4
A differentiable fold

Our goal in this chapter is to prove theorem 1.3. To this end we obtain necessary and sufficient conditions for F to be a differentiable fold. Sufficient conditions are obtained in section 4.1. Necessary conditions are obtained in section 4.2. In section 4.3, we conclude the proof of theorem 1.3. Finally, in section 4.4, we provide some geometric properties of the height function h_z when f is convex.

Throughout this chapter we assume that $f \in C^2(\mathbb{R})$, $\overline{f'(\mathbb{R})} = (a, b)$ and, for B as in theorem 3.6, $a < \lambda_1 < b \leq B$. The hypotheses above imply that hypothesis (2.7) is valid and that F, Φ, h_z are C^2 by corollaries 6.2 and 3.2, respectively.

4.1
Sufficient conditions

Add the hypothesis $f'' \geq 0$. Note that hypothesis (2.8) is now valid with $\tilde{a} = a$ and $\tilde{b} = b$. By proposition C.1, to prove that F is a differentiable fold, it suffices to

1. Prove that F is C^2, a consequence of corollary 6.2;
2. Obtain the Lyapunov-Schmidt reduction given by Φ, done in theorem 3.6, and show that Φ is a C^2-diffeomorphism, done in lemma 3.1;
3. Show that, for all $z \in H_Y$, the height function $t \mapsto h_z(t)$ satisfy $\lim_{|t| \to -\infty} h_z(t) = -\infty$, which was proved in proposition 2.14;
4. Show that, for all $z \in H_Y$, at a critical point t_0 of h_z, we have $h_z''(t_0) < 0$.

This is very close to what Berger and Podolak did in [9] to obtain an exact count of solutions for the equation $-\Delta u - f(u) = g$ for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in L^2(\Omega)$, except that, in their case, F was just C^1 (then, F was a topological fold). Nevertheless, it was still possible to differentiate each height h_z twice by observing that elliptic estimates provide $u_z'(t) \in L^\infty(\Omega)$.

We prove item 4 in the list above supposing, additionally, that $f''(0) > 0$. This actually proves the first part of theorem 1.6.

Lemma 4.1 *If $f \in C^2(\mathbb{R})$, $f'' \geq 0$, $f''(0) > 0$ and $\overline{f'(\mathbb{R})} = (a, b)$ with $a < \lambda_1 < b \leq B$ then at a critical point $u_z(t_0)$ of F we have that $h_z''(t_0) < 0$.*

We provide two proofs which rely on the fact that F is C^2. The first one obtains a formula for $\dfrac{\partial \lambda_1}{\partial t}(z, t)$ and the second one a formula for $h_z''(t)$.

There is still a third proof which uses only the fact that $u_z'(t) \in L^\infty(\Omega)$ and $F \in C^1$. This is the one which shows that h_z is twice differentiable in the case studied by Berger and Podolak, even though F is just C^1. We do not give this proof as it is more technical and out of context.

First proof: For simplicity, given $z \in H_Y$, take t near t_0 and let $(\lambda_1(t), \phi_1(t))$, $(\lambda_1(t), \phi_1^*(t))$ be some principal eigenpairs of $DF(u_z(t))$ and $DF(u_z(t))^*$, respectively. The principal eigenpairs above have C^1 dependence on t (use proposition 6.5 to obtain such pairs). Note that

$$DF(u_z(t))\phi_1(t) = \lambda_1(t)\phi_1(t).$$

Differentiating on t we have

$$\begin{aligned} & DF(u_z(t))\phi_1'(t) - f''(u_z(t))u_z'(t)\phi_1(t) = \lambda_1'(t)\phi_1(t) + \lambda_1(t)\phi_1'(t) \\ \Longrightarrow\ & (DF(u(t)) - \lambda_1(t))\phi_1'(t) - f''(u_z(t))u_z'(t)\phi_1(t) = \lambda_1'(t)\phi_1(t) \end{aligned}$$

Apply the functional $\phi_1^*(t)$ to obtain

$$\lambda_1'(t) = -\frac{\langle f''(u_z(t))u_z'(t)\phi_1(t), \phi_1^*(t)\rangle}{\langle \phi_1(t), \phi_1^*(t)\rangle} \tag{4.1}$$

We observe that the formula for $\lambda_1'(t)$ above does not depend on the choice of the eigenvectors $\phi_1(t)$ and $\phi_1^*(t)$.

