \oplus V_Y \to H_Y \oplus V_Y,$$

which is invertible if, and only if, $h_z'(t) \neq 0$. As $\Phi : Y \to X$ is a diffeomorphism, we conclude that $DF(u_z(t))$ is invertible if, and only if, $h_z'(t) \neq 0$.

When $u_0 = u_z(t_0)$ is critical, then $h_z'(t_0) = 0$. On the other hand, if $h_z'(t_0) = 0$, then $u_z(t_0)$ is a critical point of F.

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The following corollary connects proposition 3.5 and lemma 3.7.

Corollary 3.8 *A point $u_0 \in X$ is critical of F if, and only if, the principal eigenvalue $\lambda_1(u_0)$ of $DF(u_0)$ equals 0.*

This fact is a consequence of a deeper relationship between the principal eigenvalue of $DF(u_z(t))$ and the derivative $h_z'(t)$ which we prove in proposition 3.10. In order to obtain such relationship, we need the following result.

Lemma 3.9 *If $u_0 = u_z(t_0)$ is a critical point of F, then $u_z'(t_0)$ is positive and is a principal eigenfunction of $DF(u_0)$.*

Proof: Take $u_0 = u_z(t_0)$ a critial point of F. By lemma 3.7, $h_z'(t_0) = 0$. It follows that $DF(u_0)u_z'(t_0) = 0$ so that $u_z'(t_0)$ is an eigenvector associated to the smallest eigenvalue of $DF(u_0)$, by proposition 3.5, *i.e.*, it has sign. Moreover, $\langle u_z'(t_0), \phi_1^* \rangle = \langle \phi_1, \phi_1^* \rangle > 0$. It follows that $u_z'(t_0) > 0$.

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We now relate $\lambda_1(L + f'(u_z(t)), \Omega) = \lambda_1(-DF(u(z,t)), \Omega) := \lambda_1(z,t)$ and $h_z'(t)$.

Proposition 3.10 *Let $f \in C^k(\mathbb{R})$. Then, there exists a positive C^{k-1} function $(z,t) \in H_Y \times \mathbb{R} \mapsto p(z,t) \in \mathbb{R}$ such that $h_z'(t) = p(z,t)\lambda_1(z,t)$.*

Proof: As $f \in C^k(\mathbb{R})$, from corollary 3.2 we have that $h_z'(t)$ is C^{k-1}. From proposition 6.4, since $(z,t) \mapsto DF(u(z,t))$ is C^{k-1} the function $(z,t) \mapsto \lambda_1(z,t)$ is also C^{k-1}.

Take any eigenfunction $\phi_1^*(z,t)$ such that

$$DF(u_z(t))^* \phi_1^*(z,t) = \lambda_1(z,t)\phi_1^*(z,t).$$

Apply the functional $\phi_1^*(z,t)$ to the equation $DF(u_z(t))u_z'(t) = h_z'(t)\phi_1$ obtaining

$$\lambda_1(z,t)\langle u_z'(t), \phi_1^*(z,t)\rangle = \langle DF(u(z,t))u_z'(t), \phi_1^*(z,t)\rangle = h_z'(t)\langle \phi_1, \phi_1^*(z,t)\rangle.$$

As $\langle \phi_1(z,t), \phi_1^*(z,t)\rangle \neq 0$, we have that $\lambda_1(z,t)p(z,t) = h_z'(t)$ with

$$p(z,t) = \frac{\langle u_z'(t), \phi_1^*(z,t)\rangle}{\langle \phi_1(z,t), \phi_1^*(z,t)\rangle}.$$

Observe that, if $u_0 = u_z(t_0)$ is critial, then $u_z'(t_0) > 0$ by lemma 3.9, so that $p(z,t_0) > 0$. It follows that, for all $t \in \mathbb{R}$, $\lambda_1(z,t) = 0$ if, and only if, $h_z'(t) = 0$, which is another way of phrasing lemma 3.7. The continuity of $(z,t) \mapsto p(z,t)$ and the aforementioned fact suffice to prove that $p(z,t)$ is everywhere positive. By contradiction, suppose that $p(z,t) = 0$ somewhere.

