6
Technical tools

6.1
Regularity of the nonlinearity

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $X := W^{2,n}(\Omega) \cap C_0(\overline{\Omega})$ and $Y := L^n(\Omega)$.

We begin by relating the regularity of $f : \mathbb{R} \to \mathbb{R}$ to the regularity of

$$F : X \to Y \ , \quad u \mapsto -L - f(u).$$

Proposition 6.1 *If $f \in C^k(\mathbb{R})$ then the map below is $C^k(X, Y)$*

$$f : X \mapsto Y \ , \quad u \mapsto N_f(u) := f(u).$$

Proof: Set $Tv = f'(u)v$ — this is the candidate for $DN_f(u)$. First, $T : X \to Y$ is well defined and a bounded operator: indeed, $f'(u)$ is bounded and $v \in X \subset Y$. To see that $DN_f(u) = T$, we must check that

$$r(t) = \frac{N_f(u + tv) - N_f(u) - tTv}{t} \to 0 \quad \text{in } Y \ .$$

Since f is Lipschitz, say with constant M, $|r(t)| \leq M|v| + |f'(u)||v| \leq 2M|v| \in Y$ — the result now follows from dominated convergence.

Now, we use the continuous embedding $X \hookrightarrow C(\overline{\Omega})$. To show that $u \mapsto DN_f(u)$ is continuous, we have to show that

$$\|u - u_0\|_X \to 0 \implies \sup_{\|v\|_X = 1} \|(f'(u) - f'(u_0))v\|_Y \to 0.$$

Since $||u - u_0||_\infty \to 0$ and f' is continuous, use uniform continuity to get

$$\|f'(u) - f'(u_0)\|_\infty \to 0$$

and the rest is easy.

If $k \geq 2$, we consider the second derivative. Let $H(u)(v, w) = f''(u)vw$ be the candidate. We show that it is well defined. Since $u, v \in X \hookrightarrow C(\overline{\Omega})$, $f''(u)$ is bounded:

$$\|f''(u)vw\|_Y \leq C\|v\|_\infty \|w\|_Y \leq \tilde{C}\|v\|_X \|w\|_X.$$

To see that $D^2 N_f(u) = H$, we must check that

$$s(t) = \frac{DN_f(u + tw)v - DN_f(u)v - tH(v, w)}{t} \to 0 \quad \text{in } Y \ (x \in \Omega)\ .$$

The estimate here is more delicate than the one for the first derivative. Note that f' is a $C^1(\mathbb{R})$ function, so it is Lipschitz on compact sets. For $|t| < 1$ we have that $|u + tw|$ is bounded. So, there exists $M > 0$ such that,

$$|f'(u + tw) - f'(u)| \leq M|u + tw - u| = M|t|\,|w|.$$

Recall that $|f''(u)|$ is bounded. A simple computation provides

$$|s(t)| \leq M|v|\,|w| \in Y$$

and once more use the dominated convergence theorem.

Finally, we show continuity of $D^2 N_f$. We need to prove that

$$\|u - u_0\|_X \to 0 \implies \sup_{\|v\|_X = \|w\|_X = 1} \|(f''(u) - f''(u_0))vw\|_Y \to 0.$$

Again, since $||u - u_0||_\infty \to 0$ and f'' is continuous, use uniform continuity to get $\|f''(u) - f''(u_0)\|_\infty \to 0$. The inequality below ends the proof

$$\begin{aligned}
\|(f''(u) - f''(u_0))vw\|_Y &\leq \|f''(u) - f''(u_0)\|_\infty \|vw\|_Y \\
&\leq C\|f''(u) - f''(u_0)\|_\infty \|v\|_\infty \|w\|_Y \\
&\leq \tilde{C}\|f''(u) - f''(u_0)\|_\infty \|v\|_X \|w\|_X \\
&\leq \tilde{C}\|f''(u) - f''(u_0)\|_\infty.
\end{aligned}$$

The proof for $f \in C^k(\mathbb{R})$ is analogous.