As $f''(0) > 0$, there exists an interval $(-\epsilon, \epsilon) \subset \mathbb{R}$ on which $f'' > 0$. Since $u_z(t)$ is continuous and $u_z(t)|_{\partial\Omega} = 0$, there exists some measurable $B_{z,t} \subset \Omega$ with positive measure on which $-\epsilon < u_z(t) < \epsilon$. Since $f'' \geq 0$ and $f''(u_z(t)) > 0$ on $B_{z,t}$ we have that $f''(u_z(t)) \geq 0$ and is positive on the set $B_{z,t}$ where $|B_{z,t}| > 0$.

For $t = t_0$, $u_z(t_0)$ is critical, so that, from lemma 3.9 we have $u_z'(t_0) > 0$. It follows that $\lambda_1'(t_0) < 0$.

From corollary 3.8, as $h_z'(t_0) = 0$ we have that $\lambda_1(t_0) = 0$. Now, corollary 3.11 implies that $h_z''(t_0) = p(z, t)\lambda_1'(z, t)$ with $p(z, t) > 0$, so that $h_z''(t_0) < 0$.

■

Second proof: Consider the function $t \mapsto F(u_z(t)) = z + h_z(t)\phi_1$ and differentiate it twice on t to obtain, by the chain rule,

$$-Lu_z''(t) - f'(u_z(t))u_z''(t) - f''(u_z(t))u_z'(t)^2 = h_z''(t)\phi_1,$$

so that $DF(u_z(t))u_z''(t) - f''(u_z(t))u_z'(t)^2 = h_z''(t)\phi_1$.

Apply the functional $\phi_1^*(t) := \phi_1^*(z,t) \neq 0$ such that

$$DF(u_z(t))^*\phi_1^*(t) = \lambda_1(z,t)\phi_1^*(t)$$

to obtain

$$\begin{aligned} h_z''(t)\langle\phi_1, \phi_1^*(t)\rangle &= \langle DF(u_z(t))u_z''(t), \phi_1^*(t)\rangle - \langle f''(u_z(t))u_z'(t)^2, \phi_1^*(t)\rangle \\ &= \lambda_1(z,t)\langle u_z''(t), \phi_1^*(t)\rangle - \langle f''(u_z(t))u_z'(t)^2, \phi_1^*(t)\rangle \end{aligned}$$

The formula above does not depend on the choice of the eigenvector $\phi_1^*(t)$.

Now, if $t = t_0$ and $\lambda_1(z, t_0) = 0$ we have

$$h_z''(t_0) = -\frac{\langle f''(u_z(t_0))u_z'(t_0)^2, \phi_1^*(t_0)\rangle}{\langle\phi_1, \phi_1^*(t)\rangle} \tag{4.2}$$

which is negative since $u_z'(t_0) > 0$, $\phi_1^*(t_0) > 0$ and $f''(u_z(t_0)) \geq 0$ in Ω and is stricly positive in some measurable $B_{z,t_0} \subset \Omega$ with positive measure. It follows that $h_z''(t_0) < 0$.

■

What if $f''(0) = 0$? Will F still be a differentiable fold?

Lemma 4.2 *Suppose that $f \in C^2(\mathbb{R})$, $f'' \geq 0$, $f''(0) = 0$, $f'(\mathbb{R}) = (a, b)$ with $a < \lambda_1 < b \leq B$ and $f'(0) \neq \lambda_1$. Then at a critical point $u_z(t_0)$ of F we have that $h_z''(t_0) < 0$.*

Proof: The proof is divided in two cases, the first one is where f'' is not identically 0 in a closed interval containing 0, and the other one when it is.

(First case) Let $u = u_z(t) \in X$ be a critical point. Either $u \equiv 0$ or $u \neq 0$. Suppose, by contradiction that $u \equiv 0$. Then, $u_z(t) = w_z(t) + t\phi_1 \in H_Y \oplus V_Y$ with $w_z(t) \equiv 0$ and $t = 0$. Use equation (3.1) to obtain

$$h_z'(0) = \lambda_1 - \frac{\int_\Omega f'(0)u_z'(0)\phi_1^*}{\langle\phi_1, \phi_1^*\rangle} = \lambda_1 - f'(0) \neq 0.$$

By lemma 3.7, u is not a critical point.

