Pontifícia Universidade Católica
do Rio de Janeiro

Edgar Matias da Silva

Non-hyperbolic Iterated Function Systems: attractors, stationary measures, and step skew products

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Matemática of the Departamento de Matemática of PUC-Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Lorenzo Justiniano Díaz Casado

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Abstract

Silva, Edgar Matias da; Díaz, Lorenzo (Advisor). **Non-hyperbolic Iterated Function Systems: attractors, stationary measures, and step skew products**. Rio de Janeiro, 2016. 86p. PhD Thesis – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We consider iterated function systems IFS$(T_1, \dots, T_k)$ consisting of continuous self maps of a compact metric space X. We introduce the subset S_{t} of *weakly hyperbolic sequences* $\xi = \xi_0 \dots \xi_n \dots$ having the property that $\bigcap_n T_{\xi_0} \circ \dots \circ T_{\xi_n}(X)$ is a point $\{\pi(\xi)\}$. The *target set* $\pi(S_{\mathrm{t}})$ plays a role similar to the semifractal introduced by Lasota-Myjak.

Assuming that $S_{\mathrm{t}} \neq \emptyset$ (the only hyperbolic-like condition we assume) we prove that the IFS has at most one strict attractor and we state a sufficient condition guaranteeing that the strict attractor is the closure of the target set. Our approach applies to a large class of genuinely non-hyperbolic IFSs (e.g. with maps with expanding fixed points) and provides a necessary and sufficient condition for the existence of a globally attracting fixed point of the Barnsley-Hutchinson operator. We provide sufficient conditions under which the disjunctive chaos game yields the target set (even when it is not a strict attractor).

We state a sufficient condition for the asymptotic stability of the Markov operator of a recurrent IFS. For IFSs defined on $[0, 1]$ we give a simple condition for their asymptotic stability. In the particular case of IFSs with probabilities satisfying a “local injectivity” condition, we prove that if the target set has at least two elements then the Markov operator is asymptotically stable and its stationary measure is supported in the closure of the target set.

We use the results about IFSs to study several types of attractors of step skew products of non-hyperbolic type. We introduce the subset S_{t}^- of Σ_k of sequences with trivial spines (this set is a version of S_{t} for bilateral sequences) and assume that this set is non-empty. We identify a closed subset of the phase space (that is a bony-like graph) where the restriction of the skew product map is topologically mixing. We prove that, in many relevant cases, the skew product has only a Milnor attractor that coincides with this bony-like graph. Finally, as an application, we construct a robust example of a topologically mixing Milnor attractor having a complicated fiber structure (the attractor contains “disconnected bones”).

Keywords

Barnsley-Hutchinson operator; Conley and strict attractors; Markov operators; Target set; Chaos game;

Resumo

Silva, Edgar Matias da; Díaz, Lorenzo. **Sistemas de funções iteradas não-hiperbólicos: atratores, medidas estacionárias e produtos tortos simples**. Rio de Janeiro, 2016. 86p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nós consideramos sistemas de funções iteradas SFI$(T_1, \dots, T_k)$ consistindo de funções contínuas definida em um espaço métrico compacto X. Introduzimos o conjunto S_{t} de *sequências fracamente hiperbólicas* $\xi = \xi_0 \dots \xi_n \dots$ tendo a propriedade que $\bigcap_n T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X)$ é um ponto $\{\pi(\xi)\}$. O conjunto alvo $\pi(S_{\mathrm{t}})$ desempenha um papel similar ao semi-fractal introduzido por Lasota-Myjak.

Assumindo que $S_{\mathrm{t}} \neq \emptyset$ (a única condição do tipo hiperbólica que assumimos) provamos que o SFI tem no máximo um atrator estrito e estabelecemos uma condição suficiente garantindo que o atrator estrito é o fecho do conjunto alvo. Como consequência obtivemos uma condição necessária e suficiente para a existência de um ponto fixo globalmente atrator do operador de Hutchinson. Nós também estabelecemos condições sob as quais o jogo do caos disjuntivo determina o conjunto alvo (mesmo quando não existe atrator estrito).

Nós estabelecemos uma condição suficiente para estabilidade assintótica do operador de Markov de um SIF recorrente. Para SIFs no intervalo $[0, 1]$ apresentamos uma condição simples garantindo a estabilidade assintótica. No caso particular de um SFI com probabilidade satisfazendo uma "injetividade local" provamos que se o conjunto alvo tem pelo menos dois elementos então o operador de Markov é assintoticamente estável e o suporte da medida estacionária é o fecho do conjunto alvo.

Os resultados sobre SFIs são usados para estudar os diversos tipos de atratores do produto torto simples de tipo não-hiperbólico. Introduzimos o subconjunto S_{t}^- de Σ_k de sequências com espinho não trivial (este conjunto é uma versão do conjunto S_{t} para sequências bilaterais) e assumimos que este conjunto é não vazio. No espaço de fases identificamos um conjunto fechado(que é o fecho de um gráfico) onde a restrição do producto torto é topologicamente misturadora. Em muitos casos relevantes provamos que o produto torto tem apenas um atrator de Milnor que coincide com este gráfico. Finalmente, como aplicação, contruímos um exemplo robusto de um atrator de Milnor tendo uma estrutura complicada (O atrator contém *bones* desconexos).

Palavras-chave

Operador de Barnsley-Hutchinson; Atratores estritos e de Conley; Operadores de Markov; Conjunto target; Jogo do caos;

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1
Introduction

In this work we study iterated function systems (IFSs) associated to continuous self-maps $T_1, \dots, T_k$, $k \geq 2$, defined on a compact metric space (X, d) (denoted by $\mathrm{IFS}(T_1, \dots, T_k)$). In his fundamental paper (17), Hutchinson considered hyperbolic (uniformly contracting) IFSs and proved the existence and uniqueness of global attractors and stationary measures for such IFSs. We obtain similar results for genuinely non-hyperbolic IFSs having contracting and expanding regions as well as contracting and expanding fixed points. We use our constructions and results for IFSs to study Milnor attractors of their associated step skew product.

1.1
Iterated function systems

A key ingredient in this study is the so-called *Barnsley-Hutchinson operator* of an IFS $\mathfrak{F} = \mathrm{IFS}(T_1, \dots T_k)$ that associates to each subset A of X the set

$$\mathcal{B}_{\mathfrak{F}}(A) \stackrel{\text{def}}{=} \bigcup_{i=1}^{k} T_i(A). \tag{1.1.1}$$

This operator acts continuously in the space of nonempty compact subsets of X endowed with the Hausdorff metric. In the hyperbolic setting (all maps T_i are uniform contractions) the operator $\mathcal{B}_{\mathfrak{F}}$ has a unique global attractor: there exists a compact set $A_{\mathfrak{F}}$, called the *attractor of the IFS*, such that

$$\lim_{n\to\infty} \mathcal{B}^n_{\mathfrak{F}}(K) = A_{\mathfrak{F}} \quad \text{for every compact set } K \subset X,\ K \neq \emptyset,$$

see (17). Edalat (14) extended this result to *weakly hyperbolic IFSs*, that is, IFSs satisfying the following "reverse" contracting condition

$$\operatorname{diam}\Big(T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X)\Big) \to 0 \quad \text{for every} \quad \xi = \xi_0\xi_1\xi_2\cdots \in \Sigma_k^+, \tag{1.1.2}$$

where $\Sigma_k^+ \stackrel{\text{def}}{=} \{1, \dots, k\}^{\mathbb{N}}$.

In this thesis we will study a more general setting than the above one, considering genuinely non-hyperbolic IFSs. One of our goals is to describe the global and local "attractors" of $\mathcal{B}_{\mathfrak{F}}$. More precisely, we will consider so-called *strict* and *Conley attractors*. A compact set $A \subset X$ is a *strict attractor* of the

IFS $\mathfrak{F}$ if there is an open neighbourhood U of A such that

$$\lim_{n\to\infty} \mathcal{B}^n_{\mathfrak{F}}(K) = A \quad \text{for every compact set } K \subset U.$$

The *basin of attraction* of A is the largest open neighbourhood of A for which the above property holds. A strict attractor whose basin of attraction is the whole space is a *global attractor*. A compact set $S \subset X$ is a *Conley attractor* of the IFS $\mathfrak{F}$ if there exists an open neighbourhood U of S such that

$$\lim_{n\to\infty} \mathcal{B}^n_{\mathfrak{F}}(\overline{U}) = S.$$

The continuity of the Barnsley-Hutchinson operator $\mathcal{B}_{\mathfrak{F}}$ implies that Conley and strict attractors both are fixed points of $\mathcal{B}_{\mathfrak{F}}$. Note also that strict attractors are Conley attractors but the converse is not true in general. Finally, we say that the IFS $\mathfrak{F}$ is *asymptotically stable* if there is a (unique) global attractor.

The above mentioned results in (17, 14) require some sort of global contraction (hyperbolicity) of the IFS. Having in mind the definition of weakly hyperbolicity in (1.1.2), we introduce the subset $S_{\mathrm{t}} \subset \Sigma_k^+$ of *weakly hyperbolic sequences* defined by

$$S_{\mathrm{t}} \stackrel{\text{def}}{=} \Big\{\xi \in \Sigma_k^+ \colon \lim_{n\to\infty} \operatorname{diam}\Big(T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X)\Big) = 0\Big\}. \tag{1.1.3}$$

Note that for a weakly hyperbolic IFS one has $S_{\mathrm{t}} = \Sigma_k^+$. If $S_{\mathrm{t}} \neq \Sigma_k^+$ we will call the IFS *non-weakly hyperbolic*. We say that an IFS *has a weakly hyperbolic sequence* if $S_{\mathrm{t}} \neq \emptyset$. When $S_{\mathrm{t}} \neq \emptyset$ then it contains a residual subset of Σ_k^+[1]. We replace the condition *every* sequence is weakly hyperbolic by the condition *there is at least one* weakly hyperbolic sequence. The goal of this thesis is to recover results in the spirit of (17, 14) in such a setting.

We briefly sketch our main results and philosophy of our approach, postponing the precise statements. As a general principle, rephrasing Pugh-Shub principle (28), we show that “a little hyperbolicity goes a long way guaranteeing stability-like properties”. Here by a “little hyperbolicity” we understand either the almost-sure existence of weakly hyperbolic sequences or the existence of at least one, according to the case. First, assuming that the set S_{t} has “probability one”, we prove that the Markov operator is asymptotically stable (here we consider Markov measures associated to transition matrices and the particular case of Bernoulli probabilities). Second, we prove that if the Barnsley-Hutchinson operator has a unique fixed point then the IFS is asymptotically stable. Finally, in the case when X is an interval, to establish

[1]This follows using genericity standard arguments, see for instance the construction in (12, Proposition 3.15)

the stability of the Markov operator we show that it is enough to assume that there are no common fixed points for the maps of the IFS and that there exists at least one weakly hyperbolic sequence.

We now provide more details for our main results (for the precise definitions and statements see Section 2). Associated to the set S_{t} of weakly hyperbolic sequences we consider the *coding map* $\pi\colon S_{\mathrm{t}} \to X$ that projects S_{t} into the phase space X, see equation (2.1.1). The set $A_{\mathrm{t}} \stackrel{\mathrm{def}}{=} \pi(S_{\mathrm{t}})$ is called the *target* set and contains relevant dynamical information of the IFS. Assuming that $S_{\mathrm{t}} \neq \emptyset$, we prove the following results:

- The closure of the target set $\overline{A_{\mathrm{t}}}$ is a Conley attractor if and only if it is a strict attractor (Theorem 1).
- The set $\overline{A_{\mathrm{t}}}$ is the global maximal fixed point of the IFS if and only if the IFS is asymptotically stable. Moreover, the Barnsley-Hutchinson operator has a unique fixed point if and only if it is asymptotically stable (Theorem 2).

We will investigate more closely the relation between target sets and semifractals introduced in (24). An IFS $\mathfrak{F} = \mathrm{IFS}(T_1, \dots, T_k)$ is said to be *regular* if there are numbers $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ such that $\mathfrak{F}' = \mathrm{IFS}(T_{i_1}, \dots, T_{i_\ell})$ is asymptotically stable. The global attractor of $\mathfrak{F}'$ is called a *nucleus* of $\mathfrak{F}$ (an IFS may have several nuclei). By (24) for any regular IFS $\mathfrak{F}$ there exists *the* minimum fixed point of $\mathfrak{F}$, called its *semifractal* and denoted by $\mathrm{Semi}(\mathfrak{F})$. It is obtained from any nucleus of $\mathfrak{F}$ and attracts every compact set inside it, where iterations are taken with respect the Barnsley-Hutchinson operator of $\mathfrak{F}$. On the other hand, when $S_{\mathrm{t}} \neq \emptyset$, the set $\overline{A_{\mathrm{t}}}$ is a minimum fixed point that attracts every compact set inside it. This provides the following characterisation of semifractals:

- If an IFS $\mathfrak{F}$ is regular and satisfies $S_{\mathrm{t}} \neq \emptyset$ then $\mathrm{Semi}(\mathfrak{F}) = \overline{A_{\mathrm{t}}}$.

For a non-regular IFS with $S_{\mathrm{t}} \neq \emptyset$ (see Example 11.0.8) the set $\overline{A_{\mathrm{t}}}$ plays the same role as a semifractal plays for a regular IFS. We refer to Remark 4.3.5 to support this assertion.

We will also study the consequence of our approach for the so-called chaos game. The *chaos game* is an algorithm for generating fractals using random iterations of an IFS, see (3). It has probabilistic and disjunctive (deterministic) versions, see (3, 7, 9, 6). Given an initial point $x = x_0 \in X$, one considers the orbit $x_{n+1} = T_{\xi_n}(x_n)$, where the sequence $\xi \in \Sigma_k^+$ is chosen according to some probability (*probabilistic game*) or is a disjunctive sequence (*disjunctive game*).

Recall that $\xi \in \Sigma_k^+$ is *disjunctive* if its orbit (with respect to the usual left shift σ defined by $\sigma(\xi)_n = \xi_{n+1}$) is dense in Σ_k^+. The *chaos game holds* when the sequence of *tails* $(\{x_n : n \geq \ell\})_\ell$ in the Hausdorff distance converges to some attracting "fractal" (in such a case we also say that *chaos game yields the fractal*).

A natural question is how typically this game holds, where the term typical either refers to sequences in Σ_k^+ or points in the phase space X. By (7), the probabilistic chaos game holds when the fractal is a strict attractor and the initial point is in its basin of attraction. By (9), the disjunctive chaos game holds for a special class of attractors[2] and every point in the pointwise basin of attraction.

In the context of the chaos game, (24) considers IFSs whose maps are Lipschitz with constants less than or equal to 1 and have at least one uniformly contracting map. It is proved that the probabilistic chaos game starting at any point of the phase space yields the semifractal (even if the semifractal is not an attractor). In our setting, we get a similar result for the disjunctive chaos game where the fractal is the closure of the target set.

A fixed point A of the Barnsley-Hutchinson operator is *Lyapunov stable* if for every open neighbourhood V of A there is an open neighbourhood V_0 of A such that

$$\mathcal{B}^n(V_0) \subset V \quad \text{for every} \quad n \geq 0. \tag{1.1.4}$$

For instance, the set $\overline{A_{\mathrm{t}}}$ is Lyapunov stable when it is a Conley attractor or when all the maps of the IFS are Lipschitz with constants less than or equal to 1 (the existence of a contracting map is not required). See Section 4.2 for an example where $\overline{A_{\mathrm{t}}}$ is stable but is not a Conley attractor.

– When $\overline{A_{\mathrm{t}}}$ is a Lyapunov stable fixed point the *disjunctive chaos game* holds for every point in the phase space (Theorem 3).

Finally we consider IFSs from the ergodic point of view, studying the existence and uniqueness of stationary measures. Recall that given an space of finite measures $\mathfrak{M}(X)$ defined on a set X, an operator $\mathfrak{T} \colon \mathfrak{M}(X) \to \mathfrak{M}(X)$ such that

– $\mathfrak{T}$ is linear and

– $\mathfrak{T}\nu(X) = \nu(X)$ for every $\nu \in \mathfrak{M}(X)$

is called a *Markov operator*. A *stationary measure of* $\mathfrak{T}$ is a fixed point of $\mathfrak{T}$. The operator $\mathfrak{T}$ is *asymptotically stable* if it has a stationary measure ν such that $\lim \mathfrak{T}^n \mu = \nu$ for every $\mu \in \mathfrak{M}(X)$, in the weak$*$ topology. The ergodic study of IFSs deals with two main settings:

[2]Called *well-fibered* attractors, see also the strongly fibered case in (6).

- *IFSs with probabilities* given by a Bernoulli probability $\mathfrak{b}$ that assigns (positive) weights to each map;
- *Recurrent IFSs* associated to an irreducible transition matrix P inducing a Markov probability $\mathbb{P}^+$.

From the ergodic viewpoint one studies the iterations of points by an IFS (random orbits) as a Markov process and each type of IFS has associated a special type of Markov operator (associated to Bernoulli probabilities and associated to transition matrices). For a discussion see (4, 5).

Our results are summarised as follows:

- We get a sufficient condition for the asymptotically stability of a recurrent IFS and characterise its unique stationary measure (Theorem 6.2.1). For a recurrent IFS on the interval $[0,1]$ with a *splitting Markov measure* [3] $\mathbb{P}^+$ we prove that $\mathbb{P}^+(S_t) = 1$ (Theorem 5). As consequence we get the asymptotically stability of recurrent IFSs with a splitting *inverse* Markov measure (Theorem 6).
- Every injective IFS on the interval $[0,1]$ with Bernoulli probability $\mathfrak{b}$ whose target set A_t is not a *singleton* (i.e., has at least two points) is asymptotically stable and its unique stationary measure is $\pi_*\mathfrak{b}$ and $\mathrm{supp}(\pi_*\mathfrak{b}) = \overline{A_t}$. In this case, $\overline{A_t}$ is uncountable and the stationary measure is continuous (Theorem 4). For IFSs with $S_t \neq \emptyset$ we see that if the Markov operator associated to a Bernoulli probability $\mathfrak{b}$ is asymptotically stable then the support of its stationary measure is $\overline{A_t}$, even when $\mathfrak{b}(S_t) = 0$, see Proposition 6.1.4 (this proposition does not require $X = [0,1]$).

1.2
Step skew products

We now study step skew products associated to IFSs. Let $\Sigma_k = \{1,\dots,k\}^{\mathbb{Z}}$ be the set of infinite two-sided sequences of symbols in $\{1,\dots,k\}$. The *shift map* $\sigma\colon \Sigma_k \to \Sigma_k$ associates to a two-sided sequence ω the sequence $\sigma(\omega)$ defined by $(\sigma(\omega))_i = \omega_{i+1}$. For a given IFS $\mathfrak{F} = \mathrm{IFS}(T_1,\dots T_k)$, $T_i\colon X \to X$, we associate the *(step) skew product map* defined by

$$F_{\mathfrak{F}}\colon \Sigma_k \times X \to \Sigma_k \times X, \qquad F_{\mathfrak{F}}(\xi,t) \stackrel{\text{def}}{=} (\sigma(\xi), T_{\xi_0}(t)). \tag{1.2.1}$$

We say that $\mathfrak{F}$ is the underlying IFS of $F_{\mathfrak{F}}$.

[3] This is an ergodic version of the condition "the set A_t is not a singleton" and means that there is i such that the restriction of π to $[i] \cap \mathrm{supp}(\mathbb{P}^+)$ is not constant.