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Corollary 6.2 *If $f : \mathbb{R} \to \mathbb{R}$ is C^k, then $F : X \to Y$ is C^k.*

Now we state a result about the continuity of the potential $f'(u)$ when $u \in Y$ instead of X. This a consequence of proposition 6.1 and the fact that $f' : \mathbb{R} \to \mathbb{R}$ is bounded.

Corollary 6.3 *If $f \in C^1(\mathbb{R})$, then the map below is continuous*

$$\mathsf{m}_{f'} : Y \mapsto \mathcal{L}(X, Y) \ , \quad u \mapsto f'(u)$$

where $f'(u)$ is the multiplication operator in $\mathcal{L}(X, Y)$

6.2 Proof of lemma 2.8

Proof: (lemma 2.8) Take $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$a \leq \frac{f(x) - f(y)}{x - y} \leq b \ , \quad x \neq y$$

Consider functions f_δ as defined in (A.1).

Recall that

$$F : X \to Y \ , \quad u \mapsto -Lu - f(u)$$

$$F_\delta : X \to Y \ , \quad u \mapsto -Lu - f_\delta(u)$$

We want to prove that $F_\delta \to F$ uniformly.

It suffices to prove that $\delta \to 0$ implies that $f_\delta : X \to Y$, $u \mapsto f_\delta(u)$ converges uniformly to $f : X \to Y$, $u \mapsto f(u)$,

By proposition A.1, there exists some function $c : \{x > 0\} \to \{x > 0\}$ such that $\delta \to 0$ implies that $c(\delta) \to 0$ and, for all $x \in \mathbb{R}$, we have $|f(x) - f_\delta(x)| < c(\delta)$.

Take $u \in X$. By the estimate above, given $\epsilon > 0$, there exists some $\delta > 0$ such that, for all $x \in \Omega$, we have

$$|f_\delta(u(x)) - f(u(x))| < c(\delta) < \frac{\epsilon}{|\Omega|^{\frac{1}{n}}}$$

so that $\|f_\delta(u) - f(u)\|_Y < \epsilon$. Now it is easy to obtain a sequence $f_k \to f : X \to Y$ uniformly where $f_k : \mathbb{R} \to \mathbb{R}$ is C^1 (actually, C^∞).

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6.3
Regularity of the principal eigenpair

Here we assume that $X = W^{2,n}(\Omega) \cap C_0(\overline{\Omega})$ and $Y = L^n(\Omega)$ and L is an elliptic operator as defined in the introduction.

Let A be a Banach space and $q : A \to \mathcal{L}(X,Y)$ be a C^k map where $q(u)$ is a bounded potential, that is, $q(u) \in L^\infty(\Omega)$. Clearly, the map

$$T : A \to \mathcal{L}(X,Y)\ , \quad u \mapsto -L - q(u)$$

is C^k and $T(u)$ is an elliptic operator as in [13] and hence has a principal eigenvalue.

Fix $u_0 \in A$ and suppose that $|q(u)| \leq M - 1$. Proposition B.7 assures the existence of a C^k function $\lambda : B(u_0) \subset A \to \mathbb{R}$ satisfying $\lambda(u_0) = \lambda_1(T(u_0), \Omega)$. In proposition 6.4, we prove that there is a possibly smaller ball containing u_0 such that the restriction of λ to that ball satisfies $\lambda(u) = \lambda_1(T(u), \Omega)$. We will use this result to prove that the principal eigenpair of $T(u)$ has a C^k dependence on $u \in A$.

Proposition 6.4 *Let A be a Banach space and $q : A \to \mathcal{L}(X,Y)$ be a C^k map where $q(u) : X \to Y$ are uniformly bounded potentials, that is, $q(u) \in L^\infty(\Omega)$ and for some $M > 0$, $|q(u)| \leq M - 1$. Then, for all $u_0 \in A$, there exists a C^k function $\lambda : B(u_0) \subset A \to \mathbb{R}$ such that $\lambda(u) = \lambda_1(-L - q(u), \Omega)$.*