Suppose $u = u_z(t) \neq 0$. Set $\|u\|_{L^\infty(\Omega)} = \delta > 0$. Take points $p, n \in \mathbb{R}$ such that $-\delta < n < 0 < p < \delta$ and $f''(n), f''(p) > 0$. As u is continuous in $\overline{\Omega}$, δ is in its range and u equals 0 at $\partial\Omega$, then there exists some point $x_0 \in \Omega$ such that $u(x_0) = p$ or $u(x_0) = n$. It follows that, the continuous function $f'' \circ u$ is positive at $x_0 \in \Omega$. Then, there exists a ball $B(x_0) \subset \Omega$ in which $f''(u) > 0$. From equation (4.2),

$$h_z''(t) = -\frac{\int_\Omega f''(u_z(t))u_z'(t)\phi_1^*}{\langle \phi_1, \phi_1^* \rangle} \leq -\frac{\int_{B(x_0)} f''(u_z(t))u_z'(t)\phi_1^*(z,t)}{\langle \phi_1, \phi_1^* \rangle} < 0.$$

(Second case) By an argument similar to the one in the previous case, we have that $u \equiv 0$ cannot be a critical point.

Suppose that $[\alpha, \beta]$ is the maximal interval containing 0 such that, if $s \in [\alpha, \beta]$, then $f''(s) = 0$. Note that at least one of the terms α or β is finite since $\overline{f'(\mathbb{R})} = [a, b]$ with $a \neq b$.

Let $0 \neq u = u_z(t) \in X$ be a critical point of F. Either $\alpha \leq u \leq \beta$ or there exists some $x \in \Omega$ such that $u(x) > \beta$ or $u(x) < \alpha$.

If $\alpha \leq u \leq \beta$ then $f''(u) = 0$ and $f'(u) \equiv f'(0)$. From equation (3.1), it follows that $h_z'(t) = \lambda_1 - f'(0) \neq 0$, that is, u is not a critical point.

On the other hand, suppose, without loss, that $u(x) > \beta$ for some $x \in \Omega$. As u equals 0 at $\partial\Omega$, by the definition of $[\alpha, \beta]$ and the continuity of $f'' \circ u$, there exists some $x_0 \in \Omega$ such that $f''(u(x_0)) > 0$ and there exists a ball $B(x_0) \subset \Omega$ in which $f''(u) > 0$. Again, from equation (4.2),

$$h_z''(t) = -\frac{\int_\Omega f''(u_z(t))u_z'(t)\phi_1^*}{\langle \phi_1, \phi_1^* \rangle} \leq -\frac{\int_{B(x_0)} f''(u_z(t))u_z'(t)\phi_1^*(z,t)}{\langle \phi_1, \phi_1^* \rangle} < 0.$$

■

Up until now we have the following: under usual Lipschitz and asymptotic hypotheses on f, if $f'' \geq 0$ and either $f''(0) > 0$ or both $f''(0) = 0$ and $f'(0) \neq \lambda_1$, then F is a differentiable fold.

One can use proposition C.1 to prove the result below, nevertheless, there is a simple real analysis argument to that fact.

Proposition 4.3 *Under the hypotheses of lemma 4.1 or lemma 4.2, for every $z \in H_Y$, the height function h_z has a single critical point.*

Proof: By proposition 2.14, for every $z \in H_Y$, its height function h_z has some critical point.

Suppose that, for $z \in H_Y$ and $t_0 < t_1$ we have that $u_z(t_0)$ and $u_z(t_1)$ are critical points of F. By lemma 3.7, $h'_z(t_0) = h'_z(t_1) = 0$ and, by lemma 4.1, $h''_z(t_0)$ is negative. Then, there exists an interval $(t_0, t) \in \mathbb{R}$ such that, for all $s \in (t_0, t)$, we have $h'_z(s) < 0$. Take $s_0 := \inf\{s > t : h'_z(s) = 0\}$ which is well defined, because $t_1 \in \{s > t : h'_z(s) = 0\}$, and $s_0 > t > t_0$. Then, $h'_z(s) < h'_z(s_0) = 0$ for all $s \in (t_0, s_0)$. If follows that

$$\frac{h'_z(s) - h'_z(s_0)}{s - s_0} > 0,$$

which is a contradiction, since $h'_z(s_0) = 0$ and $h''_z(s_0) < 0$, by lemma 4.1.

■

4.2
Necessary conditions

We first prove that convexity is necessary for F to be a differentiable fold.