Let us introduce some definitions. The *maximal attractor* of $F_{\mathfrak{F}}$ is defined by

$$\Lambda_{F_{\mathfrak{F}}} \stackrel{\text{def}}{=} \bigcap_{n\geq 0} F_{\mathfrak{F}}^n(\Sigma_k \times X) \tag{1.2.2}$$

and the *spine of a sequence* $\xi \in \Sigma_k$ is defined by

$$I_\xi \stackrel{\text{def}}{=} \bigcap_{n\geq 0} T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-n}}(X). \tag{1.2.3}$$

Related to the subset S_{t} of Σ_k^+ in (1.1.3) we consider the *set of sequences with trivial spines* defined by

$$S_{\mathrm{t}}^- \stackrel{\text{def}}{=} \{\xi \in \Sigma_k \colon \text{the spine } I_\xi \text{ is a singleton}\} \tag{1.2.4}$$

and the coding map $\varrho\colon S_{\mathrm{t}}^- \to X$ defined by

$$\varrho\colon S_{\mathrm{t}}^- \to X, \quad \varrho(\xi) \stackrel{\text{def}}{=} \lim_{n\to\infty} T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-n}}(p). \tag{1.2.5}$$

This definition does not depend on the choice of the point $p \in X$. It follows from definition of the target set that $A_{\mathrm{t}} = \varrho(S_{\mathrm{t}}^-)$. An important dynamical object is the graph of the map ϱ,

$$\operatorname{graph}\varrho \stackrel{\text{def}}{=} \{\big(\xi, \varrho(\xi)\big)\colon \quad \xi \in S_{\mathrm{t}}^-\}. \tag{1.2.6}$$

In the context where the set S_{t}^- is nonempty we study step skew product maps from the ergodic and topological viewpoints. As in the case of IFSs, ergodic properties of $F = F_{\mathfrak{F}}$ are related to the set S_{t}^- while topological properties of F are related to A_{t}. In what follows we assume that $S_{\mathrm{t}}^- \neq \emptyset$.

One of our goals is to study *Milnor attractors* of step skew products. Milnor attractors are considered with respect some standard measure on the ambient space $\Sigma_k \times X$. In the one dimensional case when $X = [0,1]$, one considers the product measure $\mu = \mathfrak{b} \times m$, where m is the Lebesgue measure on $[0,1]$ and $\mathfrak{b}$ is some Bernoulli measure in Σ_k. In this context, we will see that, under certain conditions on the target set of the underlying IFS $\mathfrak{F}$, the ω-limit of points (ξ, x) for $F_{\mathfrak{F}}$ does not depend on (ξ, x) when ξ is disjunctive. Since the set of disjunctive sequences has full Bernoulli measure, the statements for points (ξ, x) with forward dense orbits can be used to understand the Milnor attractor of $F_{\mathfrak{F}}$.

To state these statements a bit more precisely we first obtain some results for skew products whose fiber maps are defined on general compact sets. In Theorem 7 we see that F is topologically mixing in $\overline{\operatorname{graph}\varrho}$. It also claims that if ξ is a disjunctive sequence then for every point z of the form $z = (\xi, x)$ we have that $\overline{\operatorname{graph}\varrho} \subset \omega(z)$. In some special cases (for instance when $\overline{A_{\mathrm{t}}}$ is a Lyapunov stable fixed point of the Barnsley-Hutchinson operator or when it has nonempty interior) we prove that in fact $\overline{\operatorname{graph}\varrho} = \omega(z)$.

To state ergodic properties of F, for a σ-invariant measure λ defined on Σ_k consider the (compact and F_*-invariant) set $\mathcal{M}_\lambda$ of probabilities defined on $\Sigma_k \times X$ whose marginal is λ, see (2.3.2). Theorem 8 states that if $\lambda(S_{\mathrm{t}}^-) = 1$ then $F_{*|\mathcal{M}_\lambda}$ has a *global attractor* v_λ, that is, $F_*^n \mu \to v_\lambda$ for every $\mu \in \mathcal{M}_\lambda$. Moreover, the disintegration v_λ with respect to λ is the δ-Dirac measure $\delta_{\varrho(\xi)}$, v_λ satisfies a strong form of ergodicity, and (F, v_λ) is isomorphic to (σ, λ).

We use the results above to describe the Milnor attractor (and also other types of attractors) when $X = [0,1]$. Theorem 10 claims that every Milnor attractor of F contains the set $\overline{\mathrm{graph}\,\varrho}$. Moreover, in some relevant cases there is a unique Milnor attractor that coincides with $\overline{\mathrm{graph}\,\varrho}$.

Let us recall some results in (19) that raise some interesting questions about the structure of Milnor attractors of step skew product with fiber maps defined on $[0,1]$. In (19), assuming that the fiber maps preserve the orientation, it is proved that generic step skew products of this type have a finite number of physical ergodic measures $m_1, \dots, m_n$ (each of these measures is the lift of a stationary measure of the underlying IFS). Moreover, the phase space $\Sigma_k \times [0,1]$ splits into finitely many pairs of attracting and repelling (in the fiber direction) "strips" such that every Milnor attractor is contained in the maximal invariant set in one attracting strip. For contracting (resp. repelling) strips each forward (resp. backward) locally maximal invariant set Λ_i in the strip contains the support of some physical measure m_i above (but these sets can be different) and also contains a Milnor attractor of F (although this attractor may be no related to the measure m_i). Finally, the sets Λ_i are *bony sets*, in particular, the intersection of Λ_i with a fiber is either a point or an interval, where the first case occurs almost surely. In Theorem 11 we see that Milnor attractors can be properly contained in the sets Λ_i and its fiber structure can be quite complicated: For each n there is an open set of skew products defined over Σ_4 with injective fiber maps preserving the orientation such that the intersection of the Milnor attractor with the fibers is either a point or disconnected set with at most n components. The first case occurs almost surely in Σ_4 and for a dense subset of Σ_4 the intersection consists of exactly n components which are nontrivial closed intervals.

This thesis has two main parts, in the first one (Sections 4-6) we study IFSs and in the second one (Sections 7-10) we apply the results in the first part to step skew products. It is organised as follows. In Section 2 we state the main definitions and the precise statements of our results. In Section 3 we introduce some basic definitions and notation. Section 4 is devoted to the study of different types of attractors of IFSs and to the proofs of Theorems 1, 2, and 3. In Section 5, we consider IFSs defined on the interval $[0,1]$, study the measure of S_{t} for Markov measures, and prove Theorem 5. We also get results about "probabilistic rigidity" of S_{t} (Theorem 5.2.1) and characterise separable IFSs (Theorem 5.3.1). In Section 6 we prove Theorems 4 and 6 about stability of the Markov operator. In Section 7 we start the study of step skew products with the proof of Theorem 7. In Section 8 we study the existence of global attracting measures and prove Theorem 8 and Corollary 9. Section 9 is devoted to the study of Milnor attractors and the proof of Theorem 10. In Section 10 we construct robust examples of skew product having Milnor attractors with disconnected bones and prove Theorem 11. Finally, in Section 11 we present some examples.

2
Precise statement of results

2.1
Topological properties of IFSs

Consider the set S_{t} of weakly hyperbolic sequences in (1.1.3) and define the *coding map*[1]

$$\pi \colon S_{\mathrm{t}} \to X \quad \text{by} \quad \pi(\xi) \stackrel{\text{def}}{=} \lim_{n\to\infty} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(p), \tag{2.1.1}$$

where p is any point of X. By definition of the set S_{t}, this limit always exists and is independent of $p \in X$. We introduce the *target set* $A_{\mathrm{t}} \stackrel{\text{def}}{=} \pi(S_{\mathrm{t}})$. This name is justified by the following characterisation

$$A_{\mathrm{t}} = \{x \in X \colon \text{there is } \xi \in \Sigma_k^+ \text{ with } \{x\} = \textstyle\bigcap_n T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X)\}, \tag{2.1.2}$$

see (4.1.2). The target set plays a key role in the study of strict attractors. We prove that if $S_{\mathrm{t}} \neq \emptyset$ then the IFS has at most one strict attractor. Moreover, if such a strict attractor exists then it is equal to $\overline{A_{\mathrm{t}}}$, see Proposition 4.2.3.

Theorem 1 *Consider an* IFS *defined on a compact metric space such that* $S_{\mathrm{t}} \neq \emptyset$. *Then* $\overline{A_{\mathrm{t}}}$ *is a Conley attractor if and only if it is a strict attractor.*

In (8) Barnsley and Vince consider IFSs consisting either of affine maps or of Möbius maps and introduce sufficient conditions that guarantee the existence of a unique strict attractor. The proof involves some type of local hyperbolicity in a neighbourhood of a Conley attractor, see (2, 32). We point out that Theorem 1 only requires the existence of *at least one* weakly hyperbolic sequence.

Given an IFS $\mathfrak{F}$ and its Barnsley-Hutchinson operator $\mathcal{B}_{\mathfrak{F}}$, a subset $Y \subset X$ is $\mathcal{B}_{\mathfrak{F}}$*-invariant* if $\mathcal{B}_{\mathfrak{F}}(Y) \subset Y$. The closure of any $\mathcal{B}_{\mathfrak{F}}$-invariant set contains some fixed point of $\mathcal{B}_{\mathfrak{F}}$ (see the discussion below). Therefore, since X is $\mathcal{B}_{\mathfrak{F}}$-invariant, the operator $\mathcal{B}_{\mathfrak{F}}$ always has at least one fixed point. Indeed, we have a more precise description of the fixed points of $\mathcal{B}_{\mathfrak{F}}$. Following (14), given $Y \subset X$ define the set

$$Y^* \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \mathcal{B}_{\mathfrak{F}}^n(Y). \tag{2.1.3}$$

[1] This is the standard terminology for the map π when $S_{\mathrm{t}} = \Sigma_k^+$.

If Y is $\mathcal{B}_{\mathfrak{F}}$-invariant then the set $(\overline{Y})^*$ is the *global maximal fixed point of the restriction of $\mathcal{B}_{\mathfrak{F}}$ (or of the IFS) to the subsets of $\overline{Y}$*, see Proposition 4.1.1. The next theorem generalizes (14) in two ways: it applies also to IFSs which are not weakly hyperbolic and it provides a necessary and sufficient condition for the existence of a global attractor.

Theorem 2 *Consider an IFS $\mathfrak{F}$ defined on a compact metric space X such that $S_{\mathrm{t}} \neq \emptyset$. Then the following three assertions are equivalent:*

1. $\overline{A_{\mathrm{t}}} = X^*$,
2. *the Barnsley-Hutchinson operator $\mathcal{B}_{\mathfrak{F}}$ has a unique fixed point,*
3. *X^* is a global attractor of the IFS $\mathfrak{F}$.*

We observe that the statement in Theorem 2 is sharp. Indeed, there are examples of non-weakly hyperbolic IFSs where $A_{\mathrm{t}} \subsetneq \overline{A_{\mathrm{t}}} = X^*$, see Section 11.

Let us observe that for weakly hyperbolic IFSs it holds $A_{\mathrm{t}} = X^*$, see Lemma 4.1.3 and also (14). We observe that there are IFSs that are non-weakly hyperbolic such that $A_{\mathrm{t}} = \overline{A_{\mathrm{t}}} = X^*$, see Section 11.

Theorem 3 (Disjunctive chaos game) *Consider an* IFS$(T_1, \dots, T_k)$ *defined on a compact metric space X such that $\overline{A_{\mathrm{t}}}$ is a stable fixed point of the Barnsley-Hutchinson operator. Then for every $x \in X$ and every disjunctive sequence $\xi \in \Sigma_k^+$ we have*

$$\overline{A_{\mathrm{t}}} = \bigcap_{\ell \geq 0} \overline{\{x_{n,\xi} \colon n \geq \ell\}}, \quad \textit{where} \quad x_{n,\xi} \stackrel{\text{def}}{=} T_{\xi_n} \circ \cdots \circ T_{\xi_0}(x).$$

In particular

$$\lim_{\ell \to \infty} \{x_{n,\xi} \colon n \geq \ell\} = \overline{A_{\mathrm{t}}},$$

where the limit is considered in the Hausdorff distance.

2.2
Ergodic properties of IFSs

2.2.1 IFSs with probabilities

Consider an $\mathrm{IFS}(T_1,\ldots,T_k)$ defined on a compact metric space X and strictly positive numbers $p_1,\ldots,p_k$ (called *weights*) such that $\sum_{i=1}^k p_i = 1$. We denote by $\mathfrak{b} = \mathfrak{b}(p_1,\ldots,p_k)$ the (non-trivial) Bernoulli probability measure with weights $p_1,\ldots,p_k$ defined on Σ_k^+. We denote by $\mathrm{IFS}(T_1,\ldots T_k;\mathfrak{b})$ the IFS with the corresponding Bernoulli probability and say that it is an *IFS with probabilities*.

Let $\mathcal{M}_1(X)$ be the space of Borel probability measures defined on X equipped with the weak$*$-topology. The *Markov operator* associated to the $\mathrm{IFS}(T_1,\ldots T_k;\mathfrak{b})$ is defined by

$$\mathfrak{T}_{\mathfrak{b}}\colon \mathcal{M}_1(X) \to \mathcal{M}_1(X), \qquad \mathfrak{T}_{\mathfrak{b}}\mu \overset{\text{def}}{=} \sum_{i=1}^k p_i\, T_{i*}\mu, \tag{2.2.1}$$

where $T_{i*}\mu(A) = \mu(T_i^{-1}(A))$ for every Borel set A. Note that the Markov operator $\mathfrak{T}_{\mathfrak{b}}$ is continuous. Hence, if $\mathfrak{T}_{\mathfrak{b}}$ is asymptotically stable then its attracting measure μ is stationary, that is, satisfies $\mathfrak{T}_{\mathfrak{b}}\mu = \mu$.

An IFS with probabilities $\mathrm{IFS}(T_1,\ldots T_k;\mathfrak{b})$ is called *asymptotically stable* if its Markov operator $\mathfrak{T}_{\mathfrak{b}}$ is asymptotically stable. It is a folklore result that if $\mathfrak{b}(S_{\mathrm{t}}) = 1$ then the IFS is asymptotically stable and $\pi_*\mathfrak{b}$ is the unique stationary measure, see for instance (29, 23). In Proposition 6.1.1 we prove this fact and we see that $\mathrm{supp}(\pi_*\mathfrak{b}) = \overline{A_{\mathrm{t}}}$. Note that, since that $\sigma^{-1}(S_{\mathrm{t}}) \subset S_{\mathrm{t}}$, the ergodicity of the Bernoulli measure (with positive weights) $\mathfrak{b}$ with respect to the shift implies that either $\mathfrak{b}(S_{\mathrm{t}}) = 1$ or $\mathfrak{b}(S_{\mathrm{t}}) = 0$.

A combination of Theorem 2 and Proposition 6.1.1 allows us to recover properties of hyperbolic IFSs in non-hyperbolic settings provided that the sets A_{t} and S_{t} are "big enough" (from the topological and probabilistic points of view, respectively): there are a unique global attractor and the IFS with probabilities is asymptotically stable.

Proposition 6.1.1 assumes that $\mathfrak{b}(S_{\mathrm{t}}) = 1$ (which is often difficult to verify). When $X = [0,1]$ we improve this proposition replacing the condition $\mathfrak{b}(S_{\mathrm{t}}) = 1$ by the topological condition $\#(A_{\mathrm{t}}) \geq 2$ that we call *separability* and it is quite straightforward to verify.

Theorem 4 *Consider an* $\mathrm{IFS}(T_1,\ldots T_k)$ *defined on* $[0,1]$ *such that*

– *the target set* A_{t} *has at least two elements and*

– *there is a non-trivial closed interval* $J \subset [0,1]$ *such that* $T_i(J) \subset J$ *and* $T_i|_J$ *is injective for every* $j \in \{1,\ldots,k\}$.

Then for every (non-trivial) Bernoulli probability $\mathfrak{b}$ *the* $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$ *is asymptotically stable.*

Moreover, $\pi_* \mathfrak{b}$ *is the (unique) stationary measure of* $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$*, satisfies* $\mathrm{supp}(\pi_* \mathfrak{b}) = \overline{A_{\mathrm{t}}}$*, and is continuous. As a consequence, the set* A_{t} *has no isolated points.*

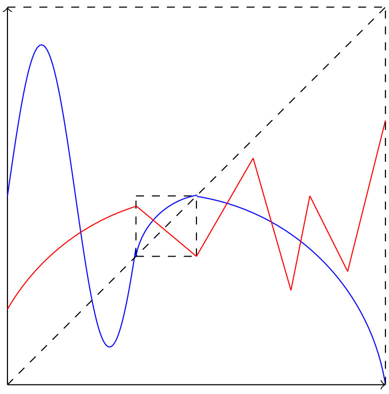

Figure 2.1: The injectivity conditions of Theorem 4

In the previous theorem, the purely topological condition $\#(A_{\mathrm{t}}) \geq 2$ depending only on $\mathrm{IFS}(T_1, \dots T_k)$ implies the asymptotic stability of the Markov operator $\mathfrak{T}_{\mathfrak{b}}$ of $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$ for any (non-trivial) Bernoulli probability $\mathfrak{b}$. Moreover, we also obtain properties of the stationary measure. The support of this stationary measure is independent of the Bernoulli probability. In Proposition 6.1.4 we state a result about the support of stationary measures that holds for general compact metric spaces: if $S_{\mathrm{t}} \neq \emptyset$ and the Markov operator associated to $\mathfrak{b}$ is asymptotically stable then the support of its stationary measure always is $\overline{A_{\mathrm{t}}}$, even when $\mathfrak{b}(S_{\mathrm{t}}) = 0$.

The asymptotic stability of an IFS with probabilities has been obtained in several contexts such as, for example, *contracting on average* (4), weakly hyperbolic (14), and *non-overlapping*[2] (31). Observe that the contexts of (4, 14) have a hyperbolic flavour. Let us also observe that (30) states the asymptotic stability of admissible IFSs consisting of circle homeomorphisms (these homeomorphisms preserve the orientation and some homeomorphism of the IFS is transitive). Note that in this case the set S_{t} is empty. Let us compare these results with Theorem 4. First, the condition to be contracting on average depends on the selected Bernoulli probability (an IFS may be contracting in average with respect to some probabilities but not with respect to *all*

[2] An IFS is called non-overlapping if the maps T_i are injective and the sets $T_i(I)$ have disjoint interiors. We will see that separability is a weak form of non-overlapping, see Theorem 5.3.1. We observe that (14) and (4) do not involve injective-like conditions of the IFS.

Bernoulli probabilities). In contrast, weak hyperbolicity, separability, non-overlapping, and admissibility conditions are topological conditions that do not involve probabilities. These conditions guarantee the asymptotic stability of the Markov operator $\mathfrak{T}_{\mathfrak{b}}$ of the IFS with respect to *any* Bernoulli probability $\mathfrak{b}$.

Note that checking the properties of weak hyperbolicity and contracting on average may be rather complicated, while the separability condition is comparably much simpler, thus Theorem 4 can also be useful in these contexts.

Finally, we refer to (25) to a recent outstanding result by Dominique Malicet about synchronization on the circle. In particular, it is proved that for an IFS on the circle, under a quite natural assumption and minimality, the Markov operator is asymptotically stable.

2.2.1.1
Recurrent IFSs

A generalization of IFSs with probabilities are the so-called *recurrent IFSs* introduced in (5), where the weights p_i are replaced by a transition matrix.