Proof: Consider the complexifications of X and Y, that is, the Banach spaces

$$X_{\mathbb{C}} := (\{u + iv : u, v \in X\}, \|u + iv\|_{X_{\mathbb{C}}} := (\|u\|_X + \|v\|_Y)^{\frac{1}{2}})$$

$$Y_{\mathbb{C}} := (\{u + iv : u, v \in Y\}, \|u + iv\|_{Y_{\mathbb{C}}} := (\|u\|_Y + \|v\|_Y)^{\frac{1}{2}}).$$

Denote the eigenvalues of $-L - q(u) + M : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ by $\lambda_i(u) + M$ ordered in a way that $|\lambda_i(u) + M| \leq |\lambda_{i+1}(u) + M|$. Observe that, $\lambda_1(u) = \lambda_1(L + q(u), \Omega)$. Also, since $|q(u)| \leq M - 1$, we have

$$1 \leq \lambda_1(u) + M < Re(\lambda_i(u)) + M\ , \quad \text{for } i > 1. \tag{6.1}$$

In conclusion, all the eigenvalues of $-L - q(u) + M$ have positive real part.

Consider the function

$$\Psi : A \to \mathcal{L}(Y_{\mathbb{C}}, Y_{\mathbb{C}})\ , \quad u \mapsto (-L - q(u) + M)^{-1}.$$

As $f \in C^k(\mathbb{R})$ implies that $u \mapsto q(u)$ is C^k, we have that Ψ is C^k.

Consider complex valued functions

$$\xi : \mathbb{C} - \{-M\} \to \mathbb{C}\ ,\ \xi(z) = \frac{1}{z+M} \quad ; \quad \xi^{-1} : \mathbb{C} - \{0\} \to \mathbb{C}\ ,\ \xi^{-1}(z) = \frac{1}{z} - M.$$

Observe that the eigenvalues of $\Psi(u)$ and $-L - q(u)$ are given respectively by

$$\tau_i(u) := \xi(\lambda_i(u)) = \frac{1}{\lambda_i(u) + M}\ , \quad \xi^{-1}(\tau_i(u)) = \lambda_i(u).$$

Note that $\sigma(\Psi(u_0)) = \tau_i(u_0) \cup \{0\}$ and, by equation 6.1, $Re(\tau_i(u)) \leq 1$.

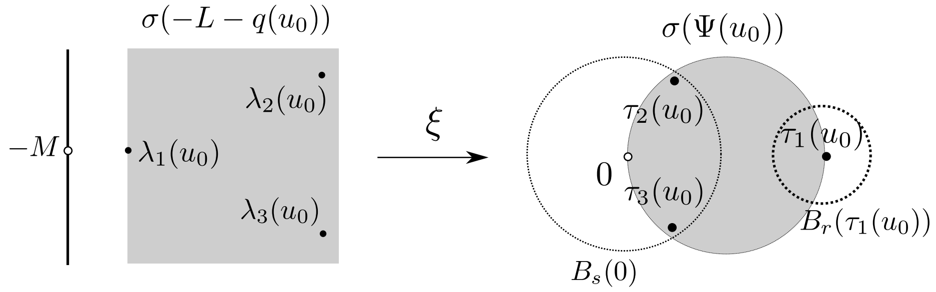

Figure 6.1: The image of $\xi_{|\sigma(-L-q(u_0))}$.

Now fix $u_0 \in A$. We provide neighbourhoods for $\tau_1(u_0)$ and $\sigma(\Psi(u_0)) - \{\tau_1(u_0)\}$ where we use proposition B.6 to obtain the desired result.

Since $Re(\lambda_2(u_0) + M) > \lambda_1(u_0) + M \geq 1$ (equation 6.1) we have that

$$0 < Re(\tau_2(u_0)) < \tau_1(u_0).$$

Take a ball $B_r(\tau_1(u_0)) \subset \mathbb{C}$ of radius

$$r = \frac{\tau_1(u_0) - Re(\tau_2(u_0))}{2}.$$

Also, for $i > 1$,

$$\lambda_1(u_0) + M < |\lambda_2(u_0) + M| \leq |\lambda_i(u_0) + M| \implies |\tau_i(u_0)| \leq |\tau_2(u_0)| < \tau_1(u_0).$$

Consider the ball $B_s(0) \subset \mathbb{C}$ of radius $s = |\tau_2(u_0)| + r/4$. and observe that $\sigma(\Psi(u_0)) - \{\tau_1(u_0)\}$ is contained in $B_s(0)$.