Proposition 4.4 *Suppose that $f \in C^2(\mathbb{R})$, $\overline{f'(\mathbb{R})} = [a, b]$ with $a < \lambda_1 < b \leq B$ where B is given in theorem 3.6. Also, suppose that*

$$\lim_{s \to -\infty} f'(s) = a \ , \quad \lim_{s \to +\infty} f'(s) = b.$$

Then, if there exists some $r \in \mathbb{R}$ such that $f''(r) < 0$, then there exists some $g \in Y$ such that the equation $F(u) = g$ for $u \in X$ has at least four solutions.

Proof: We have seen in proposition 2.16 that $\lim_{|t| \to \infty} h(z, t) = -\infty$. So it suffices to find a critical point $u_0 = u_z(t_0)$ of $F : X \to Y$ such that $h''_z(t_0) > 0$.

The idea is to use two nonlinear functionals,

$$\lambda_1 : Y \to \mathbb{R} \ , \quad u \mapsto \lambda_1(L + f'(u), \Omega)$$

$$\psi : Y \to \mathbb{R} \ , \quad u \mapsto -\langle f''(u)\phi_1(u)^2, \phi_1^*(u)\rangle$$

where $\phi_1(u)$ and $\phi_1^*(u)$ are, respectively, the positive, normalized in X and Y^* eigenfunctions of the elliptic operator $L + f'(u)$ and its adjoint $(L + f'(u))^*$.

We claim that when $\lambda_1(u) = 0$ and $u = u_z(t) \in X$, then $\psi(u)$ has the same sign as $h''_z(t)$. Note that, in this case, $u'_z(t)/\|u'_z(t)\|_X = \phi_1(z, t)$ (apply lemma 3.9). From equation (4.2)

$$h''_z(t) = -\frac{\langle f''(u_z(t))u'_z(t)^2, \phi_1^*(t)\rangle}{\langle \phi_1, \phi_1^*(t)\rangle} = \psi(u)\frac{\|u'_z(t)\|_X^2}{\langle \phi_1, \phi_1^*(t)\rangle} \ ,$$

so we claim that it suffices to find some $u = u_z(t) \in X$ such that $\lambda_1(u) = 0$ and $\psi(u) > 0$. By proposition 2.14, our hypotheses imply that $\lim_{|t|\to-\infty} h_z(t) = -\infty$. So, if we show that, for some $t_0 \in \mathbb{R}$ we have $h_z'(t_0) = 0$ and $h_z''(t_0) > 0$, then there exists some $s_0 \in \mathbb{R}$ such that the equation $h_z(t) = s_0$ has at least four solutions thus finishing the proof. In other words, we have found a critical point of $t \mapsto h_z(t)$ which is a local minimum.

Recall that, in this context, f'' changes sign, so that it is not guaranteed that ψ has always the same sign, demanding a more careful argument.

To that end, we first note that ψ and λ_1 are continuous. After that we obtain some $u \in Y$ such that $\lambda_1(u) = 0$ and $\psi(u) > 0$. Finally, with a sort of regularization argument, we obtain some $u_0 \in X$ near $u \in Y$ (in the Y norm) such that $\lambda_1(u_0) = 0$ and $\psi(u_0) > 0$ to finish the proof.

Lemma 4.5 *The maps*

$$\lambda_1 : Y \to \mathbb{R}\ , \quad u \mapsto \lambda_1(L + f'(u), \Omega)$$

$$\phi_1 : Y \to X\ , \quad u \mapsto \phi_1(L + f'(u), \Omega)$$

$$\phi_1^* : Y \to Y^*\ , \quad u \mapsto \phi_1((L + f'(u))^*, \Omega)$$

with

$$\phi_1(u) > 0\ , \quad \|\phi_1(u)\|_X = 1\ ,$$

$$\phi_1^*(u) > 0\ , \quad \|\phi_1^*(u)\|_{Y^*} = 1.$$

are continuous.

Proof: Take $u_k \to u_0$ in Y. It follows that there exists a subsequence u_{k_i} such that $f'(u_{k_i}) \to f'(u_0)$ almost everywhere. As $|f'(u_{k_i})| \leq M$, the dominated convergence theorem implies that $f'(u_{k_i})$ converges to $f'(u_0)$ in Y. As for all sequence there exists some subsequence with that property, it follows that for all $u_k \to u_0$ we have $f'(u_k) \to f'(u_0)$ in Y.

Now, use proposition 6.6 to see that the maps λ_1, ϕ_1 and ϕ_1^* above are continuous.