Then, $h_z'(t) = 0$ so that $u = u_z(t)$ is a critical point of $DF(u)$. Then, $u_z'(t) > 0$ and so $p(z,t) > 0$: a contradiction.

Up until now we have that $p(z,t) > 0$ and for regular points $u = u_z(t)$ of F, we can write

$$p(z,t) = \frac{h'(z,t)}{\lambda_1(z,t)},$$

which is uniquely defined (independently of the choice of $\phi_1^*(z,t)$). Moreover, at regular points $u_z(t)$ of F, we have that $p(z,t)$ is C^{k-1}, since $\lambda_1(z,t)$ and $h_z'(t)$ are C^{k-1} by proposition 6.4 and corollary 3.2.

We are left to check that $p(z,t)$ is C^{k-1} at a critical point $u_0 = u_{z_0}(t_0)$.

In proposition 6.5, take $A = X$ and $q(u) = f'(u) \in \mathcal{L}(X,Y)$, so that $u \in X \mapsto f'(u) \in \mathcal{L}(X,Y)$ is a C^{k-1} map by corollary 6.2. Use proposition 6.5 to conclude that, there exists a ball $B(u_{z_0}(t_0)) \subset X$ in which we can define a C^{k-1} function $u \mapsto (\phi_1(u), \phi_1^*(u))$ such that

$$DF(u_{z_0}(t_0))\phi_1(z,t_0) = \lambda_1(z_0,t_0)\phi_1(z_0,t_0)$$

$$DF(u_{z_0}(t_0))^*\phi_1^*(z_0,t_0) = \lambda_1(z_0,t_0)\phi_1^*(z_0,t_0)$$

$$DF(u)\phi_1(u) = \lambda_1(-DF(u),\Omega)\phi_1(u)$$

$$DF(u)^*\phi_1^*(u) = \lambda_1(-DF(u)^*,\Omega)\phi_1^*(u) = \lambda_1(-DF(u),\Omega)\phi_1^*(u).$$

Take a small ball $B((z_0,t_0)) \subset H_Y \times \mathbb{R}$ such that $\Phi(B(z_0 + t_0\phi_1)) \subset B(u_{z_0}(t_0))$. Since $(z,t) \in H_Y \times \mathbb{R} \mapsto u_z(t) \in X$ is C^k, by the chain rule, the map

$$(z,t) \in B((z_0,t_0)) \mapsto u_z(t) \mapsto (\phi_1(z,t), \phi_1^*(z,t))$$

is C^{k-1}. It follows that

$$(z,t) \mapsto p(z,t) = \frac{\langle u_z'(t), \phi_1^*(z,t)\rangle}{\langle \phi_1, \phi_1^*(z,t)\rangle}$$

is C^{k-1} in $B((z_0,t_0))$. We conclude that $(z,t) \mapsto p(z,t)$ is C^{k-1} is all of $H_Y \times \mathbb{R}$.

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Corollary 3.11 *If f is C^k and $j \leq k$ and $h_{z_0}'(t_0) = \ldots = h_{z_0}^j(t_0) = 0$, then $\lambda_1(z_0,t_0) = \ldots = \lambda_1^{j-1}(z_0,t_0) = 0$. Moreover, if $j < k$, then*

$$h_{z_0}^{j+1}(t_0) = p(z_0,t_0)\lambda_1^j(z_0,t_0).$$

Proof: The first part is an immediate consequence of Proposition 3.10. Moreover, the same proposition assures that $\lambda_1(z,t)$ and $p(z,t)$ are C^{k-1}. By the product rule of differentiation,

$$h_z^{j+1}(t) = \sum_{m=0}^{j} \frac{j!}{m!(j-k)!} \lambda_1^{j-m}(z,t) p^j(z,t).$$

so that $h_{z_0}^{j+1}(t_0) = \lambda_1^j(z_0,t_0) p(z_0,t_0)$.

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