To be more precise, recall that a $k \times k$ matrix $P = (p_{ij})$ is a *transition matrix* if $p_{ij} \geq 0$ for all i, j and for every i it holds $\sum_{j=1}^{k} p_{ij} = 1$. An *stationary probability vector* associated to P is a vector $\bar{p} = (p_1, \ldots, p_k)$ whose elements are non-negative real numbers and sum up to 1 and satisfies $\bar{p}\,P = \bar{p}$. The transition matrix P is called *irreducible* if for every $\ell, r \in \{1, \ldots, k\}$ there is $n = n(\ell, r)$ such that $P^n = (p^n_{ij})$ satisfies $p^n_{\ell,r} > 0$. An irreducible transition matrix has a unique stationary probability vector $\bar{p} = (p_i)$, see (18, page 100). We consider the *cylinders*

$$[a_0 \ldots a_\ell] \stackrel{\text{def}}{=} \{\omega \in \Sigma_k^+ \colon \omega_0 = a_0, \ldots, \omega_\ell = a_\ell\} \subset \Sigma_k^+$$

which is a semi-algebra that generates the Borel σ-algebra of Σ_k^+. We denote by $\mathbb{P}^+$ the *Markov measure* associated to $(P, \bar{p})$ defined on Σ_k^+, this measure is defined on the cylinders $[a_0 \ldots a_\ell]$ by

$$\mathbb{P}^+([a_0 \ldots a_\ell]) \stackrel{\text{def}}{=} p_{a_0} p_{a_0 a_1} \ldots p_{a_{\ell-1} a_\ell}.$$

Given an IFS$(T_1, \ldots T_k)$ and an irreducible transition matrix $P = (p_{ij})$, we call IFS$(T_1, \ldots T_k; \mathbb{P}^+)$ a *recurrent* IFS. We now introduce the Markov operator in this context. Consider the set $\widehat{X} \stackrel{\text{def}}{=} X \times \{1, \ldots, k\}$ with the product topology and the corresponding Borel sets. Given a subset $\widehat{B} \subset \widehat{X}$, its *$i$-section* is defined by

$$\widehat{B}_i \stackrel{\text{def}}{=} \{x \in X \colon (x, i) \in \widehat{B}\}.$$

The *i-section of a probability measure* $\widehat{\mu}$ on $\widehat{X}$ is defined on the set X by

$$\mu_i(B) \stackrel{\text{def}}{=} \widehat{\mu}(B \times \{i\}), \qquad \text{where } B \text{ is any Borel subset of } X.$$

Observe that μ_i is a finite measure on X but, in general, it is not a probability measure. Since the measure $\widehat{\mu}$ is completely defined by its sections we write $\widehat{\mu} = (\mu_1, \dots, \mu_k)$ and note that

$$\widehat{\mu}(\widehat{B}) = \sum_{j=1}^{k} \mu_j(\widehat{B}_j) \quad \text{for every Borel subset } \widehat{B} \text{ of } \widehat{X}.$$

The *(generalised) Markov operator* of recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ is defined by

$$\mathfrak{S}_{\mathbb{P}^+} \colon \mathcal{M}_1(\widehat{X}) \to \mathcal{M}_1(\widehat{X}), \quad \widehat{\mu} \mapsto \mathfrak{S}_{\mathbb{P}^+}(\widehat{\mu}), \tag{2.2.2}$$

where

$$\mathfrak{S}_{\mathbb{P}^+}(\widehat{\mu})(\widehat{B}) \stackrel{\text{def}}{=} \sum_{i,j} p_{ij} T_{j*}\mu_i(\widehat{B}_j).$$

A recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ is called *asymptotically stable* if the Markov operator $\mathfrak{S}_{\mathbb{P}^+}$ is asymptotically stable.

Given a Markov measure $\mathbb{P}^+$ there is associated its *inverse Markov measure* $\mathbb{P}^-$ defined on Σ_k^+ by

$$\mathbb{P}^-([a_0 a_1 \dots a_n]) \stackrel{\text{def}}{=} \mathbb{P}^+([a_n \dots a_1 a_0]), \quad \text{for a cylinder } [a_0 a_1 \dots a_n]. \tag{2.2.3}$$

The measure $\mathbb{P}^-$ is also Markov (see Section 3.3).

There is the following *generalised coding map* from $S_{\rm t}$ to $\widehat{X}$ defined by

$$\varpi \colon S_{\rm t} \to \widehat{X}, \qquad \varpi(\xi) \stackrel{\text{def}}{=} (\pi(\xi), \xi_0). \tag{2.2.4}$$

In Theorem 6.2.1 we see that if a recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ is such that $\mathbb{P}^-(S_{\rm t}) = 1$ and $\mathbb{P}^-$ is mixing then it is asymptotically stable and the stationary measure of $\mathfrak{S}_{\mathbb{P}^+}$ is $\varpi_* \mathbb{P}^-$, that is,

$$\mathfrak{S}^n_{\mathbb{P}^+}(\widehat{\mu}) \underset{*}{\to} \varpi_* \mathbb{P}^- \quad \text{for every } \widehat{\mu} \in \mathcal{M}_1(\widehat{X}).$$

This is a version of Proposition 6.1.1 for recurrent IFSs.

As in the case of IFSs with probabilities, when $X = [0,1]$ we can improve Theorem 6.2.1. In this proposition it is assumed that $\mathbb{P}^-(S_{\rm t}) = 1$ (verifying this assumption is in general difficult). When $X = [0,1]$ we can replace this condition by a "splitting condition" that is quite straightforward to verify.

Consider a recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ defined on a compact metric space X. A cylinder $[j_1 \dots j_s]$ is called *admissible* if $\mathbb{P}^+([j_1 \dots j_s]) > 0$.

Definition 2.2.1 (Splitting Markov measure) *Consider an IFS* $\mathfrak{F} = \mathrm{IFS}(T_1, \dots, T_k)$ *defined on* $[0,1]$ *and a non-trivial closed interval* J *of* $[0,1]$. *A Markov measure* $\mathbb{P}^+$ *defined on* Σ_k^+ splits the IFS $\mathfrak{F}$ in J *if*

– $T_i(J) \subset J$ *and* $T_i|_J$ *is injective for every* $j \in \{1, \dots, k\}$,

– *there are admissible cylinders* $[i_1 \dots i_\ell]$ *and* $[j_1 \dots j_s]$ *of* $\mathbb{P}^+$ *with* $i_1 = j_1$ *such that*

$$T_{j_1} \circ \cdots \circ T_{j_s}(I) \cap T_{i_1} \circ \cdots \circ T_{i_\ell}(I) = \emptyset$$

and

$$T_{j_1} \circ \cdots \circ T_{j_s}(I) \cup T_{i_1} \circ \cdots \circ T_{i_\ell}(I) \subset J.$$

When $J = I$ *we say that* $\mathbb{P}^+$ splits $\mathfrak{F}$.

Let T be a measure-preserving transformation on a probability space $(X, \mathbb{B}, \mu)$. Recall that (T, μ) is *ergodic* if for every measurable set A with $T^{-1}(A) = A$ it holds $\mu(A) = 0$ or $\mu(A) = 1$. Recall that (T, μ) is *mixing* if

$$\lim_{n \to \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\, \mu(B) \quad \text{for every } A, B \in \mathbb{B}.$$

A Borel measure μ on Σ_k^+ is *mixing* if the system (σ, μ) is mixing.

Next theorem states consequences of the splitting property of a Markov measure and is the main tool to get the asymptotic stability of the Markov operator.

Theorem 5 *Consider an* $\mathrm{IFS}(T_1, \dots T_k)$ *defined on the interval* $[0,1]$. *If* $\mathbb{P}^+$ *is a mixing Markov measure that splits the* IFS *in some non-trivial closed interval* J *then* $\mathbb{P}^+(S_{\mathrm{t}}) = 1$.

Next theorem gives sufficient conditions for the asymptotic stability of the Markov operator.

Theorem 6 *Consider a recurrent* $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ *defined on the interval* $[0,1]$. *Suppose that the inverse Markov measure* $\mathbb{P}^-$ *is mixing and splits the* IFS *in some non-trivial closed interval* J. *Then* $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ *is asymptotically stable and* $\varpi_* \mathbb{P}^-$ *is the stationary measure of the Markov operator* $\mathfrak{S}_{\mathbb{P}^+}$.

Note that $\mathbb{P}^+$ is mixing if and only if $\mathbb{P}^-$ is mixing. However, a splitting property for $\mathbb{P}^+$ does not imply a splitting property for $\mathbb{P}^-$ (and vice-versa).

2.3 Step skew products

Given an IFS $\mathfrak{F} = \mathrm{IFS}(T_1, \dots T_k)$, $T_i \colon X \to X$, consider its associated skew product map defined in (1.2.1)

$$F_{\mathfrak{F}} \colon \Sigma_k \times X \to \Sigma_k \times X, \qquad F_{\mathfrak{F}}(\xi, t) \stackrel{\text{def}}{=} (\sigma(\xi), T_{\xi_0}(t)).$$

For simplicity, write $F = F_{\mathfrak{F}}$. Recall the definition of the *maximal attractor* of F in (1.2.2). We say that F is *topologically mixing* if for every pair of non-trivial open sets U and V of $\Sigma_k \times X$ there is n_0 such that $F^n(V) \cap U \neq \emptyset$ for all $n \geq n_0$. Finally, we denote by $\omega_F(z)$ the ω-limit set of the point z for F.

For the next theorem, recall the definitions of the set $S_{\mathrm{t}}^- \subset \Sigma_k$ of trivial spines in (1.2.4), the coding map in (1.2.5), the graph of ϱ in (1.2.6), and the target set A_{t}. We denote by F_{t} the restriction of F to the set $\Sigma_k \times \overline{A_t}$ and by $\Lambda_{F_{\mathrm{t}}}$ the maximal attractor of this restriction.

Theorem 7 *Let F be a step skew product map associated to an* IFS *defined on a compact metric space X such that $S_{\mathrm{t}}^- \neq \emptyset$. Then the following holds:*

1. *For every $z = (\xi, x)$ such that ξ is disjunctive it holds $\overline{\mathrm{graph}\,\varrho} \subset \omega(z)$.*

2. $\Lambda_{F_{\mathrm{t}}} = \overline{\mathrm{graph}\,\varrho}$.

3. *F is topologically mixing in $\overline{\mathrm{graph}\,\varrho}$.*

4. *If either $\overline{A_{\mathrm{t}}}$ is a Lyapunov stable fixed point of the Barnsley-Hutchinson operator or has nonempty interior, then for every $z = (\xi, x)$ such that ξ is disjunctive we have $\overline{\mathrm{graph}\,\varrho} = \omega(z)$.*

Note that in the above theorem the injectivity of the underlying IFS of the skew product map is not required.

To state ergodic properties for skew products consider the projection

$$\pi_1 \colon \Sigma_k \times X \to \Sigma_k, \quad \pi_1(\omega, x) \stackrel{\text{def}}{=} \omega. \tag{2.3.1}$$

Given a probability μ in $\Sigma_k \times X$ the measure $(\pi_1)_*\mu$ is called the *marginal* of μ in Σ_k. For a σ-invariant measure λ consider the set of measures defined on $\Sigma_k \times X$ whose marginal is λ,

$$\mathcal{M}_\lambda \stackrel{\text{def}}{=} \{\mu \in \mathcal{M}_1(\Sigma_k \times X) \colon (\pi_1)_*\mu = \lambda\}. \tag{2.3.2}$$

The set $\mathcal{M}_\lambda$ is compact and F_*-invariant, see Proposition 8.1.1. We say that $F_{*|\mathcal{M}_\lambda}$ *has a global attractor* v_λ if $F_*^n\mu \to v_\lambda$ for every probability measure μ in $\mathcal{M}_\lambda$.

For the precise definitions of isomorphic preserving measures transformations and disintegration of measures see Definitions 8.1.5 and 8.1.3, respectively.

Theorem 8 (Global attractors) *Consider a skew product map* $F = F_{\mathfrak{F}}$ *associated to an* IFS $\mathfrak{F}$. *Let* λ *be an ergodic probability such that* $\lambda(S_t^-) = 1$. *Then we have the following:*

1. $F_{*|\mathcal{M}_\lambda}$ *has a global attractor* v_λ *and the disintegration of* v_λ *with respect to* λ *is the Dirac delta measure* $\delta_{\varrho(\cdot)}$.
2. (F, v_λ) *is isomorphic to* (σ, λ).
3. $\operatorname{supp} v_\lambda = \overline{\operatorname{graph}\,(\varrho_{|\operatorname{supp}\lambda})}$.
4. *For* λ*-almost every sequence* ξ *and every point* $x \in X$ *it holds*

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{F^i(\xi,x)} \to v_\lambda.$$

This theorem implies that (F, v_λ) carries all ergodic properties of (σ, λ) as entropy, mixing properties, and ergodicity, for instance.

As consequence of Theorem 8 we get a version of Elton's ergodic theorem, see (15) and (5). For the next corollary recall the definition of the map $\pi \colon S_t \to X$ in (2.1.1).

Corollary 9 *Consider a recurrent* IFS$(T_1, \ldots T_k; \mathbb{P}^+)$. *Suppose that* $\mathbb{P}^-(S_t) = 1$. *Then for* $\mathbb{P}^+$*-almost every sequence* $\xi \in \Sigma_k$ *and every point* x *it holds*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T_{\xi_{i-1}} \circ \cdots \circ T_{\xi_0}(x)) = \int f\, d(\pi_*\mathbb{P}^-),$$

for every continuous function $f \colon X \to \mathbb{R}$.

We will see in Proposition 6.3.1 that an recurrent IFS$(T_1, \ldots T_k; \mathbb{P}^+)$ satisfying $\mathbb{P}^-(S_t) = 1$ has a unique invariant measure, given by $\varpi_*\mathbb{P}^-$ (recall the definition of ϖ in (2.2.4)). We observe that the measure $\pi_*\mathbb{P}^-$ in Corollary 9 is the push-forward of the unique stationary measure by the map $\Pi(\xi, i) = x$, that is, $\Pi_*(\varpi_*\mathbb{P}^-) = \pi_*\mathbb{P}^-$. This follows from $\Pi \circ \varpi = \pi$.

Remark 2.3.1 By standard arguments, one can replace the continuous function f in Corollary 9 by any characteristic function χ_B with $\pi_*\mathbb{P}^-(\partial B) = 0$. However, since the measure $\pi_*\mathbb{P}^-$ is, in principle, unknown, it is not possible to know if the boundary of a set has zero measure. For IFSs defined on the interval we have the following. Consider a recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ defined on $[0,1]$ such that $\mathbb{P}^+$ is a mixing Markov measure that splits the IFS in some non-trivial closed interval. Then the measure $\pi_* \mathbb{P}^-$ is atom free. This fact follows from Theorem 5.1.2. Therefore the stationary measure of any interval is the limit of any empirical distribution (starting at any point $x \in [0,1]$).

A version of Elton's ergodic theorem for some infinite IFSs and Bernoulli measures was announced in (22). The same type of ergodic result was obtained in (30) (there called "Strong Law of Large Numbers") for IFSs consisting of circle homeomorphisms with at least one of them having dense orbit.

We now consider skew products whose fiber maps are defined on the interval $[0,1]$. Thus in the next definitions we consider $X = [0,1]$ and *standard measures* $\mathfrak{s} = \mathfrak{b} \times m$, where $\mathfrak{b}$ is a Bernoulli measure (with nontrivial weights) in Σ_k and m is the Lebesgue measure in $[0,1]$.

The *realm of attraction* of a set $A \subset \Sigma_k \times [0,1]$, denoted by $\delta(A)$, is the set of points z such that $\omega(z) \subset A$. Definitions 2.3.2, 2.3.3, and 2.3.4 below are given with respect a fixed standard measure $\mathfrak{s}$.

Definition 2.3.2 (Milnor attractor) A closed set A is a *Milnor attractor* of $F_{\mathfrak{F}}$ if it satisfies the following conditions:

- $\mathfrak{s}(\delta(A)) > 0$ and
- $\mathfrak{s}(\delta(A')) < \mathfrak{s}(\delta(A))$ for every closed set A' strictly contained in A.

Definition 2.3.3 (Likely limit set) The *likely limit set* A_M of $F_{\mathfrak{F}}$ is the smallest compact subset of $\Sigma_k \times [0,1]$ with the property that $\omega(z) \subset A_M$ for $\mathfrak{s}$-almost every point $z \in \Sigma_k \times [0,1]$.

The likely limit set is well defined and is the maximal Milnor attractor, see (27).

The *statistical ω-limit set* of point z, denoted by $\omega_{\text{stat}}(z)$, is the set of points $z \in \Sigma_k \times [0,1]$ such that for every neighbourhood U of z it holds

$$\limsup_{N \to \infty} \frac{1}{N} \#\{n \colon F^n(z) \in U, \, 0 \le n < N\} > 0.$$

Definition 2.3.4 (Statistical attractor) The *statistical attractor* A_{stat} is the smallest closed subset of $\Sigma_k \times [0,1]$ such that $\omega_{\text{stat}}(z) \subset A_{\text{stat}}$ for $\mathfrak{s}$-almost every point $z \in \Sigma_k \times [0,1]$.

It follows from definition that $\omega_{\text{stat}}(z) \subset \omega(z)$. Hence $A_{\text{stat}} \subset A_M$.

Theorem 10 (Milnor attractors) *Let F be a skew product map associated to an* IFS *defined on the interval $[0,1]$ such that $S_{\mathrm{t}}^{-} \neq \emptyset$. Then*

1. *The statistical attractor A_{stat} and every Milnor attractor of F contain the set $\overline{\mathrm{graph}\,\varrho}$.*

2. *If either $\overline{A_{\mathrm{t}}}$ is a Lyapunov stable fixed point of the Barnsley-Hutchinson operator or has nonempty interior, then the likely limit set A_M is equal to $\overline{\mathrm{graph}\,\varrho}$. In particular, the likely limit set is the unique Milnor attractor and*
$$A_M = A_{\mathrm{stat}} = \overline{\mathrm{graph}\,\varrho}.$$

3. *If A_{t} is not a singleton and the* IFS *is injective in some $\mathcal{B}$-invariant closed interval J, then there is a unique F-invariant measure with marginal $\mathfrak{b}$ (that we denote by $\mu_{\mathfrak{b}}$) and for $\mathfrak{b}$-almost every sequence ξ and every point $x \in [0,1]$ it holds*
$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{F^i(\xi,x)} \to \mu_{\mathfrak{b}}.$$
In particular, the measure $\mu_{\mathfrak{b}}$ is physical.

4. *Under assumptions of item (2) and (3) we have $\mathrm{supp}\,\mu_{\mathfrak{b}} = A_M$.*

The second item of the above theorem is a consequence of Theorem 7.

Let us state a remark that is an immediate consequence of Theorem 8.

Remark 2.3.5 Consider a step skew product $F\colon \Sigma_k \times [0,1] \to \Sigma_k \times [0,1]$. A *good measure* of F with respect to the standard measure $\mathfrak{s} = \mathfrak{b} \times m$ is any accumulation point of the sequence
$$\frac{1}{n}\sum_{i=0}^{n-1} F_*^i(\mathfrak{s}).$$
The *minimal attractor* of F is the closure of the union of the supports of all its good measures. See (16) for a discussion about minimal attractors.

As a consequence of Theorem 8 we have that if $\mathfrak{b}(S_{\mathrm{t}}^{-}) = 1$ then the minimal attractor of F is $\overline{\mathrm{graph}\,\varrho}$. Hence, under the hypotheses of item (2) in Theorem 10, the different notions of attractors coincide.

Finally, we give robust examples of step skew products whose fiber topological structure is much more complex than perhaps expected. We start by recalling the robust examples in (20) of a skew product with two injective and order preserving fiber maps defined on $[0,1]$ whose likely limit set A_M is the unique Milnor attractor and is equal to $\overline{\mathrm{graph}\,\varrho}$. This set is bony-like meaning that for almost every $\xi \in \Sigma_2$ the intersections $(\xi \times [0,1]) \cap \overline{\mathrm{graph}\ \varrho}$ is a point and for the remaining points $(\xi \times [0,1]) \cap \overline{\mathrm{graph}\ \varrho}$ is an interval, where the last possibility occurs densely in Σ_2. Compare also with the constructions in (12).

Kudryashov (21) poses the following question about the fiber structure of attractors: *Does the likely limit set of typical step skew products over a Markov shift intersect each fiber on a finite union of intervals and points?* (see (21, Question 2.9.3)). We use the example in (20) as a plug (see the notion of a $\mathcal{K}$-pair in Section 10) to construct robust examples of step skew products having bony-like Milnor attractors with complicated fiber structure.

More precisely, in the space of skew products $F_{\mathfrak{F}} \colon \Sigma_k \times [0,1] \to \Sigma_4 \times [0,1]$ with C^1-fiber maps, called C^1 *smooth step skew products*, we consider the distance

$$d(F_{\mathfrak{F}}, F_{\mathfrak{G}}) = \max_i d_{C^1}(f_i, g_i), \quad \mathfrak{F} = \{f_1, \dots, f_k\}, \ \mathfrak{G} = \{g_1, \dots, g_k\},$$

where d_{C^1} denotes the C^1-uniform distance.

Theorem 11 (Milnor attractors with disconnected bones) *For every $n \geq 1$ there is an open set $\mathcal{S}$ of C^1 smooth step skew product $F \colon \Sigma_4 \times [0,1] \to \Sigma_4 \times [0,1]$ over the Bernoulli shift with injective fiber maps such that every $F \in \mathcal{S}$ satisfies the following conditions:*

1. *The likely limit set A_M of F is the unique Milnor attractor A_M and F is topologically mixing in A_M.*
2. *The projection ϱ is defined $\mathfrak{b}$-almost everywhere and $A_M = \overline{\mathrm{graph}\ \varrho}$.*
3. *There is a dense subset Γ of Σ_4 such that for every sequence $\xi \in \Gamma$ the intersection of the fibre $\{\xi\} \times [0,1]$ with the Milnor attractor is a union of n disjoint non-trivial intervals.*

It is clear from the proof that one can perform a similar construction with Σ_k for every $k \geq 4$. We do not know if for $k \in \{2,3\}$ a similar construction can be done.