Now, we note that

$$\sigma(\Psi(u_0)) \subset B_r(\tau_1(u_0)) \cup B_s(0)\ , \quad B_r(\tau_1(u_0)) \cap B_s(0) = \emptyset.$$

By proposition B.6 we have that, for a small neighbourhood of $V(\Psi(u_0))$, $T \in V_1(\Psi(u_0)) \subset \mathcal{L}(Y_{\mathbb{C}}, Y_{\mathbb{C}})$ implies that $\sigma(T) \subset B_r(\tau_1(u_0)) \cup B_s(0)$.

Set γ_1 as the positively oriented parametrization of $\partial B_r(\tau_1(u_0))$. By lemma B.5, there exists a neighbourhood $V_2(\Psi(u_0)) \subset \mathcal{L}(Y_{\mathbb{C}}, Y_{\mathbb{C}})$ such that, for all $T \in V_2(\Psi(u_0))$ we have $P_{\gamma_1}(T)$ is unidimensional, that is, there exists a single eigenvalue of T contained in $B_r(\tau_1(u_0))$. Moreover, still from lemma B.5, this eigenvalue is simple. Another important property of this eigenvalue is that it is the one of largest modulus in $\sigma(T)$.

By proposition B.8, there exists $V_3(\Psi(u_0)) \subset \mathcal{L}(Y_{\mathbb{C}}, Y_{\mathbb{C}})$ in which we can define a C^k function $(\lambda_p, \phi_p) : V_3(\Psi(u_0)) \to B_r(\tau_1(u_0))$ such that $T\phi_p(T) = \lambda_p(T)\phi_p(T)$ where $\lambda_p(T)$ is a simple eigenvalue of T. Also, $\lambda_p(T)$ is the eigenvalue of largest modulus contained in $\sigma(T)$ and is C^k dependent on T.

Take $V(\Psi(u_0)) = V_1(\Psi(u_0)) \cap V_2(\Psi(u_0)) \cap V_3(\Psi(u_0))$ and note that, for all $T \in V(u_0)$ we have $\sigma(\Psi(u_0)) \subset B_r(\tau_1(u_0)) \cap B_s(0)$ and the point spectrum of $\Psi(u_0)$ has a single point contained in $B_r(\Psi(u_0))$.

Finally, take a neighbourhood $B(u_0)$ of $u_0 \in A$ such that $u \in B(u_0)$ implies that $\Psi(u) \in V(u_0)$. It follows that there exists a single eigenvalue $\tau_1(u) \in B_r(\tau_1(u_0))$ and it is also simple.

Note that the spectrum of $\Psi(u)$ for $u \in B(u_0) \subset A$ is given by $\{0\}$ and a sequence of eigenvalues converging to 0 with all of them having positive real part (equation (6.1)) . It follows that $\tau_1(u)$, the eigenvalue of largest modulus of $\Psi(u)$, must be contained in $B_r(\tau_1(u_0))$. Also, it is simple, isolated and the only eigenvalue contained in $B_r(\tau_1(u_0))$.

All the other eigenvalues are contained in $B_s(0)$. As a consequence, $|\tau_i(u)| < |\tau_1|$ and $0 < Re(\tau_i(u)) < Re(\tau_1(u))$ for all $i > 1$. It follows that the eigenvalues of $-L - q(u) + M : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ are given by $\xi^{-1}(\tau_i(u)) = \lambda_i(u)$ satisfying, for $i > 1$, $Re(\lambda_1(u)) < Re(\lambda_i(u))$, so that $\lambda_1(u)$ is the principal eigenvalue of $-L - q(u)$. Hence, $\tau_1(u) = \lambda_p(\Psi(u)) \in \mathbb{R}$.