□

Lemma 4.5 above provides the continuity of λ_1. The continuity of ψ depends on the continuity of ϕ_1 and ϕ_1^* and the continuous inclusion $X \hookrightarrow C_0(\overline{\Omega})$: we omit its proof.

For $\Omega \subset \mathbb{R}^n$, take a box, containing Ω, parallel to the cartesian axis. Define for $x = (x_1, \ldots, x_n) \in \Omega$ and $(s, l, r) \in \mathbb{R}^3$,

$$u(s,l,r)(x) := \begin{cases} l \quad , \quad x_1 \le s \\ r \quad , \quad x_1 > s \end{cases}$$

Suppose that $f'(r) > \lambda_1$. Take a sequence $l_k \to -\infty$ such that $f'(l_k) < \lambda_1$ and is decreasing. We claim that, for each $k \in \mathbb{N}$, there exists $s_k \in \mathbb{R}$ such that $\lambda_1(u(s_k, l_k, r)) = 0$. Just observe that, as Ω is bounded, for very negative s, $u(s, l_k, r) = r$, so that

$$\lambda_1(u(s, l_k, r)) = \lambda_1(-L - f'(r), \Omega) = \lambda_1 - f'(r) < 0.$$

On the other hand for very big s, $u(s, l_k, r) = l_k$ so that

$$\lambda_1(u(s, l_k, r)) = \lambda_1(-L - f'(l_k)) = \lambda_1 - f'(l_k) > 0.$$

By the continuity of $\lambda_1 : Y \to \mathbb{R}$, we obtain the sequence $\{s_k\}_k$.

Set $u_k := u(s_k, l_k, r)$. By definition, $\lambda_1(u_k) = 0$. Now we calculate $\psi(u_k)$

$$\psi(u_k) = -\int_{\{u_k = l_k\}} f''(l_k)\phi_1(u_k)^2\phi_1^*(u_k) - \int_{\{u_k = r\}} f''(r)\phi_1(u_k)^2\phi_1^*(u_k).$$

By the monotonicity of the principal eigenvalue, s_k is increasing. Note that the sequence s_k is bounded (by $\max_{x\in\Omega}|x_1|$), so that it has a (already relabelled) convergent subsequence $\{s_k\}$ with limit s_∞. We claim that

$$\min_{x\in\Omega} x_1 < s_\infty < \max_{x\in\Omega} x_1.$$

If not, the principal eigenvalue $\lambda_1(u_k) = 0$ would not converge to 0, a contradiction. For a measurable subset $A \subset \Omega$ define, for $x \in \Omega$, the function $\chi : \Omega \to \mathbb{R}$ satisfying $\chi(x) = 1$ if $x \in A$ and $\chi(x) = 0$ if $x \notin A$.

Observe that

1. $f'(u_k) \to b\chi_{\{x_1 \le s_\infty\}} + f'(r)\chi_{\{x_1 > s_\infty\}} = q_\infty$ almost everywhere. As $f'(u_k)$ is uniformly bounded, it converges in Y to $b\chi_{\{x_1 \le s_\infty\}} + f'(r)\chi_{\{x_1 > s_\infty\}}$.
2. Analogously, $f''(u_k) \to f''(r)\chi_{\{x_1 > s_\infty\}} + 0\chi_{\{x_1 \le s_\infty\}} = f''(r)\chi_{\{x_1 > s_\infty\}}$ in Y.

We note that, by proposition 6.6, the sequence of potentials $f'(u_k) : X \to Y$ converge to $q_\infty : X \to Y$.

By proposition 6.6, we have the following convergences

$$\phi_1(u_k) \to \phi_1(L + q_\infty, \Omega) = \phi_{1,\infty} > 0$$

$$\phi_1^*(u_k) \to \phi_1\big((L + q_\infty)^*, \Omega\big) = \phi_{1,\infty}^* > 0$$

where both sequences are of normalized functions in X and Y^* respectively. Now, it is easy to see that

$$\psi(u_k) \to \psi_\infty := -\int_{\{x_1 > s_\infty\}} f''(r)\phi_{1,\infty}^2 \phi_{1,\infty}^* > 0.$$

Take $k \in \mathbb{N}$ such that $\psi(u_k) > 0$. Also, there exists a ball $B(u_k) \subset Y$ such that $\psi(u) > 0$ for every $u \in B(u_k)$. There exists some $\delta > 0$ such that, for all $\epsilon \in (0, \delta)$, we have $u(s_k, l_k, r \pm \epsilon) \in B(u_k)$ and f' is decreasing in $(r - \delta, r + \delta)$, since $f''(r) < 0$.