3
Preliminaries and notation

We now establish some basic definitions and notations.

3.1
Distances

Throughout this thesis (X, d) is a compact metric space and $\mathcal{P}(X)$ denotes the power set of X. Given a point $x \in X$ and a set $A \subset X$, distance between x and A is defined by

$$d(x, A) \stackrel{\text{def}}{=} \inf\{d(x, a)\colon a \in A\}.$$

The *Hausdorff distance* between two sets $A, B \subset X$ is defined by

$$d_H(A, B) \stackrel{\text{def}}{=} \max\{h_s(A, B), h_s(B, A)\}, \quad \text{where} \quad h_s(A, B) \stackrel{\text{def}}{=} \sup_{a \in A} d(a, B).$$

Note that, in general, d_H is only a pseudo-metric defined on $\mathcal{P}(X)$. Let $\mathcal{H}(X) \subset \mathcal{P}(X)$ be the set of all non-empty compact subsets of X. Then $(\mathcal{H}(X), d_H)$ is a compact metric space, see (3).

3.2
Shifts spaces

Let $\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$ be the set of infinite two-sided sequences of symbols in $\{1, \dots, k\}$. Given a two- sided sequence ω, we define the sequence $\sigma(\omega)$ by $(\sigma(\omega))_i = \omega_{i+1}$. In this way we get a self map σ defined on Σ_k called *shift* map. The pair (Σ_k, σ) is called the *full two-sided shift.* The shift space Σ_k is a compact topological space whose basis consist of the *cylinders*:

$$[m; a_r, \dots, a_\ell] = \{\omega\colon \omega_m = a_r, \dots, \omega_{m+\ell} = a_\ell\}, \quad \text{where } m \in \mathbb{Z}. \tag{3.2.1}$$

Let $P = (p_{ij})$ be a transition matrix and $\bar{p}$ a stationary probability vector. There is a unique σ-invariant Borel probability $\mathbb{P}$ called *Markov measure* (associate to $(P, \bar{p})$), such that for every cylinder $[m; a_r, \dots, a_\ell]$ it holds

$$\mathbb{P}([m; a_r, \dots, a_\ell]) = p_{a_r} p_{a_r a_{r+1}} \dots p_{a_{\ell-1} a_\ell}.$$

3.3
Inverse Markov measures

Consider a transition matrix $P = (p_{ij})$ and a stationary probability vector $\bar{p} = (p_1, \ldots, p_k)$ of P. If all entries of $\bar{p}$ are (strictly) positive then the *inverse transition matrix* associated to $(P, \bar{p})$ is the matrix $Q_{(P,\bar{p})} = (q_{ij})$ where

$$q_{ij} \stackrel{\text{def}}{=} \frac{p_j}{p_i} p_{ji}.$$

Note that $Q = Q_{(P,\bar{p})}$ is a transition matrix and $\bar{p}$ is a stationary probability vector of $Q_{(P,\bar{p})}$. We observe that if P is primitive if and only if Q is primitive.

Denote by $\mathbb{P}^-$ the Markov measure associated to $(Q, \bar{p})$. For every cylinder $[a_0 \ldots a_\ell]$ it holds

$$\mathbb{P}^-([a_0 \ldots a_\ell]) = \mathbb{P}^+([a_\ell \ldots a_0]),$$

where $\mathbb{P}^+$ is the Markov measure associated to $(P, \bar{p})$.

Let us observe that a Markov measure $\mathbb{P}^+$ is mixing if and only if the transition matrix is *primitive*[1] (i.e. there is $n \geq 1$ such that all the entries of P^n are strictly positive), see for instance (11, page 79). As a consequence, $\mathbb{P}^-$ is mixing if and only if P is primitive.

[1] Also called *aperiodic*.

4
Attractors of iterated function systems

This chapter is devoted to the study of fixed points and the attractors of the Barnsley-Hutchinson operator of an IFS (see Sections 4.1 and 4.2). Our goal is to prove Theorems 1, 2, and 3 (see Sections 4.3, 4.4, and 4.6, respectively). We also get some topological properties of the target set A_{t} in Section 4.5.

In what follows we consider $\mathfrak{F} = \mathrm{IFS}(T_1, \ldots T_k)$ and denote by $\mathcal{B}_{\mathfrak{F}} = \mathcal{B}$ its Barnsley-Hutchinson operator, recall (1.1.1).

4.1
Fixed points for the Barnsley-Hutchinson operator

We will show that every compact invariant set A of X (i.e., $\mathcal{B}(A) \subset A$) contains some fixed point of $\mathcal{B}$. Since X is invariant for $\mathcal{B}$ this implies that $\mathcal{B}$ always has at least one fixed point. To each set A we associate the set $A^* \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \mathcal{B}^n(A)$, recall (2.1.3).

Recall that $\mathcal{H}(X)$ denotes the set consisting of all non-empty compact subsets of X. We consider in $\mathcal{H}(X)$ the Hausdorff distance d_H.

Proposition 4.1.1 (Existence of fixed points of $\mathcal{B}$) *Consider $A \in \mathcal{H}(X)$ such that $\mathcal{B}(A) \subset A$. Then A^* is a fixed point of $\mathcal{B}$. In particular, X^* is a fixed point of $\mathcal{B}$.*

Proof. The proposition follows from the next lemma and the continuity of $\mathcal{B}$.

Lemma 4.1.2 *Let (A_n) be a sequence of nested compact sets, $A_{n+1} \subset A_n$, and $A = \bigcap_{n\geq 0} A_n$. Then $d_H(A_n, A) \to 0$.*

Proof. The proof is by contradiction. If the lemma is false there are $\epsilon > 0$ and a subsequence (n_ℓ), $n_\ell \to \infty$, such that $d_H(A_{n_\ell}, A) \geq \epsilon$ for all ℓ. Since $A \subset A_n$, for each ℓ there is a point $p_\ell \in A_{n_\ell}$ such that $d(p_\ell, A) \geq \epsilon$. By compactness, taking a subsequence if necessary, we can assume that $p_\ell \to p$. As (A_n) is nested it follows that $p \in A$, contradicting that $d(p_\ell, A) \geq \epsilon$ for all ℓ. ■

To prove the proposition it is enough to apply the lemma to nested sequence $A_n = \mathcal{B}^n(A)$. ■

Now let us look more closely to the fixed point X^* of $\mathcal{B}$. For that to each $\xi \in \Sigma_k^+$ we consider its *fibre* defined by

$$I_\xi \stackrel{\text{def}}{=} \bigcap_{n\geq 0} T_{\xi_0}\circ\cdots\circ T_{\xi_n}(X), \quad \text{if } \xi=\xi_0\xi_1\ldots. \tag{4.1.1}$$

We will see in Lemma 4.1.3 that the set X^* is the union of the fibres I_ξ.

Note that every fibre is a non-empty set: just note that $(T_{\xi_0}\circ\cdots\circ T_{\xi_n}(X))_{n\in\mathbb{N}}$ is a sequence of nested compacts sets. Moreover, when X is an interval, the fibres also are intervals (may be trivial ones). With this definition, the set S_{t} of weakly hyperbolic sequences, recall (1.1.3), is given by

$$S_{\mathrm{t}} = \{\xi\in\Sigma_k^+ \colon I_\xi \text{ is a singleton}\}.$$

From the definition of the target set A_{t} in (2.1.2) it immediately follows that

$$A_{\mathrm{t}} = \bigcup_{\xi\in S_{\mathrm{t}}} I_\xi. \tag{4.1.2}$$

Recall that by definition for every set A we have

$$A^* = \bigcap_{n\geq 0}\mathcal{B}^n(A) = \bigcap_{n\geq 0}\ \bigcup_{\xi\in\Sigma_k^+} T_{\xi_0}\circ\cdots\circ T_{\xi_{n-1}}(A). \tag{4.1.3}$$

Next lemma just says that the operations " $\cup$ " and " $\cap$ " above commute.

Lemma 4.1.3 *Let $A\in\mathcal{H}(X)$ such that $\mathcal{B}(A)\subset A$. Then*

$$A^* = \bigcap_{n\geq 0}\mathcal{B}^n(A) = \bigcup_{\xi\in\Sigma_k^+}\ \bigcap_{n\geq 0} T_{\xi_0}\circ\cdots\circ T_{\xi_n}(A).$$

In particular,

$$X^* = \bigcup_{\xi\in\Sigma_k^+} I_\xi.$$

Proof. Condition $\mathcal{B}(A)\subset A$ implies that $\mathcal{B}^n(A)$ is a decreasing nested family of compact subsets and $T_i(A)\subset A$ for all $i=1,\ldots,k$. From equation (4.1.3) it follows immediately that

$$\bigcup_{\xi\in\Sigma_k^+}\ \bigcap_{n\geq 0} T_{\xi_0}\circ\cdots\circ T_{\xi_n}(A) \subset \bigcap_{n\geq 0}\ \bigcup_{\xi\in\Sigma_k^+} T_{\xi_0}\circ\cdots\circ T_{\xi_n}(A) = A^*.$$

which implies the inclusion "$\supset$".

To prove the inclusion "$\subset$" we will use the compactness the of hyperspace $(\mathcal{H}(\Sigma_k^+), d_H)$ where d_H is the Hausdorff metric defined using the following metric d on Σ_k^+,

$$d(\xi,\omega) = \frac{1}{2^{n_0}} \quad \text{where} \quad n_0=\min\{n\colon \xi_n\neq\omega_n\}.$$

The metric space (Σ_k^+, d) is compact and therefore $(\mathcal{H}(\Sigma_k^+), d_H)$ is also compact.

We now prove the in inclusion "$\subset$". Given $p \in A^*$, by definition, for each n there is a finite sequence $\xi_0^n \dots \xi_n^n$ such that

$$p \in T_{\xi_0^n} \circ \cdots \circ T_{\xi_n^n}(A).$$

Consider the sequence of cylinders $E_n = [\xi_0^n \dots \xi_n^n]$. Since $E_n \in \mathcal{H}(\Sigma_k^+)$, by compactness there is a subsequence E_{n_k} converging to some compact K. Let $\xi \in K$. We claim that $p \in T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A)$ for every n and hence $p \in \bigcup_{\xi \in \Sigma_k^+} \bigcap_{n \geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A)$ proving the lemma. To prove our claim, fix n and take k such that $n_k \geq n$ and

$$d(\xi, E_{n_k}) < \frac{1}{2^n}.$$

In particular there is $\omega \in E_{n_k}$ such that $d(\xi, \omega) < \frac{1}{2^n}$. By definition of d we get that $\xi_0 = \omega_0 \dots, \xi_n = \omega_n$. On the other hand we have $\omega_0 = \xi_0^{n_k}, \dots, \omega_n = \xi_n^{n_k}$ $(n_k \geq n)$ and

$$p \in T_{\xi_0^n} \circ \cdots \circ T_{\xi_{n_k}^{n_k}}(A),$$

which implies that $p \in T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A)$. ■

4.2 Conley and strict attractors

In this section we introduce the notion of a minimum fixed point of an IFS and prove that if $S_{\mathrm{t}} \neq \emptyset$ then the closure of the target set is a minimum fixed point of $\mathcal{B}$. We also characterise strict attractors for IFSs with $S_{\mathrm{t}} \neq \emptyset$.

4.2.1 Minimal fixed points and minimum fixed point

Note that the set X^* is the *maximum fixed point* (ordered by inclusion) of the map $\mathcal{B}$, meaning that if K is another fixed point of $\mathcal{B}$ then $K \subset X^*$. A natural question is about the existence of a *minimum fixed point* Y of $\mathcal{B}$, meaning that if K is any fixed point of $\mathcal{B}$ then $Y \subset K$. By definition, maximum and minimum fixed points are unique. We see that, in general, may no exist a minimum fixed point. Observe that an application of Zorn's lemma immediately provides a *minimal fixed point* for $\mathcal{B}$, that is, a fixed point that does not contain properly another fixed point. Note that, by definition, a minimum fixed point is minimal, but the converse is not true in general.

To get a simple example of an IFS without a minimum fixed point just consider the IFS(T_1, T_2) defined on the interval $[0, 1]$ with $T_1(x) = x$ and $T_2(x) = 1 - x$. For each $x \in [0, 1]$, the set $\{x, 1 - x\}$ is a fixed point of $\mathcal{B}$.

Clearly, the set $\{x, 1-x\}$ is minimal. It is also obvious, that there is not a fixed point contained in all fixed points. Thus the minimal fixed points cannot be minimum fixed points.

In the previous example we have $S_{\mathrm{t}} = \emptyset$. Next proposition shows that the condition $S_{\mathrm{t}} \neq \emptyset$ guarantees the existence of a minimum fixed point. For the next result recall the characterisation of the set A_t in (4.1.2).

Proposition 4.2.1 *Suppose that $S_{\mathrm{t}} \neq \emptyset$. Then $\overline{A_{\mathrm{t}}}$ is the minimum fixed point of $\mathcal{B}$.*

Proof. We need to see that $\mathcal{B}(\overline{A_{\mathrm{t}}}) = \overline{A_{\mathrm{t}}}$ and $\overline{A_{\mathrm{t}}} \subset K$ for every compact set K with $\mathcal{B}(K) = K$.

To prove the second assertion, fix any compact set K that is fixed point of $\mathcal{B}$ and take any point $p \in A_{\mathrm{t}}$. By the characterisation of A_t in (4.1.2) there is a sequence ξ such that

$$\{p\} = \bigcap_{n\geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X) \supset \bigcap_{n\geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(K). \tag{4.2.1}$$

Since the last intersection is non-empty and contained in K it follows $p \in K$. This implies that A_{t} (and hence $\overline{A_{\mathrm{t}}}$) is contained in K.

To see that $\mathcal{B}(\overline{A_{\mathrm{t}}}) = \overline{A_{\mathrm{t}}}$ note that the continuity of the maps T_i implies that for p as in (4.2.1) and every $i = 1, \dots, k$ it holds

$$\{T_i(p)\} = \bigcap_{n\geq 0} T_i \circ T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X).$$

This implies that $\mathcal{B}(A_t) \subset A_t$. Hence, by continuity of the maps T_i, $\mathcal{B}(\overline{A_t}) \subset \overline{A_t}$. By definition this implies that $(\overline{A_{\mathrm{t}}})^* \subset (\overline{A_{\mathrm{t}}})$. By Proposition 4.1.1, $(\overline{A_{\mathrm{t}}})^*$ is a fixed point of $\mathcal{B}$. The minimality property proved before implies that $\overline{A_{\mathrm{t}}} \subset (\overline{A_{\mathrm{t}}})^*$. This ends the proof of the proposition. ∎

Note that in the proof of the proposition we obtained the following.

Scholium 4.2.2 *Given an* IFS$(T_1, \dots T_k)$ *with $S_{\mathrm{t}} \neq \emptyset$ it holds $T_i(A_t) \subset A_t$.*

4.2.1.1
Characterisation of strict attractors

The proposition below claims that an IFS with a weakly hyperbolic sequence has at most one strict attractor and describes such an attractor.

Proposition 4.2.3 *Consider an* IFS *defined on a compact metric space such that $S_{\mathrm{t}} \neq \emptyset$. Then there exists at most one strict attractor. If such a strict attractor exists then it is equal to $\overline{A_{\mathrm{t}}}$.*

Proof. If there are no strict attractor we are done. Otherwise assume that there is a strict attractor K. Since K is a fixed point of $\mathcal{B}$, Proposition 4.2.1 implies that $\overline{A_t} \subset K$. Since by definition of a strict attractor the set K attracts every compact set in a neighbourhood of it, the minimum fixed point $\overline{A_t}$ is attracted by K. Therefore $\overline{A_t} = K$, proving the proposition. ■

The following example shows that there are IFSs with $S_t \neq \emptyset$ without strict attractors. In this example $\overline{A_t}$ is stable (recall (1.1.4)).

Example 4.2.4 Consider the maps T_1, T_2: $[0, 2] \to [0, 2]$ depicted in Figure 4.1 and defined by

- $T_1(x) = \frac{1}{3}x$ and
- T_2: $[0, 2] \to [0, 2]$ is the piecewise-linear map defined by $T_2(x) = \frac{1}{3}x + \frac{2}{3}$ for $x \in [0, 1]$ and $T_2(x) = x$ for $x \in [1, 2]$.

Let $\mathcal{C}$ be the standard ternary Cantor set in the interval $[0, 1]$. We claim that $S_t \neq \emptyset$, $\overline{A_t} = \mathcal{C}$, and $\mathcal{C}$ is not a strict attractor. Indeed, $\mathcal{C}$ is not a Conley attractor. We now prove these assertions.

First, as T_1 is a contraction $\bar{1} \in S_t$, where $\bar{1}$ is the sequence whose terms are all equal to 1.

For the second assertion, consider the auxiliary IFS(f_1, f_2) where $f_1 = T_{1|[0,1]}$ and $f_2 = T_{2|[0,1]}$. Note that $\mathcal{C}$ is the attractor of IFS(f_1, f_2) (see, for instance, Example 1 in (17, Section 3.3)). In particular, the set $\mathcal{C}$ is the unique fixed point of the Barnsley-Hutchinson operator $\mathcal{B}$ of IFS(T_1, T_2) contained in $[0, 1]$. Since $[0, 1]$ is $\mathcal{B}$-invariant, by Propositions 4.1.1 and 4.2.1 we have $\overline{A_t} \subset [0, 1]^* \subset [0, 1]$ and therefore $\overline{A_t} = \mathcal{C}$.

To see that $\mathcal{C}$ is not a strict attractor, just note that every open neighbourhood of $\mathcal{C}$ necessarily contains an interval of the form $[1, \delta)$. Since $T_2(x) = x$ for all $x \in [1, \delta)$ the assertion follows.

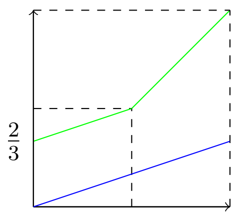

Figure 4.1: The set $\overline{A_t}$ is not a Conley attractor

4.3
Proof of Theorem 1

Since every strict attractor is a Conley attractor, to prove the theorem 1 it is enough to see that given an IFS$(T_1, \dots T_k)$ such that $\overline{A_t}$ is a non-empty Conley attractor then $\overline{A_t}$ is a strict attractor. We need the following preparatory lemma:

Lemma 4.3.1 *Consider sequences (A_n) of compact sets in $\mathcal{H}(X)$ and (p_n) of points in X with $A_n \to A$ and $p_n \to p$ in the Hausdorff distance d_H. Then*

$$d(p, A) = \lim_{n \to \infty} d(p_n, A_n).$$

Proof. We use the following "triangular" inequality: given a point q and two compact sets A and B it holds

$$d(q, A) \leq d(q, B) + d_H(A, B).$$

Consider the sequences (A_n) and (p_n) in the lemma. Applying twice the "triangular" inequality above we get

$$d(p, A) \leq d(p, p_n) + d(p_n, A) \leq d(p, p_n) + d(p_n, A_n) + d_H(A_n, A).$$

By hypothesis, $d(p, p_n) \to 0$ and $d_H(A_n, A) \to 0$. We conclude that

$$d(p, A) \leq \liminf_n d(p_n, A_n). \tag{4.3.1}$$

Applying again twice the "triangular" inequality, we get

$$d(p_n, A_n) \leq d(p_n, p) + d(p, A_n) \leq d(p_n, p) + d(p, A) + d_H(A, A_n).$$

This implies that

$$\limsup_n d(p_n, A_n) \leq d(p, A). \tag{4.3.2}$$

Equations (4.3.1) and (4.3.2) imply the lemma. ■

We are now ready to prove the theorem. Since $\overline{A_t}$ is a Conley attractor it has an open neighbourhood U such that $\mathcal{B}^n(\overline{U}) \to \overline{A_t}$. To prove that $\overline{A_t}$ is a strict attractor we need to check that for every compact set $K \in \mathcal{H}(\overline{U})$ it holds $\mathcal{B}^n(K) \to \overline{A_t}$. For that it is enough to see that for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds

$$d_H(\overline{A_t}, \mathcal{B}^n(K)) = \max\{h_s(\overline{A_t}, \mathcal{B}^n(K)), h_s(\mathcal{B}^n(K), \overline{A_t})\} \leq \epsilon. \tag{4.3.3}$$

By hypothesis, $\mathcal{B}^n(\overline{U}) \to \overline{A_t}$. Thus there is n_0 such that for every $n \geq n_0$ we have

$$h_s(\mathcal{B}^n(\overline{U}), \overline{A_t}) \leq \epsilon.$$

Therefore, for every $n \geq n_0$,

$$h_s(\mathcal{B}^n(K), \overline{A_t}) \leq h_s(\mathcal{B}^n(\overline{U}), \overline{A_t}) \leq \epsilon.$$

Hence to prove (4.3.3) it remains to see that $h_s(\overline{A_t}, \mathcal{B}^n(K)) \leq \epsilon$ for every n sufficiently large. This is proved in the next lemma.

