5
A topological fold

Take some convex function $f : \mathbb{R} \to \mathbb{R}$ with Lipschitz bounds a and b such that

$$a \leq \frac{f(x) - f(y)}{x - s} \leq b \leq B \tag{5.1}$$

where B is as in theorem 3.6. Suppose that

$$a = \lim_{s \to -\infty} \frac{f(s)}{s} < \lambda_1 < \lim_{s \to +\infty} \frac{f(s)}{s} = b. \tag{5.2}$$

In the first section we prove a counting result with f as above. In the second section we define strict convexity at a point $s \in \mathbb{R}$ and suppose that f is strictly convex at 0 to prove that F is a topological fold.

5.1
A count of solutions for convex Lipschitz nonlinearities

Theorem 5.1 *Suppose that f is convex (but not necessarily strictly convex) and satisfies hypotheses* (5.1) *and* (5.2). *Then, for every $z \in H_Y$ there exists $t_z \in \mathbb{R}$ such that the equation $F(u) = -Lu - f(u) = z + t\phi_1$ for $u \in X$ has the following count of solutions:*

1. *no solution if $t > t_z$,*
2. *exactly one solution or a curve of solutions if $t = t_z$,*
3. *exactly two solutions or one solution and a curve of solutions if $t < t_z$.*

In order prove theorem 5.1, we approximate F by functions

$$F_\delta : X \to Y \ , \quad u \mapsto -Lu - f_\delta(u)$$

$\delta > 0$, where f_δ is a regularization of f defined by equation (A.1).

Proof: For $F_{\delta,t}(w) = F_\delta(w + t\phi_1)$, define

$$PF_{\delta,t} : H_X \to H_Y \ , \quad w \mapsto PF_\delta(w + t\phi_1).$$

Given $z \in H_Y$, the fiber of F_δ related to z is defined by

$$u_{\delta,z}(t) = w_{\delta,z}(t) + t\phi_1 := (PF_{\delta,t})^{-1}(z) + t\phi_1.$$

From equation (2.6), the height function of F_δ related to z is given by

$$h_{\delta,z}(t) := \langle F_\delta(u_{\delta.z}(t)), \phi_1^* \rangle = \lambda_1 t - \frac{\langle f_\delta(u_{\delta,z}(t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle}.$$

We prove that, given $z \in H_Y$, $\delta \to 0$ implies that $h_{\delta,z}(t) \to h_z(t)$ uniformly. Suppose $\|\phi_1^*\|_{Y^*} = 1$. By the Hölder inequality and the Lipschitz bounds on f_δ

$$\begin{aligned}
&|h_{\delta,z}(t) - h_z(t)| \langle \phi_1, \phi_1^* \rangle \\
=&|\langle f_\delta(u_{\delta,z}(t)) - f(u_z(t)), \phi_1^* \rangle| \\
\leq&\|f_\delta(u_{\delta,z}(t)) - f(u_z(t))\|_Y \, \|\phi_1^*\|_{Y^*} \\
\leq&\|f_\delta(u_{\delta,z}(t)) - f_\delta(u_z(t))\|_Y + \|f_\delta(u_z(t)) - f(u_z(t))\|_Y \\
\leq&(|a| + |b|) \, \|u_{\delta,z}(t) - u_z(t)\|_Y + \|f_\delta(u_z(t)) - f(u_z(t))\|_Y \\
\leq&(|a| + |b|) \, \|w_{\delta,z}(t) - w_z(t)\|_Y + \|f_\delta(u_z(t)) - f(u_z(t))\|_Y
\end{aligned}$$

By lemma 2.8, $\delta \to 0$ implies that $\|f_\delta(u_z(t)) - f(u_z(t))\|_Y \to 0$ uniformly on t. We are left to prove that, for fixed $z \in H_Y$, $\delta \to 0$ implies that $\|w_{\delta,z}(t) - w_z(t)\|_Y \to 0$ uniformly on t.