Now we obtain $\lambda_1(L + q(u), \Omega)$ as a composition of C^k functions

$$u \in B(u_0) \mapsto \Psi(u) \in \mathcal{L}(Y_{\mathbb{C}}, Y_{\mathbb{C}}) \mapsto \lambda_p(\Psi(u)) \in B_r(\tau_1(u_0)) \cap \mathbb{R} \mapsto \xi^{-1}(\lambda_p(\Psi(u)))$$

where $\xi^{-1}(\lambda_p(\Psi(u))) = \lambda_1(T(u), \Omega)$.

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We use propositions 6.4 and B.7 to show that, given $u_0 \in A$, there is a ball $B(u_0) \subset A$ where we can define a function

$$u \mapsto \big(\lambda_1(u), \phi_1(T(u),\Omega), \phi_1(T(u)^*,\Omega)\big)$$

which is as regular as $q : A \to \mathcal{L}(X,Y)$.

Take $u_0 \in A$. Set $\phi_1(T(u_0),\Omega) > 0$ with $\|\phi_1(T(u_0),\Omega)\|_X = 1$. Analogously, take $\phi_1(T(u_0)^*,\Omega) > 0$ with $\|\phi_1(T(u_0)^*,\Omega)\|_{Y^*} = 1$. Consider affine subspaces

$$W_{\phi_1(u_0)} := \phi_1(T(u_0),\Omega) + \langle \phi_1(T(u_0)^*,\Omega)\rangle^{\perp} \cap X$$

$$W_{\phi_1^*(u_0)} := \phi_1(T(u_0)^*,\Omega) + \langle \phi_1(T(u_0),\Omega)\rangle^{\perp} \cap Y^*.$$

Proposition 6.5 *Let A be a Banach space and $q : A \to \mathcal{L}(X,Y)$ be a C^k map where $q(u) : X \to Y$ are uniformly bounded potentials $q(u) \in L^\infty(\Omega)$, $|q(u)| \leq M$. Then, for every $u_0 \in A$, there exists a neighbourhood $B(u_0) \subset A$ in which is defined a C^k function*

$$(\lambda, \phi, \phi^*) : B(u_0) \to \mathbb{R} \times W_{\phi_1(u_0)} \times W_{\phi_1^*(u_0)}\ , \quad u \mapsto \big(\lambda(u), \phi(u), \phi^*(u)\big).$$

such that, for $T(u) = -L - q(u) : X \to Y$, we have

$$\lambda(u) := \lambda_1(T(u),\Omega)\ , \quad T(u)\phi(u) = \lambda(u)\phi(u)\ , \quad T(u)\phi^*(u) = \lambda(u)\phi^*(u).$$

Proof: If q is C^k, then $u \in A \mapsto T(u)$ is C^k. Given $u_0 \in A$, $T(u_0)$ has a simple isolated eingenvalue $\lambda_1(u_0)$ to which one can associate an eigenfunction $\phi_1(u_0) > 0$ (theorem 2.2).

By proposition B.7, there exists a neighbourhood $B(T(u_0)) \subset \mathcal{L}(X,Y)$ and a C^∞ function $T \mapsto (\lambda(T), \phi(T), \phi^*(T^*))$ satisfying

$$T(u_0)\phi_1(u_0) = \lambda_1(u_0)\phi_1(u_0)\ ,\ \phi_1(u_0) > 0$$

$$T(u_0)^*\phi_1(u_0) = \lambda_1(u_0)\phi_1^*(u_0)\ ,\ \phi_1^*(u_0) > 0$$

$$T\phi(T) = \lambda(T)\phi(T)\ , \quad T^*\phi^*(T) = \lambda(T)\phi^*(T^*).$$

Proposition 6.4 implies that, for a possibly smaller ball in $B(T(u_0))$, $\lambda(T(u)) = \lambda_1(T(u),\Omega)$ so that $\phi(T(u))$ and $\phi(T(u)^*)$ are eigenfunctions of $T(u)$ and $T(u)^*$ (repectively) associated to the principal eigenvalue $\lambda_1(T(u),\Omega)$ and thus, have sign.

As $u \in A \mapsto T(u)$ is C^k, the triple $(\lambda_1(T(u),\Omega), \phi(T(u)), \phi(T(u)^*)$ is locally C^k on $u \in B(u_0)$.