Lemma 4.3.2 *For every $K \in \mathcal{H}(X)$ it holds $\lim_{n\to\infty} h_s(\overline{A_t}, \mathcal{B}^n(K)) = 0$.*

Proof. The proof is by contradiction. Assume that there are a compact set $K \in \mathcal{H}(X)$ and a sequence (n_ℓ) such that $h_s(\overline{A_t}, \mathcal{B}^{n_\ell}(K)) > \epsilon$ for every ℓ. Note that for each ℓ there is $p_{n_\ell} \in \overline{A_t}$ with $d(p_{n_\ell}, \mathcal{B}^{n_\ell}(K)) > \epsilon$. By compactness we can assume that $p_{n_\ell} \to p^* \in \overline{A_t}$ and that $\mathcal{B}^{n_\ell}(K) \to \widehat{K}$. By Lemma 4.3.1,

$$d(p^*, \widehat{K}) \geq \epsilon. \tag{4.3.4}$$

We now derive a contradiction from this inequality. By construction, there is ℓ_0 such that

$$h_s(B^{n_\ell}(K), \widehat{K}) < \frac{\epsilon}{2}, \quad \text{for all } \ell \geq \ell_0. \tag{4.3.5}$$

Take $q \in B_{\frac{\epsilon}{2}}(p^*) \cap A_t$ and note that there is a sequence $\omega = \omega_0\, \omega_1 \ldots \in S_t$ such that

$$\bigcap_{n\geq 0} T_{\omega_0} \circ \cdots \circ T_{\omega_n}(X) = \{q\}.$$

Therefore there is m_0 such that

$$T_{\omega_0} \circ \cdots \circ T_{\omega_{m-1}}(K) \subset T_{\omega_0} \circ \cdots \circ T_{\omega_{m-1}}(X) \subset B_{\frac{\epsilon}{2}}(p^*) \quad \text{for every } m \geq m_0.$$

Since $T_{\omega_0} \circ \cdots \circ T_{\omega_{m-1}}(K) \subset \mathcal{B}^m(K)$, for every ℓ big enough we have $\mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*) \neq \emptyset$.

Note that for every ℓ sufficiently large $\mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*) \neq \emptyset$ and equation (4.3.5) holds. Hence for every $z \in \mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*)$ we have $d(z, \widehat{K}) < \frac{\epsilon}{2}$ and $d(z, p^*) < \frac{\epsilon}{2}$. Hence $d(p^*, \widehat{K}) < \epsilon$ contradicting (4.3.4). This ends the proof of the lemma. ■

The proof of the theorem is now complete. ■

Scholium 4.3.3 *If U is a neighbourhood of $\overline{A_t}$ such that $\mathcal{B}^n(\overline{U}) \to \overline{A_t}$ then every compact subset of $\overline{U}$ also satisfies $\mathcal{B}^n(K) \to \overline{A_t}$.*

We have the following corollary that allows us to stablish a connection between the set $\overline{A_{\rm t}}$ and semifractals.

Corollary 4.3.4 *Consider an IFS such that $S_t \neq \emptyset$. Then*

$$\lim_{n\to\infty} \mathcal{B}^n(K) = \overline{A_{\rm t}}, \quad \textit{for every compact set } K \subset \overline{A_{\rm t}}.$$

Proof. The statement is an immediate consequence of Lemma 4.3.2 and the invariance of $\overline{A_{\rm t}}$. ■

Remark 4.3.5 Combining Propositions 4.2.1 and Corollary 4.3.4 one gets the following: if $S_{\rm t} \neq \emptyset$ then set $\overline{A_{\rm t}}$ is a minimum fixed point that attracts every compact set inside it.

4.4 Proof of Theorem 2

Suppose that the set $A_{\rm t}$ is non-empty. We need to prove the equivalence of the following three assertions:

1. $\overline{A_{\rm t}} = X^*$;
2. the Barnsley-Hutchinson operator $\mathcal{B}$ has a unique fixed point;
3. X^* is a global attractor (a strict attractor whose basin is the whole space).

The equivalence 1 $\Leftrightarrow$ 2 follows immediately from Proposition 4.2.1 ("the minimum fixed point $\overline{A_{\rm t}}$ is equal to the maximum fixed point X^*").

The implication 3 $\Rightarrow$ 2 follows noting that if K is a fixed point of $\mathcal{B}$ and since X^* is a global attractor then $K = \mathcal{B}^n(K) \to X^*$ and thus $K = X^*$.

To prove 1 $\Rightarrow$ 3 note that, by Lemma 4.1.2, $X^* = \lim_{n\to\infty} \mathcal{B}^n(X)$ and thus X^* is a Conley attractor. Then if $\overline{A_t} = X^*$ we have that $\overline{A_{\rm t}}$ is a Conley attractor, by Theorem 1 and Scholium 4.3.3 this set is a strict attractor whose basin is the whole space. ■

4.5
Structure of the target set

The main result of this section is Proposition 4.5.1 about the topological structure of the target set $A_{\rm t}$. This result will be used in Section 5.3.

We begin by observing that, in general, the set $A_{\rm t}$ is not necessarily closed. The IFS in Example 4.2.4 illustrates this case. In this example $\overline{A_{\rm t}}$ is the ternary Cantor set $\mathcal{C}$ in $[0,1]$, thus $1 \in \overline{A_{\rm t}}$. We claim that $1 \notin \mathrm{A_t}$ and thus $\mathrm{A_t}$ is not closed. Recall the definitions of $\mathrm{IFS}(T_1,T_2)$ and $\mathrm{IFS}(f_1,f_2)$ in this example and consider their natural associated projections π_T and π_f, see (2.1.1). Arguing by contradiction, if $1 \in A_{\rm t}$ then there is a sequence $\xi \in S_{\rm t}$ with $\pi_T(\xi) = 1$. In this case we also have $\pi_f(\xi) = 1$. It is easy to check that $\xi = \bar{2}$ and that $\bar{2} \notin S_{\rm t}$, where $\bar{2} = (\xi_i = 2)$. This gives a contradiction.

Proposition 4.5.1 *Consider* $\mathrm{IFS}(T_1,\dots,T_k)$ *defined on a compact set* X *such that* $A_{\rm t} \neq \emptyset$.

1. *Assume that the IFS is injective in* $\mathrm{A_t}$. *Then either* $\mathrm{A_t}$ *is a singleton or* $\mathrm{A_t}$ *has no isolated points (thus it is infinite).*
2. *Assume that the maps* T_i *are open. Then either* $\mathrm{A_t}$ *has empty interior or* $\mathrm{int}(A_{\rm t}) \subset A_{\rm t} \subset \overline{\mathrm{int}(A_{\rm t})}$.

We observe that in the proof of the first item of the proposition we only use the injectivity of the maps T_i on $A_{\rm t}$.

Let us also observe that if the maps T_i are not injective then the set $A_{\rm t}$ can be finite with more than one element. The maps depicted in Figure 4.2 give an example of this case, where $A_{\rm t} = \{0, \frac{1}{2}, 1\}$.

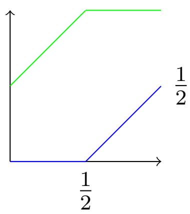

Figure 4.2: $\#(A_{\rm t}) = 3$

Remark 4.5.2 Every injective $\mathrm{IFS}(T_1,\dots,T_k)$ defined on $[0,1]$ satisfies the hypotheses in the second part of Proposition 4.5.1.

Proof. We prove the first item in the proposition. If $A_{\rm t}$ is a singleton we are done. Otherwise $\#(A_{\rm t}) \geq 2$. To see that every $p \in A_{\rm t}$ is not isolated we check that for every every neighbourhood V of p the set $A_{\mathrm{t}} \cap V$ contains at least two points. By definition of A_{t}, there is a finite sequence $\xi_0 \dots \xi_n$ such that

$$T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X) \subset V.$$

In particular,

$$T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A_{\mathrm{t}}) \subset V.$$

Since $T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A_{\mathrm{t}}) \subset A_{\mathrm{t}}$ (recall Scholium 4.2.2) and $T_{\xi_0} \circ \cdots \circ T_{\xi_n}$ is one-to-one in A_t, we have that V contains at least two points of A_t, proving the first part of the proposition.

We now prove the second item of the proposition. If A_{t} has empty interior we are done. Thus we can assume that $\mathrm{int}(A_{\mathrm{t}}) \neq \emptyset$. Since $\mathrm{int}(A_{\mathrm{t}}) \subset A_{\mathrm{t}}$ it only remains to see that $A_{\mathrm{t}} \subset \overline{\mathrm{int}(A_{\mathrm{t}})}$. Take a point $x \in A_{\mathrm{t}}$ and any open neighbourhood V of x. By definition of A_{t} there is a finite sequence $\xi_0 \dots \xi_n$ such that

$$T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X) \subset V.$$

By $T_i(A_{\mathrm{t}}) \subset A_{\mathrm{t}}$ it follows

$$T_{\xi_0} \circ \cdots \circ T_{\xi_n}(\mathrm{int}(A_{\mathrm{t}})) \subset V \cap A_{\mathrm{t}}.$$

Since $\mathrm{int}(A_{\mathrm{t}})$ is an open set and the maps T_i are open, then $T_{\xi_0} \circ \cdots \circ T_{\xi_n}(\mathrm{int}(A_{\mathrm{t}}))$ is a non-empty and open subset of $V \cap A_{\mathrm{t}}$, thus $V \cap \mathrm{int}(A_{\mathrm{t}}) \neq \emptyset$. Since this holds for every neighbourhood of x we get that $x \in \overline{\mathrm{int}(A_{\mathrm{t}})}$. The proof of the proposition is now complete. ∎

4.6
Proof of Theorem 3

Suppose that $\overline{A_{\mathrm{t}}}$ is stable. We need to prove that given any disjunctive sequence ξ and any point x it holds

$$\overline{A_{\mathrm{t}}} = \bigcap_{\ell \geq 0} \overline{\{x_{n,\xi} \colon n \geq \ell\}}, \quad \text{where} \quad x_{n,\xi} \stackrel{\text{def}}{=} T_{\xi_n} \circ \cdots \circ T_{\xi_0}(x).$$

To simplify notation write

$$Y_\ell \stackrel{\text{def}}{=} \{x_{n,\xi} \colon n \geq \ell\}.$$

For the inclusion "$\subset$" take any point $p \in A_{\mathrm{t}}$ and fix $\ell \geq 0$. We need to see that for every neighbourhood V of p it holds

$$V \cap Y_\ell \neq \emptyset. \tag{4.6.1}$$

By definition of A_{t} there is a finite sequence $c_0 \dots c_r$ such that

$$T_{c_r} \circ \cdots \circ T_{c_0}(X) \subset V. \tag{4.6.2}$$

We can assume that $r \geq \ell$. Since ξ has dense orbit there is m_1 such that

$$\xi_{m_1} = c_0,\, \xi_{m_1+1} = c_1,\, \dots,\, \xi_{m_1+r} = c_r.$$

Therefore, from (4.6.2) it follows

$$x_{m_1+r,\xi} = T_{\xi_{m_1+r}} \circ \cdots \circ T_{\xi_{m_1}} \circ T_{\xi_{m_1-1}} \circ \cdots \circ T_{\xi_0}(x) \in V.$$

Since $m_1 + r \geq \ell$ we have that $V \cap Y_\ell \neq \emptyset$, proving (4.6.1).

We now prove the inclusion "$\subset$". Take any neighbourhood V of $\overline{A_{\mathrm{t}}}$. Since $\overline{A_{\mathrm{t}}}$ is stable it has a neighbourhood $V_0 \subset V$ such that $\mathcal{B}^n(V_0) \subset V$ for every $n \geq 0$. Since ξ is a disjunctive sequence and $A_{\mathrm{t}} \subset V_0$ there is $n_0 \in \mathbb{N}$ such that $x_{n_0,\xi} \in V_0$. Hence $Y_{n_0} \subset V$ and thus $\overline{Y_{n_0}} \subset \overline{V}$. As the sequence of sets $(\overline{Y_\ell})$ is nested, we have that $\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{V}$. Since this holds for every neighbourhood V of $\overline{A_{\mathrm{t}}}$ we conclude that

$$\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{A_{\mathrm{t}}}.$$

Finally, as $\overline{Y_\ell}$ is a nested sequence of compact sets, from Lemma 4.1.2 and the definition of a Hausdorff limit, it follows $Y_\ell \longrightarrow \overline{A_{\mathrm{t}}}$, where the convergence is in the Hausdorff distance.

5
Abundance of trivial fibres

In this chapter, for IFSs defined on $[0,1]$, we study the measure of S_{t} for Markov measures and prove Theorem 5, see Section 5.1. In Section 5.2 we prove a result about probabilistic rigidity of the set S_{t}: under quite general conditions, if S_{t} intersects the support of a Markov measure it has full probability. Finally, in Section 5.3 we characterise separable IFSs.

5.1
Proof of Theorem 5

Given an IFS$(T_1, \dots T_k)$ defined on $I = [0,1]$ we need to see that every mixing Markov measure that splits the IFS in some non-trivial closed interval J satisfies $\mathbb{P}^+(S_{\mathrm{t}}) = 1$.

Recall the definition of the fibre I_ξ of a sequence in Σ_k^+ in (4.1.1). Given $x \in [0,1]$ we consider the set of sequences whose fibres contain x defined by

$$\Sigma_x \stackrel{\text{def}}{=} \{\xi \in \Sigma_k^+ \colon x \in I_\xi\}. \tag{5.1.1}$$

Lemma 5.1.1 *Suppose that $\mathbb{P}^+(\Sigma_x) = 0$ for all $x \in [0,1]$. Then $\mathbb{P}^+(S_{\mathrm{t}}) = 1$.*

Proof. Note that if $\xi \notin S_{\mathrm{t}}$ then its fibre I_ξ is a non-trivial interval and hence contains a rational point. This implies that

$$(S_{\mathrm{t}})^c = \Sigma_k^+ \setminus S_{\mathrm{t}} \subset \bigcup_{x \in \mathbb{Q} \cap [0,1]} \Sigma_x.$$

This union is countable and each set Σ_x satisfies $\mathbb{P}^+(\Sigma_x) = 0$, thus $\mathbb{P}^+(S_{\mathrm{t}}) = 1$.
■

In view of Lemma 5.1.1, to see that $\mathbb{P}^+(S_{\mathrm{t}}) = 1$ it is sufficient to show the following:

Theorem 5.1.2 *Consider an* IFS *defined on $I = [0,1]$ and a mixing Markov measure $\mathbb{P}^+$ that splits the* IFS *in some non-trivial interval J. Then $\mathbb{P}^+(\Sigma_x) = 0$ for all $x \in [0,1]$.*

Proof. By the splitting hypothesis there is a pair of admissible cylinders $[i_1 \dots i_\ell]$ and $[j_1 \dots j_s]$ with $i_1 = j_1$ such that

$$T_{j_1} \circ \cdots \circ T_{j_s}(I) \cap T_{i_1} \circ \cdots \circ T_{i_\ell}(I) = \emptyset, \quad \text{and}$$
$$T_{j_1} \circ \cdots \circ T_{j_s}(I) \cup T_{i_1} \circ \cdots \circ T_{i_\ell}(I) \subset J. \tag{5.1.2}$$

Next claim restates the splitting condition:

Claim 5.1.3 *There are admissible cylinders* $[\xi_0 \dots \xi_{N-1}]$ *and* $[\omega_0 \dots \omega_{N-1}]$ *such that* $\xi_0 = \omega_0$, $\xi_{N-1} = \omega_{N-1}$,

$$T_{\xi_0} \circ \cdots \circ T_{\xi_{N-1}}(I) \cap T_{\omega_0} \circ \cdots \circ T_{\omega_{N-1}}(I) = \emptyset \quad \textit{and}$$
$$T_{\xi_0} \circ \cdots \circ T_{\xi_{N-1}}(I) \cup T_{\omega_0} \circ \cdots \circ T_{\omega_{N-1}}(I) \subset J.$$

Proof. Consider $j_1, \dots, j_s$ and $i_1, \dots, i_\ell$ as in (5.1.2). Since $\mathbb{P}^+$ is mixing there is n_0 such that for every $n \geq n_0$ there are admissible cylinders of the form $[i_\ell c_1 \dots c_{n-1} 0]$ and $[j_s d_1 \dots d_{n-1} 0]$. Take now $n_1, n_2 \geq n_0$ and admissible cylinders $[i_\ell c_1 \dots c_{n_1} 0]$ and $[j_s d_1 \dots d_{n_2} 0]$ such that $n_1 + \ell = n_2 + s$. Let $N = \ell + n_1 + 1$. Then the cylinders

$$[\xi_0 \dots \xi_{N-1}] = [i_1 \dots i_\ell c_1 \dots c_{n_1} 0] \quad \text{and} \quad [\omega_0 \dots \omega_{N-1}] = [j_1 \dots i_s d_1 \dots d_{n_2} 0]$$

are admissible and satisfy the intersection and union properties in the claim. To see why this is so note that $T_{c_1} \circ \cdots \circ T_{c_{n_1}} \circ T_0(I) \subset I$ and $T_{d_1} \circ \cdots \circ T_{d_{n_2}} \circ T_0(I) \subset I$. ■

We now fix $x \in I$ and prove that $\mathbb{P}^+(\Sigma_x) = 0$. For that fix N, the admissible cylinders $[\xi_0 \dots \xi_{N-1}]$ and $[\omega_0 \dots \omega_{N-1}]$ in the claim, and for $j \geq 1$ define the sets

$$\Sigma_x^j \stackrel{\text{def}}{=} \{[a_0 \dots a_{jN-1}] \subset \Sigma_k^+ : x \in T_{a_0} \circ \cdots \circ T_{a_{jN-1}}(I)\} \quad \text{and} \quad S_x^j \stackrel{\text{def}}{=} \bigcup_{C \in \Sigma_x^j} C. \tag{5.1.3}$$

Note that by definition $S_x^{j+1} \subset S_x^j$ and that for each $j \geq 1$ it holds $\Sigma_x \subset S_x^j$. Hence

$$\Sigma_x \subset \bigcap_{j \geq 1} S_x^j.$$

Therefore

$$\mathbb{P}^+(\Sigma_x) \leq \mathbb{P}^+\Big(\bigcap_{j \geq 1} S_x^j\Big) = \lim_{j \to \infty} \mathbb{P}^+(S_x^j).$$

Hence the assertion $\mathbb{P}^+(\Sigma_x) = 0$ in the theorem follows from the next proposition:

Proposition 5.1.4 $\lim_{j \to \infty} \mathbb{P}^+(S_x^j) = 0$.

Proof. Suppose, for instance, that the cylinders in the claim satisfy

$$0 < \mathbb{P}^+([\xi_0 \dots \xi_{N-1}]) \leq \mathbb{P}^+([\omega_0 \dots \omega_{N-1}]). \tag{5.1.4}$$

The first inequality follows form the admissibility of $[\xi_0 \dots \xi_{N-1}]$.