Note that, for all $t \in \mathbb{R}$, $PF_t(w_z(t)) = z$ and $PF_{\delta,t}(w_{\delta,z}(t)) = z$, so that

$$\begin{aligned}
0 &= \|PF_{\delta,t}(w_{\delta,z}(t)) - PF_t(w_z(t))\| \\
&\geq \|PF_{\delta,t}(w_{\delta,z}(t)) - PF_{\delta,t}(w_z(t))\|_Y - \|PF_{\delta,t}(w_z(t)) - PF_t(w_z(t))\|_Y.
\end{aligned}$$

Recall that $PF_{\delta,t}(w_z(t)) = PF_\delta(w_z(t) + t\phi_1) = PF_\delta(u_z(t))$, so that

$$\begin{aligned}
\|PF_\delta(u_z(t)) - PF(u_z(t))\|_Y &= \|PF_{\delta,t}(w_z(t)) - PF_t(w_z(t))\|_Y \\
&\geq \|PF_{\delta,t}(w_{\delta,z}(t)) - PF_{\delta,t}(w_z(t))\|_Y.
\end{aligned}$$

By propositon A.1, $f_\delta : \mathbb{R} \to \mathbb{R}$ satisfies

$$a \leq \frac{f_\delta(x) - f_\delta(y)}{x - y} \leq b \leq B \ , \quad x \neq y.$$

By proposition 2.5, $(PF_{\delta,t})^{-1}$ is Lipschitz with constant C not depending on δ

and t. It follows that, for all $t \in \mathbb{R}$ and $\delta > 0$,

$$\|PF_{\delta,t}(w_{\delta,z}(t)) - PF_{\delta,t}(w_z(t))\|_Y \geq C\|w_{\delta,z}(t) - w_z(t)\|_X,$$

and we have

$$\|PF_\delta(u_z(t)) - PF(u_z(t))\|_Y \geq C\|w_{\delta,z}(t) - w_z(t)\|_X.$$

As $P : Y \to Y$ is linear continuous and, by lemma 2.8, $\delta \to 0$ implies that $F_\delta \to F$ uniformly, we have $\|w_{\delta,z}(t) - w_z(t)\|_X \to 0$ uniformly.

Clearly, $f_\delta \in C^\infty(\mathbb{R})$ so that the height functions $h_{\delta,z}$ are $C^\infty(\mathbb{R})$. By proposition 2.14, $\lim_{|t|\to\infty} h_{\delta.z}(t) = -\infty$ so that it has some critical point t_z. By lemma A.3, f_δ is strictly convex, so that the heights $h_{\delta,z}$ are strictly concave for $t < t_{\delta,z}$ (proposition 4.8). It follows that the height h_z is concave up until its lowest point of local maximum, say, t_z. We conclude that, for each $g \in \{F(u_z(t)) : t \in \mathbb{R}\}$ there exists a single t such that $t < t_z$ and $F(u_z(t)) = g$.

On the other hand, $h_{\delta,z}(t)$ is decreasing for $t \geq t_{\delta,z}$, so that h_z is also decreasing for $t \geq t_z$. It follows that, for every $g \in \{F(u_z(t)) : t \in \mathbb{R}\}$, there exists either a segment $[t_1, t_2]$ such that for $t \in [t_1, t_2]$ we have $F(u_z(t)) = g$ or a single point $t \geq t_z$ for which $F(u_z(t)) = g$.

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5.2 An exact count of solutions

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex and satisfies (5.1) and (5.2). Define, for all $s \in \mathbb{R}$, the set

$$D(s) = \{d \in \mathbb{R} : \text{for all } r \in \mathbb{R},\ f(r) \geq f(s) + d(r - s)\}.$$

As f is convex, $D(s) \neq \emptyset$ and is bounded for each $s \in \mathbb{R}$. Define

$$d_-(s) := \inf D(s)\ , \quad d_+(s) := \sup D(s).$$

Definition 5.2 *A real function f is strictly convex at $s_0 \in \mathbb{R}$ if there exists some neighbourhood around s_0 at which f is convex and, for all s_1, s_2 in that neighbourhood such that $s_1 < s_0 < s_2$, we have $d_+(s_1) < d_-(s_2)$.*