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Proposition 6.6 *Let $L : X \to Y$ be an elliptic operator. Let $\{q_k\}_k \in L^\infty(\Omega)$, $|q_k| \leq M$. If $\|q_k - q\|_Y \to 0$ then $\lambda_1(L + q_k, \Omega) \to \lambda_1(L + q, \Omega)$ and $\|\phi_1(L + q_k, \Omega) - \phi_1(L + q, \Omega)\|_X \to 0$ where $\phi_1(L + q_k, \Omega) > 0$ and $\|\phi_1(L + q_k, \Omega)\|_X = 1$. Moreover, $\|\phi^*_{1,k} - \phi^*_1\|_{Y^*} \to 0$ where $\phi^*_{1,k} := \phi_1((L+q_k)^*, \Omega) > 0$, $\phi^*_1 := \phi_1((L + q)^*, \Omega) > 0$ and $\|\phi^*_{1,k}\|_{Y^*} = \|\phi^*_1\|_{Y^*} = 1$.*

Proof: Without loss, suppose that $L : X \to Y$ is invertible. Set $\lambda_{1,k} := \lambda_1(L + q_k, \Omega)$ and $\phi_1(L + q_k, \Omega) = \phi_{1,k}$. The bound $|q_k| \leq M$, implies that $\lambda_1(L, \Omega) - M \leq \lambda_{1,k} \leq \lambda_1(L, \Omega) + M$. Together with the compact inclusion, $X \hookrightarrow L^\infty(\Omega)$ we obtain convergent subsequences

$$\{\lambda_{1,k_i}\}_i \ , \quad \{\phi_{1,k_i}(L + q_{k_i}, \Omega)\}_i := \{\phi_{k_i}\}_i.$$

Now, the already relabelled subsequences $\{\phi_{1,i}\}_i$, $\{\lambda_{1,i}\}_i$ and $\{q_i\}_i$ converge, respectively, to $\phi_{1,\infty}$, $\lambda_{1,\infty}$ and q, being the first convergence in $L^\infty(\Omega)$, the second one in $\mathbb{R}$ and the third one in Y. Clearly,

$$(-L - q_i)\phi_{1,i} = \lambda_{1,i}\phi_{1,i} \implies \phi_{1,i} = -L^{-1}(q_i + \lambda_{1,i})\phi_{1,i} \to L^{-1}\Big((q + \lambda_{1,\infty})\phi_{1,\infty}\Big).$$

so that $\phi_{1,i} \to \tilde{\phi}_{1,\infty}$ in X, and hence in $L^\infty(\Omega)$. Then we have that $\tilde{\phi}_{1,\infty} = \phi_{1,\infty}$. Observe that $\phi_{1,\infty} \neq 0$ because $\|\phi_{1,i}\|_X = 1 \to \|\phi_{1,\infty}\|_X = 1$.

Finally,

$$L\phi_{1,i} + (q_i + \lambda_{1,k})\phi_{1,i} \to L\phi_{1,\infty} + (q + \lambda_{1,\infty})\phi_{1,\infty} = 0$$

so that $\phi_{1,i} > 0$ and converges uniformly to $\phi_{1,\infty}$ implying that $\phi_{1,\infty} \geq 0$. By [13, theorem 2.3], if $\lambda_{1,\infty} \neq \lambda_1(L+q, \Omega)$, then $\phi_{1,\infty}$ would change sign. It follows that $\lambda_{1,\infty}$ is the principal eigenvalue of $L + q$ and $\phi_{1,\infty} > 0$ with $\|\phi_{1,\infty}\|_X = 1$, that is, $\phi_{1,\infty}$ is a principal eigenfunction of $-L - q$.

Now we prove that every subsequence of the original sequence $\{\lambda_{1,k}\}_k$ converges to $\lambda_{1,\infty}$. Suppose that there exists a subsequence $\{\lambda_{1,i}\}_i$ (already relabeled) converging to $\tilde{\lambda}_{1,\infty}$. By the same reasoning as above we conclude that $\tilde{\lambda}_{1,\infty}$ is an eigenvalue of $L + q$ with a positive eigenfunction associated to it. By [13, theorem 2.3], we conclude that $\lambda_{1,k} \to \lambda_{1,\infty} = \lambda_1(L + q, \Omega)$.