For all ϵ such that $0 < \epsilon < \delta$ and $x \in \{x = (x_1, \ldots x_n) \in \mathbb{R}^n : x_1 \geq s_k\}$, we have

$$-f'(u(s_k, l_k, r - \epsilon)) < -f'(u_k) < -f'(u(s_k, l_k, r + \epsilon))$$

so that, by [13, proposition 2.1] (monotonicity of the principal eigenvalue)

$$\lambda_1(u(s_k, l_k, r - \epsilon)) < \lambda_1(u_k) = 0 < \lambda_1(u(s_k, l_k, r + \epsilon)).$$

Take $u_+, u_- \in B(u_k) \cap X$ close to $u(s_k, l_k, r + \epsilon)$ and $u(s_k, l_k, r - \epsilon)$ in the norm of Y respectively, such that $\lambda_1(u_+) > 0$ and $\lambda_1(u_-) < 0$. As they belong to $B(u_k)$, we have $\psi((1-t)u_- + tu_+) > 0$ for all $t \in [0, 1]$, and $\lambda_1((1-t)u_- + tu_+)$ changes sign as t goes from 0 to 1. It follows that there exists some point $u \in X$ such that $\lambda_1(u) = 0$ and $\psi(u) > 0$.

The case $f'(r) < \lambda_1$ and is handled in a similar fashion taking a sequence $l_k \to +\infty$.

If $f'(r) = \lambda_1$ and $f''(r) < 0$, then begin with the function $u(x) = r \in Y$ so that $\psi(u) > 0$ and $\lambda_1(u) = 0$ and obtain the desired $u_0 \in X$ as we did before.

■

Now we prove that, if $f'' \geq 0$ and $f''(0) = 0$, then $f'(0) \neq \lambda_1$ is also necessary for F to be a differentiable fold. It is very easy, indeed. Just note that, if both $f'(0) = \lambda_1$ and $f''(0) = 0$, then at $u = w_z(t) + t\phi_1 \equiv 0$ we have

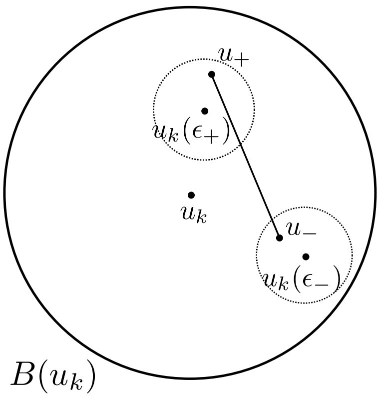

Figure 4.1: Here, $u_k(\epsilon_+) = u(s_k, l_k, r+\epsilon)$ and $u_k(\epsilon_-) = u(s_k, l_k, r-\epsilon)$.

that $t = 0$, so that

$$h_z'(t) = \lambda_1 - \lambda_1 \frac{\int_\Omega u_z'(0)\phi_1^*}{\langle \phi_1, \phi_1^* \rangle} = \lambda_1 - \lambda_1 = 0,$$

that is, $u \equiv 0$ is a critical point. Also, $h_z''(0) = 0$ since $f''(u) \equiv 0$, that is, the point $u \equiv 0$ is critical but is not a fold point. Use equivalence C.1 to conclude that F is not a differentiable fold.

We finish this section with a simple example. Suppose that $f(0) = 0$, $f'' \geq 0$, and that there exists an interval $[\alpha, \beta]$ containing 0 such that $f''|_{[\alpha,\beta]} \equiv 0$. Suppose that $f'(0) = \lambda_1$ and, without loss, that $\alpha = 0$. We claim that the segment $t\phi_1$ for $t \in [0, \beta/\max\phi_1]$ satisfies $F(t\phi_1) = 0$, that is, equation $F(u) = 0$ has a segment of solutions. Observe that $f|_{[0,\beta]}(s) = s\lambda_1$ and that, for all $t \in [0, \beta/\max\phi_1]$, we have $t\phi_1 \in [0, \beta]$. Hence, for all $t \in [0, \beta/\max\phi_1]$

$$F(t\phi_1) = -Lt\phi_1 - f(t\phi_1) = t\lambda_1\phi_1 - t\lambda_1\phi_1 = 0.$$

4.3 An equivalence statement

We resume what we did in this chapter with the following theorem.