Theorem 5.3 *Suppose that f is convex and satisfies hypotheses* (5.1) *and* (5.2). *If either f is strictly convex in an interval around* 0 *or is strictly convex at* 0 *with $d_-(0), d_+(0) \neq \lambda_1$ then, F is a topological fold.*

Proof: By theorem 2.4, there exists an homeomorphism Φ such that, for all $z \in H_Y$, $(F \circ \Phi)(z + t\phi_1) = z + h(z,t)\phi_1$. Also, $\lim_{|t|\to\infty} h_z(t) = -\infty$. We need to prove that every local extreme point of h_z is a strict maximum point. By theorem 5.1, we only need to exclude the possibility of appearing plateaus in the graph of the height functions h_z, that is, a segment on which h_z is constant.

Suppose by contradiction that, given $z \in H_Y$, there exists a segment $[t_-, t_+]$ for which, for all $t \in [t_-, t_+]$, $F(u_z(t)) = z + h_z(t)\phi_1 = z + c\phi_1$. Suppose that f is strictly convex at a neighbourhood around 0.

For simplicity, set $u(t) = w(t) + t\phi_1 := u_z(t) = w_z(t) + t\phi_1$. Define $L^\infty(\Omega)$ potentials

$$V_{t_0,t}(x) := \begin{cases} \dfrac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} & , \ \{x \in \Omega \ : \ u(t)(x) \neq u(t_0)(x)\} \\ d_-(u(t)) & , \ \{x \in \Omega \ : \ u(t)(x) = u(t_0)(x)\} \end{cases} .$$

It follows that $V_{t_0,t}$ has the same bounds as the quotient $(f(s)-f(s_0))/(s-s_0)$. Note that, for all $t_0 < t$ with $t_0, t \in [t_-, t_+]$, we have

$$(L + V_{t_0,t})(u(t) - u(t_0)) = 0.$$

By proposition 3.5, we have that $u(t) - u(t_0)$ is a principal eigenfunction associated to the elliptic operator $L + V_{t_0,t} : X \to Y$, so that it has sign. We claim that it has the same sign as $t - t_0$. As $\langle w(t) - w(t_0), \phi_1^* \rangle = 0$,

$$\langle u(t) - u(t_0), \phi_1^* \rangle = \langle (t - t_0)\phi_1, \phi_1^* \rangle = (t - t_0)\langle \phi_1, \phi_1^* \rangle.$$

Since $\langle \phi_1, \phi_1^* \rangle > 0$ we have proved our claim.

As a consequence, if $t_- \leq t_1 < t_2 < t_3 < t_4 \leq t_+$ we have that

$$(L + V_{t_1,t_2})(u(t_2) - u(t_1)) = 0 \ , \quad (L + V_{t_2,t_3})(u(t_3) - u(t_2)) = 0 \, ,$$

$$(L + V_{t_3,t_4})(u(t_4) - u(t_3)) = 0$$

so that $\lambda_1(L + V_{t_1,t_2}, \Omega) = 0 = \lambda_1(L + V_{t_3,t_4}, \Omega)$, by proposition 3.5. Moreover,

$$u(t_1) < u(t_2) < u(t_3) < u(t_4). \tag{5.3}$$

We prove that $V_{t_3,t_4} \geq V_{t_1,t_2}$ with $V_{t_3,t_4} \neq V_{t_1,t_2}$ so that

$$\lambda_1(L + V_{t_1,t_2}, \Omega) = 0 < \lambda_1(L + V_{t_3,t_4}, \Omega),$$

which is a contradiction. Note that the potentials V_{t_1,t_2} and V_{t_3,t_4} are continuous functions on $\overline{\Omega}$ from inequalities (5.3) and the definition of $V_{t_0,t}$. Clearly,

$$d_-(u(t_1)) \leq V_{t_1,t_2} \leq d_+(u(t_2)) \ , \quad d_-(u(t_3)) < V_{t_3,t_4} < d_+(u(t_4)).$$

Since $u(t_2) < u(t_3)$, as f is strictly convex around 0, we have that $d_+(u(t_2)) < d_-(u(t_3))$ at some open subset of Ω, that is, $V_{t_1,t_2} \leq V_{t_3,t_4}$ with $V_{t_1,t_2} \neq V_{t_3,t_4}$ so that $\lambda_1(L + V_{t_1,t_2}, \Omega) > \lambda_1(L + V_{t_3,t_4}, \Omega)$.