Lastly, we prove that every subsequence of $\{\phi_{1,k}\}$ converges to $\phi_{1,\infty}$. Suppose that there exists a subsequence $\{\phi_{1,i}\}_i$ (already relabeled) that converges to $\tilde{\phi}_{1,\infty}$. By the reasoning we did before, $\tilde{\phi}_{1,\infty}$ would be some eigenfunction associated to $\lambda_{1,\infty}$. By [13, theorem 2.3], we would have $\tilde{\phi}_{1,\infty} = c\phi_{1,\infty}$ for some $c \neq 0$. But, $\tilde{\phi}_{1,\infty} > 0$ and $\|\tilde{\phi}_{1,\infty}\|_X = 1$, that is, $c = 1$. It follows that $\phi_{1,k} \to \phi_{1,\infty} = \phi_1(L+q,\Omega)$ with $\phi_{1,\infty} > 0$ and $\|\phi_{1,\infty}\|_X = 1$.

Now, we prove that $\phi^*_{1,k} \to \phi^*_1 := \phi_1(L^* + q, \Omega) > 0$.

Consider the sequence of positive, normalized (in Y^*) eigenfunctions $\{\phi^*_{1,k}\}$ such that $(-L^* - q_k)\phi^*_{1,k} = \lambda_{1,k}\phi^*_{1,k}$. From what we have seen before, $q_k \to q$ implies that $\lambda_{1,k} \to \lambda_{1,\infty} := \lambda_1(L+q,\Omega)$.

Then,

$$\begin{aligned}
\|(L^*+q)\phi^*_{1,k} + \lambda_{1,\infty}\phi^*_{1,k}\|_{X^*} &\leq \|(L^*+q+\lambda_{1,k})\phi^*_{1,k}\|_{X^*} + |\lambda_{1,\infty} - \lambda_{1,k}|\,\|\phi^*_{1,k}\|_{X^*} \\
&\leq \|(L^*+q-L^*-q_k)\phi^*_{1,k}\|_{X^*} + |\lambda_{1,\infty} - \lambda_{1,k}| \\
&\leq \|(L-q_k)^* - (L-q)^*\| + |\lambda_{1,\infty} - \lambda_{1,k}| \\
&= \|(L-q_k) - (L-q)\| + |\lambda_{1,\infty} - \lambda_{1,k}| \to 0.
\end{aligned}$$

Fix $z_k := (L^* + q + \lambda_{1,\infty})\phi^*_{1,k} \in \langle \phi^*_1 \rangle^\perp$, which converges to 0. By observing that

$$\phi^*_{1,k} = w_k + t_k\phi^*_1 \in \langle \phi^*_1 \rangle^\perp \cap Y^* \oplus \langle \phi^*_1 \rangle = Y^*,$$

we have

$$z_k = (L^* + q + \lambda_{1,\infty})(w_k + t_k\phi^*_1) = (L^* + q + \lambda_{1,\infty}I)w_k \to 0 \quad \text{in } X.$$

The Fredholm alternative assures that

$$L + q + \lambda_{1,\infty}I : \langle \phi^*_1 \rangle^\perp \cap Y^* \to \langle \phi^*_1 \rangle^\perp$$

is an isomorphism, so that

$$(L^* + q + \lambda_{1,\infty}I)^{-1} z_k = w_k \to 0 \quad \text{in } Y.$$

Since $\|w_k + t_k\phi^*_1\|_{Y^*} = 1$ we have that $|t_k| \to 1$, that is, $\phi^*_{1,k} = w_k + t_k\phi^*_1 \to \phi^*_1$ or $\phi^*_{1,k} \to -\phi^*_1$ in the Y^* norm.

Since $\phi^*_{1,k} > 0$, we have that its limit is greater than or equal to 0. Then its limit must be ϕ^*_1.

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