Define for $j \geq 1$ the family of cylinders

$$E^j \stackrel{\text{def}}{=} \{[a_0 \dots a_{jN-1}] \subset \Sigma_k^+ \colon \sigma^{iN}([a_0 \dots a_{jN-1}]) \cap [\xi_0 \dots \xi_{N-1}] = \emptyset,\ i = 0, \dots, j-1\}$$

and their union

$$Q^j \stackrel{\text{def}}{=} \bigcup_{C \in E^j} C.$$

Note that by definition $Q^{j+1} \subset Q^j$. Let

$$Q^\infty \stackrel{\text{def}}{=} \bigcap_{j \geq 1} Q^j = \{\omega \in \Sigma^+ \colon \sigma^{iN}(\omega) \cap [\xi_0 \dots \xi_{N-1}] = \emptyset \text{ for all } i \geq 0\}.$$

Recall that the mixing property of $(\sigma, \mathbb{P}^+)$ implies the ergodicity of $(\sigma^N, \mathbb{P}^+)$. Thus the Birkhoff's ergodic theorem implies that $\mathbb{P}^+(Q^\infty) = 0$. Therefore condition $Q^{j+1} \subset Q^j$ implies that

$$\lim_{j \to \infty} \mathbb{P}^+(Q^j) = 0.$$

In view of this property, the proposition follows from the next lemma.

Lemma 5.1.5 $\mathbb{P}^+(S_x^j) \leq \mathbb{P}^+(Q^j)$ *for all* $j \geq 1$.

Proof. For each $j \geq 1$ consider the auxiliary substitution function $F_j : \Sigma_x^j \to E^j$ defined as follows. For each cylinder $[\alpha_0 \dots \alpha_{jN-1}] \in \Sigma_x^j$ we consider its sub-cylinders $[\alpha_0 \dots \alpha_{N-1}], [\alpha_N \dots \alpha_{2N-1}], \dots, [\alpha_{(j-1)N} \dots \alpha_{jN-1}]$ and use the following concatenation notation

$$[\alpha_0 \dots \alpha_{jN-1}] = [\alpha_0 \dots \alpha_{N-1}] * [\alpha_N \dots \alpha_{2N-1}] * \dots * [\alpha_{(j-1)N} \dots \alpha_{jN-1}].$$

In a compact way, we write

$$C = C_0 * C_1 * \dots * C_{j-1}$$

where the cylinder C has size jN and each cylinder C_i has size N. With this notation we define F_j by

$$F_j(C) \stackrel{\text{def}}{=} F_j(C_0 * C_1 * \dots * C_{j-1}) = C_0' * C_1' * \dots * C_{j-1}',$$

where $C_i' = C_i$ if $C_i \neq [\xi_0 \dots \xi_{N-1}]$ and $C_i' = [\omega_0 \dots \omega_{N-1}]$ otherwise.

Claim 5.1.6 *For every* $j \geq 1$ *it holds* $\mathbb{P}^+(C) \leq \mathbb{P}^+(F_j(C))$ *for every* $C \in \Sigma_x^j$.

Proof. Recalling that $\omega_0 = \xi_0$ and $\omega_{N-1} = \xi_{N-1}$, from equation (5.1.4) we immediately get the following: For every $m, s \geq 0$ and every pair of cylinders $[a_0 \dots a_s]$ and $[b_0 \dots b_m]$ it holds

1. $\mathbb{P}^+([a_0 \dots a_s \xi_0 \dots \xi_{N-1} b_0 \dots b_m]) \leq \mathbb{P}^+([a_0 \dots a_s \omega_0 \dots \omega_{N-1} b_0 \dots b_m])$,
2. $\mathbb{P}^+([\xi_0 \dots \xi_{N-1} b_0 \dots b_m]) \leq \mathbb{P}^+([\omega_0 \dots \omega_{N-1} b_0 \dots b_m])$, and
3. $\mathbb{P}^+([a_0 \dots a_s \xi_0 \dots \xi_{N-1}] \leq \mathbb{P}^+([a_0 \dots a_s \omega_0 \dots \omega_{N-1}]$.

The inequality $\mathbb{P}^+(C) \leq \mathbb{P}^+(F_j(C))$ now follows from the definition of F_j. ■

Claim 5.1.7 *The map F_j is injective for every $j \geq 1$.*

Proof. Fix $j \geq 1$. Given cylinders $C, \widetilde{C} \in \Sigma_x^j$, using the notation above write $C = C_0 * C_1 * \dots * C_{j-1}$ and $\widetilde{C} = \widetilde{C}_0 * \widetilde{C}_1 * \dots * \widetilde{C}_{j-1}$. Then

$$F_j(C) = C'_0 * C'_1 * \dots * C'_{j-1} \quad \text{and} \quad F_j(\widetilde{C}) = \widetilde{C}'_0 * \widetilde{C}'_1 * \dots * \widetilde{C}'_{j-1}.$$

Suppose that $F_j(C) = F_j(\widetilde{C})$. Then $C'_i = \widetilde{C}'_i$ for all $i = 0, \dots, N-1$. If $C \neq \widetilde{C}$ there is a first i such that $C_i \neq \widetilde{C}_i$. Then either $C_i = [\xi_0 \dots \xi_{N-1}]$ and $\widetilde{C}_i = [\omega_0 \dots \omega_{N-1}]$ or vice-versa. Let us assume that the first case occurs.

If $i = 0$ then the definition of Σ_x^j implies that

$$x \in T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I),$$

contradicting Claim 5.1.3. Thus we can assume that $i > 0$ and define the cylinder

$$[\eta_0 \dots \eta_{(i-1)N-1}] \stackrel{\text{def}}{=} C_0 * C_1 * \dots * C_{i-1} = \widetilde{C}_0 * \widetilde{C}_1 * \dots * \widetilde{C}_{i-1}.$$

Write $(i-1)N - 1 = r$. By the definition of Σ_x^j in (5.1.3) we have

$$x \in T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I). \quad (5.1.5)$$

Since for every i we have that $T_i(J) \subset J$ and $T_i|_J$ is injective, the intersection and union inclusion properties in Claim 5.1.3 implies that

$$T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I) = \emptyset,$$

contradicting (5.1.5). Thus $C = \widetilde{C}$ and proof of the claim is complete. ■

To prove that $\mathbb{P}^+(S_x^j) \leq \mathbb{P}^+(Q^j)$ note that

$$\mathbb{P}^+(S_x^j) \underset{\text{(a)}}{=} \sum_{C \in \Sigma_x^j} \mathbb{P}^+(C) \underset{\text{(b)}}{\leq} \sum_{C \in \Sigma_x^j} \mathbb{P}^+(F_j(C)) \underset{\text{(c)}}{=} \mathbb{P}^+\Big(\bigcup_{C \in \Sigma_x^j} F_j(C)\Big) \underset{\text{(d)}}{\leq} \mathbb{P}^+(Q^j),$$

where (a) follows from the disjointness of the cylinders $C \in \Sigma_x^j$, (b) from Claim 5.1.6, (c) from the injectivity of F_j (Claim 5.1.7), and (d) from $F_j(C) \in E_j \subset Q^j$. The proof of the lemma is now complete. ■

This completes the proof of the proposition. ■

The proof of Theorem 5.1.2 (i.e., $\mathbb{P}^+(\Sigma_x) = 0$) is now complete. ■

The proof of Theorem 5 is now complete. ■

5.2 Probabilistic rigidity

In this section we see that under quite general conditions the hypothesis $S_t \cap \text{supp}(\mathbb{P}^+) \neq \emptyset$ implies that $\mathbb{P}^+(S_t) = 1$. Recall the definition of the projection π in (2.1.1).

Theorem 5.2.1 *Consider an injective* IFS$(T_1, \dots T_k)$ *defined on* $I = [0,1]$. *Let* $\mathbb{P}^+$ *be a mixing Markov measure defined on* Σ_k^+ *with transition matrix* $P = (p_{ij})$.

– *If there is* $i \in \{1, \dots, k\}$ *such that* π *is not constant in* $[i] \cap \text{supp}(\mathbb{P}^+)$ *then* $\mathbb{P}^+(S_t) = 1$. *In particular,*

$$\#\pi\Big(S_t \cap \text{supp}(\mathbb{P}^+)\Big) \geq k+1 \quad \Longrightarrow \quad \mathbb{P}^+(S_t) = 1.$$

– *If the maps* T_i *have no common fixed points and for every* i *and* j, *with* $i \neq j$, *there is* $m \in \{1, \dots, k\}$ *with* $p_{mi}\, p_{mj} > 0$. *Then*

$$S_t \cap \text{supp}(\mathbb{P}^+) \neq \emptyset \quad \Longleftrightarrow \quad \mathbb{P}^+(S_t) = 1.$$

Proof. To prove the first item of the theorem note that by hypothesis there is i such that π is not constant in $[i] \cap \text{supp}(\mathbb{P}^+)$. Hence $\xi, \omega \in [i] \cap \text{supp}(\mathbb{P}^+) \cap S_t$ such that $\pi(\xi) \neq \pi(\omega)$. Thus there are s and ℓ such that

$$T_{\xi_0} \circ \cdots \circ T_{\xi_s}(I) \cap T_{\omega_0} \circ \cdots \circ T_{\omega_\ell}(I) = \emptyset.$$

As $\xi, \omega \in [i] \cap \text{supp}(\mathbb{P}^+)$ the cylinders $[\xi_0 \dots \xi_s]$ and $[\omega_0 \dots \omega_\ell]$ are both admissible and satisfy $\xi_0 = \omega_0 = i$. This means that $\mathbb{P}^+$ splits the IFS. Hence, by Theorem 5, $\mathbb{P}^+(S_t) = 1$.

For the second part of the first item, just note that if $\#\pi\Big(S_t \cap \text{supp}(\mathbb{P}^+)\Big) \geq k+1$ then from the pigeonhole principle there is i such that π is not constant in $[i] \cap \text{supp}(\mathbb{P}^+)$.

The implication $(\Leftarrow)$ in the second item of the theorem is immediate. For the implication $(\Rightarrow)$ we need the following lemma.

Lemma 5.2.2 *For every $\xi \in S_t \cap \operatorname{supp}(\mathbb{P}^+)$ there is $\omega \in S_t \cap \operatorname{supp}(\mathbb{P}^+)$ such that $\pi(\xi) \neq \pi(\omega)$.*

Proof. Fix $\xi \in S_t$. By definition of S_t we have that

$$\{\pi(\xi)\} = \bigcap_{n\geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(I).$$

As the maps T_i have no common fixed points there is i_0 such that $T_{i_0}(\pi(\xi)) \neq \pi(\xi)$. The definition of an irreducible matrix implies that there is an admissible cylinder of the form $[i_0 i_1 \dots i_m \xi_0]$. Let

$$r \stackrel{\text{def}}{=} \max\Big\{\ell \in \{0, \dots, m\} : T_{i_\ell}(\pi(\xi)) \neq \pi(\xi)\Big\} \geq 0.$$

Consider the concatenation $\omega = i_r \dots i_m * \xi$. Note that, by definition, $\pi(\zeta) = T_{\zeta_0}(\pi(\sigma(\zeta)))$ for every $\zeta \in S_t$. Hence

$$\pi(\omega) = T_{i_r} \circ \cdots \circ T_{i_m}(\pi(\xi)).$$

By definition of r,

$$T_{i_m}(\pi(\xi)) = \cdots = T_{i_{r+1}}(\pi(\xi)) = \pi(\xi).$$

Therefore

$$\pi(\omega) = T_{i_r}(\pi(\xi)) \neq \pi(\xi).$$

It remains to see that $\omega \in S_t \cap \operatorname{supp}(\mathbb{P}^+)$, for that just note that the cylinder $[i_0 \dots i_m \xi_0]$ is admissible and $\xi \in \operatorname{supp}(\mathbb{P}^+)$. This ends the proof of the lemma. ■

Take sequences ξ and ω as in Lemma 5.2.2. By definition of π,

$$\{\pi(\xi)\} = \bigcap_{n\geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(I) \quad \text{and} \quad \{\pi(\omega)\} = \bigcap_{n\geq 0} T_{\omega_0} \circ \cdots \circ T_{\omega_n}(I).$$

As $\pi(\xi) \neq \pi(\omega)$ there are ℓ and s such that

$$T_{\xi_0} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_{\omega_0} \circ \cdots \circ T_{\omega_s}(I) = \emptyset. \tag{5.2.1}$$

Note that the cylinders $[\xi_0 \dots \xi_\ell]$ and $[\omega_0 \dots \omega_s]$ are admissible. If $\xi_0 = \omega_0$ we are done. Otherwise, $\xi_0 \neq \omega_0$ and by hypothesis there is m such that $p_{m\xi_0} > 0$ and $p_{m\omega_0} > 0$. This implies that the cylinders $[m\xi_0 \dots \xi_\ell]$ and $[m\omega_0 \dots \omega_s]$ are both admissible. Since the maps T_i are injective it follows from (5.2.1)

$$T_m \circ T_{\xi_0} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_m \circ T_{\omega_0} \circ \cdots \circ T_{\omega_s}(I) = \emptyset.$$

Therefore $\mathbb{P}^+$ splits the IFS and by Theorem 5 we have $\mathbb{P}^+(S_t) = 1$. This ends the proof of the theorem. ■

5.3 Separability

In this section we give some characterisations of a separable IFS. Note that item (2) in the next theorem means that the IFS is separable.

Theorem 5.3.1 *Consider an* IFS$(T_1, \dots T_k)$ *defined on* $I = [0,1]$. *Suppose that there is some non-trivial closed interval* J *such that* $T_i(J) \subset J$ *and* $T_i|_J$ *is injective for every* $j \in \{1, \dots, k\}$. *Then the following assertions are equivalent:*

1. *The maps of the IFS have no common fixed points and* $S_t \neq \emptyset$.
2. *The target set* A_t *has at least two elements.*
3. *There are finite sequences* $\xi_1 \dots \xi_\ell$ *and* $\omega_1 \dots \omega_s$ *such that*

$$T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) = \emptyset \quad \textit{and}$$
$$T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cup T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) \subset J.$$

4. *The maps of the IFS have no common fixed point and* $\mathbb{P}^+(S_t) = 1$ *for every mixing Markov measure* $\mathbb{P}^+$ *whose support is the whole* Σ_k^+.

Proof. To prove the implication (1) ⇒ (2) note that since $S_t \neq \emptyset$ there is $p \in A_t$. Since the maps of the IFS have no common fixed point there is i such that $T_i(p) \neq p$. The invariance of A_t implies that $T_i(p) \in A_t$. Thus $\{p, T_i(p)\} \subset A_t$ and we are done.

To see that (2) ⇒ (3) we need the following claim:

Claim 5.3.2 $\#(A_t \cap \mathrm{int}(J)) \geq 2$.

Proof. Since $T_i(J) \subset J$ for every i we have that $\mathcal{B}(J) \subset J$. Hence Propositions 4.1.1 and 4.2.1 implies that $A_t \subset J$. The claim follows from Proposition 4.5.1. ■

Take two different points $p, q \in A_t \cap \mathrm{int}\, J$ and consider disjoint neighbourhoods U and V of p and q, respectively, such that $U \cup V \subset J$. By the definition of A_t there are sequences ξ and ω such that

$$\{p\} = \bigcap_{n\geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(I) \quad \text{and} \quad \{q\} = \bigcap_{n\geq 0} T_{\omega_0} \circ \cdots \circ T_{\omega_n}(I).$$

Hence there are n_0 and m_0 such that $T_{\xi_0} \circ \cdots \circ T_{\xi_{n_0}}(I) \subset U$ and $T_{\omega_0} \circ \cdots \circ T_{\omega_{m_0}}(I) \subset V$. Since $U \cap V = \emptyset$ we get the implication (2) ⇒ (3).

To prove (3) $\Rightarrow$ (4) consider the finite sequences $\xi_1 \dots \xi_\ell$ e $\omega_1 \dots \omega_s$ in item (3). Clearly the condition in (3) prevents the existence of a common fixed point. On the other hand, since $T_1(J) \subset J$ and $T_1|_J$ is injective, we have that

$$T_1 \circ T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_1 \circ T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) = \emptyset \quad \text{and}$$
$$T_1 \circ T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cup T_1 \circ T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) \subset J.$$

Thus every mixing Markov measure with full support $\mathbb{P}^+$ splits the IFS in J. Now Theorem 5 implies that $\mathbb{P}^+(S_{\mathrm{t}}) = 1$ and we are done.

The implication (4) $\Rightarrow$ (1) is immediate. ■

6
Asymptotic stability on measures

In this chapter we prove Theorems 4 and 6 in Sections 6.1 and 6.2, respectively.

6.1
Stationary measures for IFSs with probabilities in [0,1]

In this section we prove Theorem 4. For that we consider a separable $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$ defined on $I = [0,1]$, its Markov operator $\mathfrak{T} = \mathfrak{T}_{\mathfrak{b}}$, and its coding map π in (2.1.1), we see that for every probability measure $\mu \in \mathcal{M}_1(I)$ it holds

$$\lim_{n\to\infty} \mathfrak{T}^n \mu = \pi_* \mathfrak{b} \quad \text{(asymptotic stability)}.$$

The main step of the proof of the theorem is the next proposition that states a sufficient condition for the asymptotic stability of an IFS with probabilities.

Proposition 6.1.1 *Consider an* $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$ *with probabilities defined on a compact metric space* X*. Suppose that* $\mathfrak{b}(S_{\mathfrak{t}}) = 1$*. Then for every probability measure* $\mu \in \mathcal{M}_1(X)$ *it holds*

$$\lim_{n\to\infty} \mathfrak{T}_{\mathfrak{b}}^n \mu = \pi_* \mathfrak{b}.$$

In particular, $\mu_{\mathfrak{b}} \stackrel{\text{def}}{=} \pi_* \mathfrak{b}$ *is the unique stationary measure of* $\mathrm{IFS}(T_1, \dots T_k; \mathfrak{b})$*. Furthermore,* $\mathrm{supp}(\mu_{\mathfrak{b}}) = \overline{A_{\mathfrak{t}}}$.

We postpone the proof of Proposition 6.1.1 and deduce the theorem from it.

6.1.1
Proof of Theorem 4

In view of Proposition 6.1.1 it is sufficient to prove that $\mathfrak{b}(S_{\mathfrak{t}}) = 1$ and the measure $\pi_* \mathfrak{b}$ is continuous. Since the IFS is separable and every Bernoulli measure (with strictly positive weights) is a mixing Markov measure, Theorem 5.3.1 implies that $\mathfrak{b}(S_{\mathfrak{t}}) = 1$. To see that $\pi_* \mathfrak{b}$ is continuous we need to prove that $\pi_* \mathfrak{b}(\{x\}) = 0$ for every $x \in [0,1]$. Take $x \in [0,1]$ and recall the definition of the set Σ_x in (5.1.1). Since $\pi^{-1}(x) \subset \Sigma_x$ we have that

$$\pi_* \mathfrak{b}(\{x\}) = \mathfrak{b}(\pi^{-1}(x)) \leq \mathfrak{b}(\Sigma_x) = 0,$$

where the last equality follows from Theorem 5.1.2. The proof of Theorem 4 is now complete. ■

6.1.2 Proof of Proposition 6.1.1

We assume that $\mathfrak{b} = \mathfrak{b}(p_1, \dots, p_k)$ and write $\mathfrak{T} = \mathfrak{T}_{\mathfrak{b}}$. We begin by proving two auxiliary lemmas:

Lemma 6.1.2 *For every stationary measure of* $\mathfrak{T}$ *it holds* $\mathcal{B}(\mathrm{supp}(\mu)) \subset \mathrm{supp}(\mu)$.