Theorem 4.6 *Suppose that $f \in C^2(\mathbb{R})$, $\overline{f'(\mathbb{R})} = [a, b]$ and, for B as in theorem 3.6, $a < \lambda_1 < b \leq B$. Then, the map F is a differentiable fold if, and only if, $f'' \geq 0$ and either $f''(0) > 0$ or both $f''(0) = 0$ and $f'(0) \neq \lambda_1$.*

Proof: ($\Longrightarrow$) Suppose that F is a differentiable fold. By proposition C.1, we must have that, at every critical point $u_z(t_0)$ of F, $h_z''(t_0) < 0$. By proposition

4.4, we have that $f'' \geq 0$. If $f''(0) > 0$, then by lemma 4.1, we have that $h_z''(t_0) < 0$. On the other hand, if $f''(0) = 0$, then we must have that $f'(0) \neq \lambda_1$, otherwise 0 would be such that $h_z'(0) = 0 = h_z''(0)$ as shown at the end of section 4.2 that is, 0 is a critical point of h_z but not a fold point.

($\Longleftarrow$) Since $f'' \geq 0$ and $\overline{f'(\mathbb{R})} = [a, b]$ with $a < \lambda_1 < b$, we have that

$$\lim_{s \to -\infty} f'(s) = a \ , \quad \lim_{s \to +\infty} f'(s) = b$$

so that, by proposition 2.14, $\lim_{|t| \to \infty} h_z(t) = -\infty$. Now, by lemmas 4.1 and 4.2, both the conditions $f''(0) > 0$ and $f''(0) = 0$ with $f'(0) \neq \lambda_1$ imply that at a critical point t_0 of h_z, we have that $h_z''(t_0) < 0$. Apply proposition C.1 and conclude that F is a differentiable fold.

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4.4 Geometry of heights for C^2 convex nonlinearities

This section contains some important results that we use in chapter 5.

Here we say something about the behaviour of $h_z'(t)$ at infinity and discuss the sign of h_z'' and the monotonicity of λ_1. In this section, B is given by theorem 3.6.

Proposition 4.7 *Suppose that $f \in C^1(\mathbb{R})$ and*

$$\lim_{s \to -\infty} \frac{f(s)}{s} = a < \lambda_1 < \lim_{s \to +\infty} \frac{f(s)}{s} = b \leq B$$

Then, given $z \in H_Y$, we have

$$\lim_{t \to -\infty} h_z'(t) = \lambda_1 - a \ , \quad \lim_{t \to +\infty} h_z'(t) = \lambda_1 - b.$$

Proof: Recall that, from equation (3.2),

$$-Lu_z'(t) - f'(u_z(t))u_z'(t) = h_z'(t)\phi_1.$$

Note that $h_z'(t)$ is bounded since $DF(u_z(t))$ is uniformly bounded and, from theorem 3.6, $u_z'(t)$ is bounded in X ($\Phi(z + t\phi_1) = u(z, t) = u_z(t)$ is Lipschitz). Take a sequence $t_k \to +\infty$. There exists a subsequence (already relabelled) such that $h_z'(t_k) \to h_\infty$ and $u_z'(t_k) \to u_\infty$ in $C_0(\overline{\Omega})$.

Observe that $f'(u_z(t_k)) \to b$ in Ω (recall that $u_z(t_k)/t_k \to \phi_1$ by lemma 2.13) and is bounded by B. It follows that

$$f'(u_z(t_k))u_z'(t_k) \to bu_\infty \text{ , and } \quad |f'(u_z(t_k))u_z'(t_k)| \leq b \max_{k\in\mathbb{N}} \|u_z'(t_k)\|_{C_0(\overline{\Omega})}.$$

so that, by the dominated convergence theorem, $f'(u_z(t_k))u_z'(t_k) \to bu_\infty$ in Y.

Apply $L^{-1} : Y \to X$ to the equation in the beginning of the proof to obtain

$$u_z'(t_k) = L^{-1}\Big(f'(u_z(t_k))u'(z,t_k)\Big) + \frac{h_z'(t_k)}{\lambda_1}\phi_1$$

with the right hand side converging to $bL^{-1}u_\infty + h_\infty\phi_1$ in X, that is, $u_z'(t_k)$ converges in X. As $u_z'(t_k) \to u_\infty$ in Y, we have that $u_z'(t_k) \to u_\infty$ in X.