Now, suppose that f is strictly convex at 0 and that $d_+(0), d_-(0) \neq \lambda_1$. Suppose that there exist sequences of positive numbers $\{p_k\}_k$ and negative numbers $\{n_k\}_k$ such that $p_k, n_k \to 0$ and f is strictly convex at p_k and n_k for every $k \in \mathbb{N}$.

There exists some $k \in \mathbb{N}$ such that the images of neither $u(t_2)$ nor $u(t_3)$ contains p_k or n_k(otherwise, we would have $u(t_2) = u(t_3) = 0$). Without loss, suppose that there exists some $x_0 \in \Omega$ such that $u(t_3)(x_0) = n_k$. Note that $d_-(n_k + s) > d_+(n_k - s)$ for all $s \in \mathbb{R}$, since f convex in $\mathbb{R}$ and strictly convex at n_k. Since $u(t_2)(x_0) < u(t_3)(x_0)$, it follows that $d_+(u(t_2)) < d_-(u(t_3))$ at a heighbourhood around x_0. Then, we have that $V_{t_1,t_2} \leq V_{t_3,t_4}$ and, at a neighbourhood around x_0, $V_{t_1,t_2} < V_{t_3,t_4}$, so that $\lambda_1(L + V_{t_1,t_2}, \Omega) = 0 > \lambda_1(L + V_{t_3,t_4}, \Omega)$, which is a contradiction.

Finally, suppose that there exists some interval containing 0 such that f is not strictly convex at any point. Let $[\alpha, \beta]$ be the maximal interval containing 0 with that property. It follows that

$$\frac{f(x) - f(y)}{x - y} = c \neq \lambda_1(L, \Omega) \ , \quad x, y \in [\alpha, \beta].$$

First, suppose that there exists some $t_0 \in [t_-, t_+]$ such that $\alpha \leq u(t_0) \leq \beta$. Then, by equation (5.3), for some $t \in (t_-, t_+)$ we have $\alpha < u(t) < \beta$. It follows that, for some $\tilde{t} > t$, $\alpha < u(t) < u(\tilde{t}) < \beta$. That is, for all $s \in [t, \tilde{t}]$ we have $\alpha < u(s) < \beta$. Now, suppose that

$$-L(u(\tilde{t}) - u(t)) - V_{t,\tilde{t}}(u(\tilde{t}) - u(t)) = -L(u(\tilde{t}) - u(t)) - c(u(\tilde{t}) - u(t)) = 0$$

that is, $c = \lambda_1(L, \Omega)$ which is a contradiction.

If there is no $t \in [t_-, t_+]$ such that $u(t) \in [\alpha, \beta]$. Take $t_1 < t_2 < t_3 < t_4$ as before and suppose, without loss, that there exists some $x \in \Omega$ such that $u(t_3)(x) \geq \beta$. Note that f is strictly convex at β. By the continuity of $u(t_3)$ and the Dirichlet condition on $u(t_3)$, there exists some $x_0 \in \Omega$ such that $u(t_3)(x_0) = \beta$. Note that $u(t_2) < u(t_3)$ so that $d_+(u(t_2)(x_0)) < d_-(u(t_3)(x_0))$, so that $V_{t_1,t_2} \leq V_{t_3,t_4}$ and $V_{t_1,t_2} \neq V_{t_3,t_4}$. This is a contradiction since

$$\lambda_1(L + V_{t_1,t_2}, \Omega) = 0 > \lambda_1(L + V_{t_3,t_4}, \Omega) = 0.$$

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