Proof. It is sufficient to show that $T_i(\mathrm{supp}(\mu)) \subset \mathrm{supp}(\mu)$ for every i. Given $x \in \mathrm{supp}(\mu)$ take a neighborhood V of $T_i(x)$. By the choice of x, $\mu(T_i^{-1}(V)) > 0$. Since μ is a stationary measure we have

$$\mu(V) = p_1 \mu(T_1^{-1}(V)) + \cdots + p_k \mu(T_k^{-1}(V)) \geq p_i \mu(T_i^{-1}(V)) > 0,$$

proving the lemma. ■

Lemma 6.1.3 *Consider the* $\mathrm{IFS}(T_1, \dots T_k)$. *Then for every sequence* (μ_n) *of probabilities of* $\mathcal{M}_1(X)$ *and every* $\omega \in S_{\mathfrak{t}}$ *it holds*

$$\lim_{n\to\infty} T_{\omega_0 *} \dots {}_* T_{\omega_n *} \mu_n = \delta_{\pi(\omega)}.$$

Proof. Consider a sequence of probabilities (μ_n) and $\omega \in S_{\mathfrak{t}}$. Fix any $g \in C^0(X)$. Then given any $\epsilon > 0$ there is δ such that

$$|g(y) - g \circ \pi(\omega)| < \epsilon \quad \text{for all } y \in X \text{ with } d(y, \pi(\omega)) < \delta.$$

Since $\omega \in S_{\mathfrak{t}}$ there is n_0 such that $d(T_{\omega_0} \circ \cdots \circ T_{\omega_n}(x), \pi(\omega)) < \delta$ for every $x \in X$ and every $n \geq n_0$. Therefore for $n \geq n_0$ we have

$$\begin{aligned}\left| g \circ \pi(\omega) - \int g \, dT_{\omega_0 *} \dots {}_* T_{\omega_n *} \mu_n \right| &= \left| \int g \circ \pi(\omega) \, d\mu_n - \int g \circ T_{\omega_0} \circ \cdots \circ T_{\omega_n}(x) \, d\mu_n \right| \\ &\leq \int |g \circ \pi(\omega) - g \circ T_{\omega_0} \circ \cdots \circ T_{\omega_n}(x)| \, d\mu_n \leq \epsilon.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \int g \, dT_{\omega_0 *} \dots {}_* T_{\omega_n *} \mu_n = g \circ \pi(\omega)$$

Since this holds for every continuous map g the lemma follows. ■

We will show that $\lim_{n\to\infty} \mathfrak{T}^n\nu = \pi_*\mathfrak{b}$ for every $\nu \in \mathcal{M}_1(X)$. In particular, by the continuity of $\mathfrak{T}$, $\mathfrak{T}\pi_*\mathfrak{b} = \pi_*\mathfrak{b}$.

Note that from the definition of the Markov operator in (2.2.1), for every $\nu \in \mathcal{M}_1(X)$ and every continuous map $f \in C^0(X)$ it holds

$$\int f d(\mathfrak{T}^n\nu) = \sum_{\xi_0,\dots,\xi_{n-1}} p_{\xi_0}p_{\xi_1}\dots p_{\xi_{n-1}} \int f\, dT_{\xi_0 *}T_{\xi_1 *}\dots{}_* T_{\xi_{n-1}*}\nu. \tag{6.1.1}$$

Fixed $\nu \in \mathcal{M}_1(X)$ consider the sequence of functions $F_n : \Sigma_k^+ \to \mathbb{R}$ defined by

$$F_n(\xi) \stackrel{\text{def}}{=} \int f\, dT_{\xi_0 *}T_{\xi_1 *}\dots{}_* T_{\xi_{n-1}*}\nu.$$

Since the map F_n is constant in the cylinders $[\xi_0,\dots,\xi_{n-1}]$, it is a measurable function. From this property, equation (6.1.1), and the definition of the Bernoulli measure $\mathfrak{b}$ we have

$$\int f\, d(\mathfrak{T}^n\nu) = \int F_n\, d\mathfrak{b}.$$

By hypothesis $\mathfrak{b}(S_{\mathrm{t}}) = 1$, thus applying Lemma 6.1.3 to the constant sequence $\mu_n = \nu$ we have that

$$\lim_{n\to\infty} F_n(\xi) = f\circ\pi(\xi) \quad \text{for } \mathfrak{b}\text{ -a.e. } \xi. \tag{6.1.2}$$

Since $|F_n(\xi)| \leq \|f\|$, from (6.1.2) using the dominated convergence theorem we get

$$\lim_{n\to\infty}\int f\, d(\mathfrak{T}^n\nu) = \lim_{n\to\infty}\int F_n\, d\mathfrak{b} = \int f\circ\pi\, d\mathfrak{b} = \int f d\pi_*\mathfrak{b}.$$

Since the previous equality holds for every continuous map f it follows that $\pi_*\mathfrak{b}$ is an attracting measure.

It remains to see that $\mathrm{supp}(\pi_*\mathfrak{b}) = \overline{A_{\mathrm{t}}}$. For that note the following equalities

$$\pi_*\mathfrak{b}(A_{\mathrm{t}}) = \mathfrak{b}(\pi^{-1}(A_{\mathrm{t}})) = \mathfrak{b}(S_{\mathrm{t}}) = 1 \tag{6.1.3}$$

that imply $\mathrm{supp}(\pi_*\mathfrak{b}) \subset \overline{A_{\mathrm{t}}}$.

To get $\mathrm{supp}(\pi_*\mathfrak{b}) \supset \overline{A_{\mathrm{t}}}$ recall that, by Proposition 4.1.1, every $\mathcal{B}$-invariant compact set contains a fixed point of $\mathcal{B}$. By Lemma 6.1.2 we have $\mathcal{B}(\mathrm{supp}(\pi_*\mathfrak{b})) \subset \mathrm{supp}(\pi_*\mathfrak{b})$. Hence $\mathrm{supp}(\pi_*\mathfrak{b}))$ contains a fixed point of $\mathcal{B}$. As $\overline{A_{\mathrm{t}}}$ is a minimum fixed point of $\mathcal{B}$ (see Proposition 4.2.1) this implies that $\overline{A_{\mathrm{t}}} \subset \mathrm{supp}(\pi_*\mathfrak{b})$. Thus $\mathrm{supp}(\pi_*\mathfrak{b}) = \overline{A_{\mathrm{t}}}$, completing the proof of the proposition.
■

The previous proposition provides a (unique) stationary measure whose support is $\overline{A_{\mathrm{t}}}$. To prove that the support of this measure is the closure of the target we use the characterisation of the stationary measure in (6.1.3). Next proposition claims that the support of the stationary measure of an asymptotically stable Markov operator of an IFS with $S_{\mathfrak{t}} \neq \emptyset$ always is $\overline{A_{\mathfrak{t}}}$, even when $\mathfrak{b}(S_{\mathfrak{t}}) = 0$ (recall that either $\mathfrak{b}(S_{\mathfrak{t}}) = 1$ or $\mathfrak{b}(S_{\mathfrak{t}}) = 0$).

Proposition 6.1.4 *Consider an* $\mathrm{IFS}(T_1, \dots, T_k; \mathfrak{b})$ *with probabilities defined on a compact metric space whose Markov operator* $\mathfrak{T}_{\mathfrak{b}}$ *is asymptotically stable and let* μ *be its stationary measure. If* $S_{\mathfrak{t}} \neq \emptyset$ *then* $\mathrm{supp}(\mu) = \overline{A_{\mathfrak{t}}}$.

Proof. The inclusion $\mathrm{supp}(\mu) \supset \overline{A_{\mathfrak{t}}}$ follows from Lemma 6.1.2. To prove the inclusion "$\subset$" take any point $p \in \mathrm{supp}(\mu)$ and an open neighbourhood V of p. We need to see that $V \cap A_{\mathfrak{t}} \neq \emptyset$. For this take any point $x \in A_{\mathfrak{t}}$. Since $\mathfrak{T} = \mathfrak{T}_{\mathfrak{b}}$ is asymptotically stable Alexandrov's theorem (see (10, page 60)) implies that

$$\liminf_n \mathfrak{T}^n \delta_x(V) \geq \mu(V) > 0.$$

Hence there is n_0 such that $\mathfrak{T}^{n_0}\delta_x(V) > 0$. By definition of the Markov operator we have that

$$\mathfrak{T}^{n_0}\delta_x(V) = \sum_{\xi_0,\dots,\xi_{n_0-1}} p_{\xi_0} p_{\xi_1} \dots p_{\xi_{n_0-1}} T_{\xi_0 *} T_{\xi_1 *} \dots {}_* T_{\xi_{n_0-1} *} \delta_x(V).$$

Therefore there is a finite sequence $\xi_0 \dots \xi_{n_0-1}$ such that

$$\delta_x(T^{-1}_{\xi_{n_0-1}} \circ \dots \circ T^{-1}_{\xi_0}(V)) > 0$$

and thus $x \in T^{-1}_{\xi_{n_0-1}} \circ \dots \circ T^{-1}_{\xi_0}(V)$. The invariance of $A_{\mathfrak{t}}$ now implies that $V \cap A_{\mathfrak{t}} \neq \emptyset$, proving the proposition. ∎

6.2 Stationary measures for recurrent IFSs in [0,1]

In this section we prove Theorem 6. For that we consider a recurrent $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ defined on a compact metric space X, where $\mathbb{P}^+$ is the Markov probability associated to $(P = (p_{i,j}), \bar{p} = (p_i))$. We also consider the set $\widehat{X} = X \times \{1, \dots, k\}$ and the (generalised) Markov operator $\mathfrak{S} = \mathfrak{S}_{\mathbb{P}^+}$ (see (2.2.2)) and the generalised coding map $\varpi \colon S_{\mathfrak{t}} \to \widehat{X}$ given by $\varpi(\xi) \stackrel{\text{def}}{=} (\pi(\xi), \xi_0)$ (see (2.2.4)) of the IFS. A final ingredient is the inverse Markov measure $\mathbb{P}^-$ associated to $\mathbb{P}^+$ defined in (2.2.3).

To prove Theorem 6 we need to see that every $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ such that the inverse Markov measure $\mathbb{P}^-$ is mixing and splits the IFS in some non-trivial closed interval J satisfies

$$\lim_{n\to\infty} \mathfrak{S}^n(\hat{\mu}) = \varpi_*\mathbb{P}^- \quad \text{for every } \hat{\mu} \in \mathcal{M}_1([0,1] \times \{1,\dots,k\}).$$

The main step of the proof of Theorem 6 is the following result.

Theorem 6.2.1 *Consider a recurrent* $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ *defined on a compact metric space* X *such that* $\mathbb{P}^-$ *is mixing and* $\mathbb{P}^-(S_{\mathrm{t}}) = 1$. *Then*

$$\lim_{n\to\infty} \mathfrak{S}^n(\widehat{\mu}) = \varpi_*\mathbb{P}^- \quad \textit{for every } \widehat{\mu} \in \mathcal{M}_1(\widehat{X}).$$

In particular, $\varpi_*\mathbb{P}^-$ *is the unique stationary measure of* $\mathfrak{S}$.

Proof. Given a function $\widehat{f}\colon \widehat{X} \to \mathbb{R}$ we define its i-section $f_i\colon X \to \mathbb{R}$ by $f_i(x) \stackrel{\text{def}}{=} \widehat{f}(x,i)$ and write $\widehat{f} = \langle f_1, \dots, f_k\rangle$. We need to see that for every measure $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\widehat{X})$ and every continuous function $\widehat{f} = \langle f_1, \dots, f_k\rangle \in C^0(\widehat{X})$ it holds

$$\lim_{n\to\infty} \int \widehat{f}\, d\mathfrak{S}^n(\widehat{\mu}) = \int \widehat{f}\, d\varpi_*\mathbb{P}^-. \tag{6.2.1}$$

By definition, it follows that

$$\int \widehat{f}\, d\widehat{\mu} = \sum_{i=1}^{k} \int f_i\, d\mu_i, \quad \text{where} \quad \widehat{\mu} = (\mu_1, \dots, \mu_k),$$

and hence

$$\int \widehat{f}\, d\mathfrak{S}^n(\widehat{\mu}) = \sum_{j=1}^{k} \int f_j\, d(\mathfrak{S}^n(\widehat{\mu}))_j, \quad \mathfrak{S}^n(\widehat{\mu}) = \big((\mathfrak{S}^n(\widehat{\mu}))_1, \dots, (\mathfrak{S}^n(\widehat{\mu}))_k\big). \tag{6.2.2}$$

To get the convergence of the integrals of the sum in (6.2.2) we need a preparatory lemma. First, denote by $\|g\|$ the uniform norm of a continuous function $g\colon X \to \mathbb{R}$.

Lemma 6.2.2 *Consider* $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\widehat{X})$ *such that* $\mu_i(X) > 0$ *for every* $i \in \{1, \dots, k\}$. *Then for every* $g \in C^0(X)$ *it holds*

$$\limsup_n \left| \int g\, d(\mathfrak{S}^n(\widehat{\mu}))_j - \int_{[j]} (g \circ \pi)\, d\mathbb{P}^- \right| \le k\, \|g\|\, \max_i |\mu_i(X) - p_i|,$$

where $\bar{p} = (p_1, \dots, p_k)$ *is the unique stationary vector of* P.

Proof. Take $\widehat{\mu} \in \mathcal{M}_1(\widehat{X})$ as in the statement of the lemma and for each i define the probability measure $\overline{\mu}_i$

$$\overline{\mu}_i(B) \stackrel{\text{def}}{=} \frac{\mu_i(B)}{\mu_i(X)}, \quad \text{where } B \text{ is a Borel subset of } X.$$

A straightforward calculation and the previous definition imply that

$$\begin{aligned}(\mathfrak{S}^n(\widehat{\mu}))_j &= \sum_{\xi_1,\dots,\xi_n} p_{\xi_n\xi_{n-1}}\dots p_{\xi_2\xi_1}p_{\xi_1 j}\, T_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\mu_{\xi_n} \\ &= \sum_{\xi_1,\dots,\xi_n} \mu_{\xi_n}(X)\, p_{\xi_n\xi_{n-1}}\dots p_{\xi_2\xi_1}p_{\xi_1 j} T_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n}.\end{aligned}$$

Thus given any $g \in C^0(X)$ we have that

$$\int g\, d(\mathfrak{S}^n(\widehat{\mu}))_j = \sum_{\xi_1,\dots,\xi_n} \mu_{\xi_n}(X)\, p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \int g\, dT_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n}.$$

Let

$$L_n \stackrel{\text{def}}{=} \left| \int g\, d(\mathfrak{S}^n(\widehat{\mu}))_j - \int_{[j]} (g\circ\pi)\, d\mathbb{P}^- \right|$$

and write $\mu_{\xi_n}(X) = (\mu_{\xi_n}(X) - p_{\xi_n}) + p_{\xi_n}$. Then

$$\begin{aligned} L_n \le & \left| \sum_{\xi_1,\dots,\xi_n} (\mu_{\xi_n}(X) - p_{\xi_n})\, p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \int g\, dT_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n} \right| \\ & + \left| \sum_{\xi_1,\dots,\xi_n} p_{\xi_n}\, p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \int g\, dT_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n} - \int_{[j]} (g\circ\pi)\, d\mathbb{P}^- \right| \\ & \le \max_i |\mu_i(X) - p_i| \|g\| \sum_{\xi_1,\dots,\xi_n} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \\ & + \left| \sum_{\xi_1,\dots,\xi_n} p_{\xi_n}\, p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \int g\, dT_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n} - \int_{[j]} (g\circ\pi)\, d\mathbb{P}^- \right|. \end{aligned}$$

Note that $\sum_{\xi_1,\dots,\xi_{n-1}} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j}$ is the entry (ξ_n, j) of the matrix P^n. Hence

$$\sum_{\xi_1,\dots,\xi_n} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} = \sum_{\xi_n=1}^{k} \sum_{\xi_1,\dots,\xi_{n-1}} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \le k.$$

Therefore

$$\max_i |\mu_i(X) - p_i| \|g\| \sum_{\xi_1,\dots,\xi_n} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \le k\, \|g\|\, \max_i |\mu_i(X) - p_i|. \tag{6.2.3}$$

We now estimate the second parcel in the sum above. Observe that equation (6.2.3) and the following claim imply the lemma.

Claim 6.2.3 *For every continuous function g it holds*

$$\lim_{n\to\infty} \sum_{\xi_1,\dots,\xi_n} p_{\xi_n} p_{\xi_n\xi_{n-1}}\dots p_{\xi_1 j} \int g\, dT_{j*}T_{\xi_1 *}\dots_* T_{\xi_{n-1}*}\overline{\mu}_{\xi_n} = \int_{[j]} (g\circ\pi)\, d\mathbb{P}^-.$$

Proof. Consider the sequence of functions given by

$$G_n : \Sigma_k^+ \to \mathbb{R}, \quad G_n(\xi) \stackrel{\text{def}}{=} \int g\, dT_{\xi_0 *} T_{\xi_1 *} \dots {}_* T_{\xi_{n-1} *} \overline{\mu}_{\xi_n}.$$

By definition, for every n the corresponding map G_n is constant in the cylinders $[\xi_0, \dots, \xi_n]$ and thus it is measurable. By definition of $\mathbb{P}^{\pm}$, for every j we have that

$$p_{\xi_n} p_{\xi_n \xi_{n-1}} \dots p_{\xi_2 \xi_1} p_{\xi_1 j} = \mathbb{P}^+([\xi_n \xi_{n-1} \dots \xi_1 j]) = \mathbb{P}^-([j \xi_1 \xi_2 \dots \xi_n]).$$

Hence

$$\sum_{\xi_1, \dots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g\, dT_{j*} T_{\xi_1 *} \dots {}_* T_{\xi_{n-1} *} \overline{\mu}_{\xi_n} = \int_{[j]} G_n \, d\mathbb{P}^-.$$

It follows from the hypothesis $\mathbb{P}^-(S_t) = 1$ and Lemma 6.1.3 that

$$\lim_{n \to \infty} G_n(\xi) = g \circ \pi(\xi) \quad \text{for } \mathbb{P}^-\text{-almost every } \xi. \tag{6.2.4}$$

Now note that $|G_n(\xi)| \leq \|g\|$ for every $\xi \in \Sigma_k^+$. From (6.2.4), using the dominated convergence theorem, we get

$$\lim_{n \to \infty} \int_{[j]} G_n \, d\mathbb{P}^- = \int_{[j]} (g \circ \pi) \, d\mathbb{P}^-,$$

ending the proof of the claim. ■

The proof of the lemma is now complete. ■

To prove the theorem observe that since $\mathbb{P}^-$ is mixing the transition matrix P associated to $\mathbb{P}^+$ is primitive, recall Section 3.3. Take $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\widehat{X})$. Note that by definition of the Markov operator

$$\Big((\mathfrak{S}\widehat{\mu})_1(X), \dots, (\mathfrak{S}\widehat{\mu})_k(X) \Big) = \widehat{p}\, P, \qquad \text{where } \widehat{p} = (\mu_1(X), \dots, \mu_k(X)).$$

Hence for every $n \geq 1$

$$\Big((\mathfrak{S}^n \widehat{\mu})_1(X), \dots, (\mathfrak{S}^n \widehat{\mu})_k(X) \Big) = \widehat{p}\, P^n. \tag{6.2.5}$$

By the Perron-Frobenius theorem, see for instance (26, page 64), we have that the stationary vector $\bar{p} = (p_1, \dots, p_k)$ is positive[1] and

$$\lim_{n \to \infty} \widehat{p}\, P^n = \bar{p} \quad \text{for every probability vector } \widehat{p}.$$

Hence (6.2.5) gives n_0 such that the vector $\Big((\mathfrak{S}^{n_1} \widehat{\mu})_1(X), \dots, (\mathfrak{S}^{n_1} \widehat{\mu})_k(X) \Big)$ is

[1] A vector $v = (v_1, \dots, v_k)$ is said positive if $v_i > 0$ for all i.

positive for every $n_1 \geq n_0$. Therefore we can apply Lemma 6.2.2 to the measure $\mathfrak{S}^{n_1}(\widehat{\mu})$ for every $n_1 \geq n_0$, obtaining for every $g \in C^0(X)$ the inequality

$$\limsup_n \left| \int g \, d(\mathfrak{S}^{n+n_1}(\widehat{\mu}))_j - \int_{[j]} (g \circ \pi) \, d\,\mathbb{P}^- \right| \leq k \, \|g\| \, \max_i |(\mathfrak{S}^{n_1}\widehat{\mu})_i(X) - p_i|.$$

It follows from the definition of lim sup and the previous inequality that

$$\limsup_n \left| \int g \, d(\mathfrak{S}^{n}(\widehat{\mu}))_j - \int_{[j]} (g \circ \pi) \, d\,\mathbb{P}^- \right| \leq k \, \|g\| \, \max_i |(\mathfrak{S}^{n_1}\widehat{\mu})_i(X) - p_i|$$

for every $n_1 \geq n_0$. By (6.2.5) and the Perron-Frobenius theorem we get

$$\lim_{n_1 \to \infty} \max_i |(\mathfrak{S}^{n_1}\widehat{\mu})_i(X) - p_i| = 0.$$

Therefore

$$\lim_{n \to \infty} \int g \, d(\mathfrak{S}^{n}(\widehat{\mu}))_j = \int_{[j]} (g \circ \pi) \, d\,\mathbb{P}^- \quad \text{for every } g \in C^0(X). \tag{6.2.6}$$

To get equation (6.2.1), write $\widehat{f} = \langle f_1, \ldots, f_k \rangle$, apply (6.2.6) to the maps f_i, and use (6.2.2) to get

$$\lim_{n \to \infty} \int \widehat{f} \, d\,\mathfrak{S}^n(\hat{\nu}) \underset{(6.2.2)}{=} \sum_{j=1}^{k} \lim_{n \to \infty} \int f_j \, d\,(\mathfrak{S}^n(\hat{\nu}))_j \underset{(6.2.6)}{=} \sum_{j=1}^{k} \int_{[j]} (f_j \circ \pi) \, d\,\mathbb{P}^-.$$

Now observing that $f_j \circ \pi(\xi) = \widehat{f} \circ \varpi(\xi)$ for every $\xi \in [j]$, we conclude that

$$\lim_{n \to \infty} \int \widehat{f} \, d\,\mathfrak{S}^n(\hat{\nu}) = \sum_{j=1}^{k} \int_{[j]} \widehat{f} \circ \varpi \, d\,\mathbb{P}^- = \int \widehat{f} \, d\varpi_* \mathbb{P}^-.$$

proving (6.2.1) and ending the proof of the theorem. ■

6.2.1
Proof of Theorem 6

Since $\mathbb{P}^-$ is mixing and splits the IFS in some non-trivial interval it follows from Theorem 5 that $\mathbb{P}^-(S_t) = 1$. Thus the theorem follows from Theorem 6.2.1.