Finally, making $k \to \infty$, we have

$$-Lu_z'(t_k) - f'(u_z(t_k))u_z'(t_k) = h_z'(t_k)\phi_1 \to -Lu_\infty - bu_\infty = h_\infty\phi_1$$

so that u_∞ is parallel to ϕ_1. As $u_z'(t_k) = w_z'(t_k) + \phi_1 \to c\phi_1 = u_\infty$, we have that $c = 1$, that is, $u_\infty = \phi_1$. Then we obtain $-L\phi_1 - b\phi_1 = (\lambda_1 - b)\phi_1 = h_\infty\phi_1$.

As $\{t_k\}_k$ is an arbitrary sequence, it follows that $t \to +\infty$ implies that $\|w_z'(t)\|_X \to 0$, $\|u_z'(t) - \phi_1\|_X \to 0$ and $h_z'(t) \to \lambda_1 - b$.

The case $t \to -\infty$ is handled similarly with $h_z'(t) \to \lambda_1 - a$.

■

Proposition 4.8 *Suppose that $f'' \geq 0$, $f''(0) > 0$ and that $\overline{f'(\mathbb{R})} = [a,b]$, $a < \lambda_1 < b \leq B$. For every $z \in H_Y$, the corresponding function $t \mapsto h_z(t)$ is concave up until its single critical point t_0.*

Proof: First we prove that, for $t < t_0$, the maximum principle is valid for $DF(u_z(t))$. Note that $t \mapsto \lambda_1(DF(u_z(t)))$ is a continuous function that reaches 0 only once, more precisely, at $t = t_0$. As we have seen in the first proof of lemma 4.1, $\lambda_1'(z,t_0) < 0$, so that it is positive for $t < t_0$ and negative for $t > t_0$. It follows that the maximum principle is valid for $DF(u_z(t))$ if $t < t_0$ [13, theorem 1.1].

Recall that $F(u_z(t)) = z + h_z(t)\phi_1$. Differentiate it twice on t to obtain

$$DF(u_z(t))u_z''(t) = -Lu_z''(t) - f'(u_z(t))u_z''(t) = f''(u_z(t))u_z'(t)^2 + h_z''(t)\phi_1.$$

Suppose that, for some $t_- < t_0$, $h_z''(t_-) \geq 0$. It follows that the right hand side is positive. By the maximum principle $u_z''(t_-) \geq 0$. The strong maximum principle implies that either $u_z'(t_-) > 0$ or $u_z''(t_-) \equiv 0$. As the right hand side is not identically zero and $DF(u_z(t_-))$ is bijective, it follows that $u_z''(t_-) > 0$. This is a contradiction with the fact that $u_z''(t_-) = w_z''(t_-) \in H_X$ (recall that $u_z'(t) = w_z'(t) + \phi_1 \in H_X \oplus V_X$ and $w \in H_X \implies \langle w, \phi_1^* \rangle = 0$).

We conclude that, $h_z''(t) < 0$ for every $t \leq t_0$.

■

Lemma 4.9 *Let $f'' > 0$, $f'(\mathbb{R}) = (a,b)$ and $a < \lambda_1 < b \leq B$. Given a fiber u_z let $u_0 = u_z(t_0)$ be the single critical point of F contained in it. Then, for all $t \leq t_0$ we have $u_z'(t) > 0$.*

Proof: If $t = t_0$, $DF(u_0)u_z'(t_0) = 0$ so that, by lemma 3.9, $u_z'(t_0) > 0$. For $t < t_0$ we have that $\lambda_1(z,t) > 0$ so that the maximum principle is valid for $DF(u_z(t))$ and $h_z'(t) = \lambda_1(z,t)p(z,t) > 0$. Then,

$$DF(u_z(t))u_z'(t) = h_z'(t)\phi_1 > 0.$$

An application of the maximum principle and the strong maximum principle provides $u_z'(t) > 0$.

■

Proposition 4.10 *Under the hypothesis of lemma 4.9, the restriction of $t \mapsto \lambda_1(z,t)$ to $(-\infty, t_0)$ is strictly decreasing.*

Proof: Recall equation (4.1)

$$\lambda_1'(z,t) = -\frac{\langle f''(u_z(t))u_z'(t)\phi_1(z,t), \phi_1^*(z,t)\rangle}{\langle \phi_1(z,t), \phi_1^*(z,t)\rangle}.$$

By lemma 4.9, for $t \leq t_0$ we have $\lambda_1'(z,t) < 0$.

■