6.3
Uniqueness of the stationary measure in the irreducible case

In Proposition 6.3.1 we state a result that does not involve the mixing condition of the probability $\mathbb{P}^-$. For that we consider the subset $\mathcal{M}_{\bar{p}}(\widehat{X})$ of

$\mathcal{M}_1(\widehat{X})$ defined by

$$\mathcal{M}_{\bar{p}}(\widehat{X}) \stackrel{\text{def}}{=} \{\widehat{\mu} = (\mu_1, \dots, \mu_k) \colon \mu_i(X) = p_i \text{ for every } i\},$$

where $\bar{p} = (p_1, \dots, p_k)$ is the stationary vector of the irreducible transition matrix P associated to $\mathbb{P}^+$. The set $\mathcal{M}_{\bar{p}}(\widehat{X})$ is invariant by $\mathfrak{S}_{\mathbb{P}^+}$ and contains all stationary measures of $\mathrm{IFS}(T_1, \dots, T_k; \mathbb{P}^+)$. For the first assertion observe that given any $\widehat{\mu} \in \mathcal{M}_{\bar{p}}(\widehat{X})$ by the definition of $\mathfrak{S}_{\mathbb{P}^+}$ we have

$$(\mathfrak{S}_{\mathbb{P}^+}\widehat{\mu})_j = \sum_{i=1}^{k} p_{ij}\, T_{j*}\mu_i \quad \text{for every } j.$$

Thus

$$(\mathfrak{S}_{\mathbb{P}^+}\widehat{\mu})_j(X) = \sum_{i=1}^{k} p_{ij}\mu_i(T_j^{-1}(X)) = \sum_{i=1}^{k} p_{ij}\mu_i(X) = \sum_{i=1}^{k} p_i p_{ij} = p_j$$

and hence $\mathfrak{S}_{\mathbb{P}^+}(\widehat{\mu}) \in \mathcal{M}_{\bar{p}}(\widehat{X})$.

For the second assertion note that a measure $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\widehat{X})$ is stationary if and only if

$$\mu_j = \sum_{i=1}^{k} p_{ij}\, T_{j*}\mu_i \quad \text{for every } j.$$

If $\widehat{\mu} = (\mu_1, \dots, \mu_k)$ is stationary then $(\mu_1(X), \dots, \mu_k(X))$ is the stationary probability vector for the transition matrix P of $\mathbb{P}^+$. Thus $\mu_i(X) = p_i$ for every i.

A corollary of Lemma 6.2.2 is the following proposition.

Proposition 6.3.1 *Consider a recurrent* $\mathrm{IFS}(T_1, \dots T_k; \mathbb{P}^+)$ *defined on a compact metric space* X *such that* $\mathbb{P}^-(S_{\mathrm{t}}) = 1$*. Then*

$$\lim_{n\to\infty} \mathfrak{S}^n(\widehat{\nu}) = \varpi_*\mathbb{P}^- \quad \textit{for every} \quad \widehat{\nu} \in \mathcal{M}_{\bar{p}}(\widehat{X}).$$

In particular, $\varpi_*\mathbb{P}^-$ *is the unique stationary measure of* $\mathfrak{S}$*.*

Proof. Consider $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_{\bar{p}}(\widehat{X})$ and note that $\mu_i(X) = p_i$. Lemma 6.2.2 implies that for every continuous function g it holds

$$\lim_{n\to\infty} \int g\, d(\mathfrak{S}^n(\widehat{\mu}))_j = \int_{[j]} (g \circ \pi)\, d\,\mathbb{P}^-. \tag{6.3.1}$$

Consider a continuous map $\widehat{f} = \langle f_1, \dots, f_k \rangle$. We apply (6.3.1) to the maps f_i and use equation (6.2.2) to get

$$\lim_{n\to\infty}\int \widehat{f}\, d\,\mathfrak{S}^n(\hat{\nu}) \underset{(6.2.2)}{=} \sum_{j=1}^{k} \lim_{n\to\infty}\int f_j\, d\,(\mathfrak{S}^n(\hat{\nu}))_j \underset{(6.3.1)}{=} \sum_{j=1}^{k}\int_{[j]} (f_j\circ\pi)\, d\,\mathbb{P}^-.$$

Observing that $f_j\circ\pi(\xi) = \widehat{f}\circ\varpi(\xi)$ for every $\xi\in[j]$, we conclude that

$$\lim_{n\to\infty}\int \widehat{f}\, d\,\mathfrak{S}^n(\hat{\nu}) = \sum_{j=1}^{k}\int_{[j]} \widehat{f}\circ\varpi\, d\,\mathbb{P}^- = \int \widehat{f}\, d\varpi_*\mathbb{P}^-,$$

proving the proposition. ■

7
Transitive invariant sets of step skew products

In this chapter we prove Theorem 7.

7.1
Preliminary topological properties

Recall that the *spine of a sequence* $\xi \in \Sigma_k$ is defined by

$$I_\xi \stackrel{\text{def}}{=} \bigcap_{n\geq 0} T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-n}}(X).$$

in (1.2.3). The next simple proposition characterises the maximal attractor Λ_F of F, see in (1.2.2), in term of the set of spines.

Proposition 7.1.1 $\Lambda_F = \bigcup_{\omega\in\Sigma_k} \{\omega\} \times I_\omega$.

Proof. Consider a point (ω, x) such that $x \in I_\omega$ and fix $n \geq 1$. By definition there is $y \in [0, 1]$ such that $x = T_{\omega_{-1}} \circ \cdots \circ T_{\omega_{-n}}(y)$. Then

$$(\omega, x) = F^n\Big(\sigma^{-n}(\omega), y\Big),$$

which implies that $(\omega, x) \in \Lambda_F$, proving the inclusion "$\supset$".

Conversely, take any $(\omega, x) \in \Lambda_F$. By definition, for every $n \geq 1$ there is $(\overline{\omega}(n), \overline{x}(n)) \in \Sigma_k \times X$ such that $F^n(\overline{\omega}, \overline{x}) = (\omega, x)$. Noting that $\overline{\omega} = \sigma^{-n}(\omega)$ we conclude that

$$x \in T_{\omega_{-1}} \circ \cdots \circ T_{\omega_{-n}}(X).$$

Since this holds for every $n \geq 1$ it follows that $x \in I_\omega$, proving the proposition.
■

Recall the definition of the projection in (1.2.5),

$$\varrho\colon S_{\mathrm{t}}^- \to X, \quad \varrho(\xi) = \lim_{n\to\infty} T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-n}}(p)$$

and that it does not depend on the choice of $p \in X$.

By definition, $F(\overline{\text{graph}\,\varrho}) \subset \overline{\text{graph}\,\varrho}$ and thus it is contained in Λ_F. Recalling that $A_{\mathrm{t}} = \varrho(S_{\mathrm{t}}^-)$, see (4.1.2), we have that graph $\varrho \subset S_{\mathrm{t}}^- \times A_{\mathrm{t}}$.

7.2 Proof of Theorem 7

In what follows we denote by $\mathcal{D}$ the set of disjunctive sequences of Σ_k. We now prove the first item of the theorem claiming that $\overline{\operatorname{graph}\varrho} \subset \omega(z)$ for every $z=(\xi,x)\in\mathcal{D}\times X$. Let $z=(\xi,x)\in\mathcal{D}\times X$. Since $\omega(z)$ is closed it is enough to see that $\operatorname{graph}\varrho\subset\omega(z)$. Take a point $(\beta,\varrho(\beta))\in\operatorname{graph}\varrho$ and consider a sequence n_ℓ such that $\sigma^{n_\ell}(\xi)\in[-\ell;\beta_{-\ell}\dots\beta_\ell]$ for every $\ell\geq 0$ (here we use that $\xi\in\mathcal{D}$). Hence $\sigma^{n_\ell-\ell}(\xi)\in[0;\beta_{-\ell}\dots\beta_\ell]$. Let

$$x_\ell \stackrel{\text{def}}{=} f_{\xi_{n_\ell-\ell}}\circ\cdots\circ f_{\xi_0}(x).$$

By definition of S_{t}^- and since $\beta\in S_{\mathrm{t}}^-$ it follows that

$$\lim_{\ell\to\infty} T_{\beta_{-1}}\circ\cdots\circ T_{\beta_{-\ell}}(x_\ell)=\varrho(\beta).$$

Hence $\lim_{\ell\to\infty}F^{n_\ell}(\xi,x)=(\beta,\varrho(\beta))$, proving that $\operatorname{graph}\varrho\subset\omega(z)$.

To prove the remainder itens in the the teorem we need two preparatory lemmas. Given a sequence $\xi^+\in\Sigma_k^+$ let

$$[\xi^+]\stackrel{\text{def}}{=}\{\omega\in\Sigma_k\colon \omega_i=\xi_i^+\quad\text{for every } i\geq 0\}.$$

Lemma 7.2.1 *Consider any open set $[-\ell;b_{-\ell}\dots b_\ell]\times J\subset\Sigma_k\times X$ such that*

$$A_{\mathrm{t}}\cap\Big((T_{b_{-1}}\circ\cdots\circ T_{b_{-\ell}})^{-1}(J)\Big)\neq\emptyset.$$

Then for every open set $Q\subset\Sigma_k^+$ there is $n_0=n_0(Q)$ such that for every $n\geq n_0$ there is a sequence $\xi^{+,n}\in Q$ such that

$$F^n\Big([\xi^{+,n}]\times X\Big)\subset[-\ell;b_{-\ell}\dots b_\ell]\times J.$$

Proof. By hypothesis there is $q\in(T_{b_{-1}}\circ\cdots\circ T_{b_{-\ell}})^{-1}(J)\cap A_{\mathrm{t}}$. Thus, by definition of A_{t}, there is a sequence $c_1,\dots c_p$ such that

$$T_{c_1}\circ\cdots\circ T_{c_p}(X)\subset(T_{b_{-1}}\circ\cdots\circ T_{b_{-\ell}})^{-1}(J).$$

Hence

$$T_{b_{-1}}\circ\cdots\circ T_{b_{-\ell}}\circ T_{c_1}\circ\cdots\circ T_{c_p}(X)\subset J. \tag{7.2.1}$$

Since the one-sided shift σ is topologically mixing, there is $m_0>0$ such that for each $n\geq m_0$ there is a sequence $\omega^{+,n}\in Q$ such that

$$\sigma^n(\omega^{+,n})\in[c_p\dots c_1b_{-\ell}\dots b_\ell]\subset\Sigma_k^+. \tag{7.2.2}$$

For $n \geq m_0 + \ell + p$ we define the one-sided sequence

$$\xi^{+,n} \stackrel{\text{def}}{=} \omega^{+,n-(\ell+p)}.$$

Since $n-(\ell+p) \geq m_0$ this sequence is well defined. Take $n \geq m_0+\ell+p$ and any $\zeta \in [\xi^{+,n}]$. By construction, for every $x \in X$ we have that

$$F^{n-\ell-p}(\zeta, x) \in [0; c_p \dots c_1 b_{-\ell} \dots b_\ell] \times X.$$

Therefore from equations (7.2.1) and (7.2.2) it follows that

$$F^n(\zeta, x) = F^{\ell+p}(F^{n-\ell-p}(\zeta, x)) \in [-\ell; b_{-\ell} \dots b_\ell] \times J,$$

proving the lemma. ∎

Lemma 7.2.2 *Let Ω be a compact subset of $\Sigma_k \times \overline{A_t}$ such that $F(\Omega) = \Omega$. Then $\Omega \subset \overline{\operatorname{graph} \varrho}$.*

Proof. Let $(\xi, y) \in \Omega$ and $V = [-\ell; \xi_{-\ell} \dots \xi_\ell] \times J$ be an open set with $(\xi, y) \in V$. Since $F(\Omega) = \Omega$ we have that $F^{-\ell}(\xi, y) \cap \Omega \neq \emptyset$. This implies that

$$(T_{\xi_{-1}} \circ \dots \circ T_{\xi_{-\ell}})^{-1}(y) \cap \overline{A_t} \neq \emptyset \quad \Longrightarrow \quad A_t \cap T^{-1}_{\xi_{-\ell}} \circ \dots \circ T^{-1}_{\xi_{-1}}(J) \neq \emptyset.$$

Applying Lemma 7.2.1 to the open set $Q = \Sigma_k^+$ we get a sequence ξ^+ and $n \geq 0$ such that

$$F^n\Big([\xi^+] \times X\Big) \subset [-\ell; \xi_{-\ell} \dots \xi_\ell] \times J.$$

As $\Big([\xi^+] \times X\Big) \cap \operatorname{graph} \varrho \neq \emptyset$ and $\operatorname{graph} \varrho$ is F-invariant, we conclude that

$$\operatorname{graph} \varrho \cap [-\ell; \xi_{-\ell} \dots \xi_\ell] \times J \neq \emptyset.$$

Thus we have $\Omega \subset \overline{\operatorname{graph} \varrho}$. ∎

To prove item (2). Recall that F_t is the restriction of F to the set $\Sigma_k \times \overline{A_t}$. Clearly, $\Lambda_t \subset \Sigma_k \times \overline{A_t}$ and since Λ_t is a maximal invariant set we have that $F(\Lambda_t) = F_t(\Lambda_t) = \Lambda_t$, and by Lemma 7.2.2, $\Lambda_t \subset \overline{\operatorname{graph} \varrho}$. For the converse inclusion, take any point $z = (\xi, x) \in \Lambda_t$, such that ξ is disjunctive. Then by the first item of the theorem and the F-invariance of Λ_t we have that

$$\overline{\operatorname{graph} \varrho} \subset \omega(z) \subset \Lambda_t,$$

proving the item.

We now prove item 3. The fact that F is topologically mixing in $\overline{\operatorname{graph} \varrho}$ is stated in the next lemma.

Lemma 7.2.3 *Consider any pair of open sets*

$$V = [-\ell; b_{-\ell} \dots b_\ell] \times J \quad \textit{and} \quad U = [-m; a_{-m} \dots a_m] \times I$$

intersecting $\overline{\mathrm{graph}\,\varrho}$. *Then there is* n_0 *such that*

$$F^n(U \cap \overline{\mathrm{graph}\,\varrho}) \cap V \neq \emptyset, \quad \textit{for every } n \geq n_0.$$

Proof. We first state a claim that will also be used in the proof of the third item of the theorem. Note that the first part of the claim implies the first part of item (2).

Claim 7.2.4 *For every* $(\xi, q) \in \overline{\mathrm{graph}\,\varrho}$ *it holds*

$$\overline{A_{\mathrm{t}}} \cap T^{-1}_{\xi_{-\ell}} \circ \cdots \circ T^{-1}_{\xi_{-1}}(q) \neq \emptyset.$$

Proof. Take $(\xi, q) \in \overline{\mathrm{graph}\,\varrho}$. Since $\Lambda_{\mathrm{t}} = \overline{\mathrm{graph}\,\varrho}$, we have that $F(\overline{\mathrm{graph}\,\varrho}) = \overline{\mathrm{graph}\,\varrho}$. This implies that $F^{-\ell}(\xi, q) \cap \overline{\mathrm{graph}\,\varrho} \neq \emptyset$. By definition we have that $\overline{\mathrm{graph}\,\varrho} \subset \Sigma_k \times \overline{A_{\mathrm{t}}}$ which implies $(T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-\ell}})^{-1}(q) \cap \overline{A_t} \neq \emptyset$. ∎

By hypothesis, the set $V = [-\ell; b_{-\ell} \dots b_\ell] \times J$ contains a point $(\xi, q) \in \overline{\mathrm{graph}\,\varrho}$. Hence, by the previous claim, $(T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-\ell}})^{-1}(q) \cap \overline{A_t} \neq \emptyset$ and thus

$$A_{\mathrm{t}} \cap T^{-1}_{\xi_{-\ell}} \circ \cdots \circ T^{-1}_{\xi_{-1}}(J) \neq \emptyset.$$

This allows to apply Lemma 7.2.1 to the sets $V = [-\ell; b_{-\ell} \dots b_\ell] \times J$ and $Q = [a_0 \dots a_m]$ of Σ_k^+. By Lemma 7.2.1, there is $m_0 = m_0(Q)$ such that for every $n \geq m_0$ there is a sequence $\xi^{+,n} \in Q$ such that

$$F^n\Big([\xi^{+,n}] \times X\Big) \subset [-\ell; b_{-\ell} \dots b_\ell] \times J = V, \quad \text{for all} \quad n \geq m_0.$$

Claim 7.2.5 $(\mathrm{graph}\,\varrho) \cap U \cap \Big([\xi^{+,n}] \times X\Big) \neq \emptyset$ *for every* $n \geq \max\{m, m_0\}$.

Proof. By hypothesis, $U = [-m; a_{-m} \dots a_m] \times I$ and there is $(\zeta, a) \in U \cap \mathrm{graph}\,\varrho$ and $\zeta \in S^-_{\mathrm{t}}$. Consider the sequence $\gamma = \zeta^-.\xi^{+,n} \in \Sigma_k$ (that is γ is the sequence whose negative part is ζ^- and whose positive part if $\xi^{+,n}$). By construction,

$$\gamma \in S^-_{\mathrm{t}} \cap [\xi^{+,n}] \cap [-m; a_{-m} \dots a_m]$$

and $\varrho(\gamma) = a$. Hence $(\gamma, \varrho(\gamma))$ is in the intersection set in the claim. ∎

Using the claim we have

$$F^n(U \cap \mathrm{graph}\,\varrho) \cap V \neq \emptyset \quad \text{for all} \quad n \geq n_0 = \max\{m, m_0\},$$

proving the lemma. ■

We now prove the last item of the theorem. Note that by the first item of the theorem we need only to show that $\omega(z) \subset \overline{\text{graph } \varrho}$, provided that $z = (\xi, x)$ where ξ is disjunctive. We need to consider the two possibilities in this item.

First suppose that $\overline{A_t}$ has non-empty interior. Take any point $z = (\xi, x) \in \mathcal{D} \times X$. Since the interior of $\overline{A_t}$ is not empty and ξ is disjunctive there is n_0 such that $\bar{z} = F^{n_0}(z) \in \Sigma_k \times \overline{A_t}$. By the invariance of $\overline{A_t}$ we have that $F^n(z) \in \Sigma_k \times \overline{A_t}$ for every $n \geq n_0$, which implies that $\omega(\bar{z}) \subset \Sigma_k \times \overline{A_t}$. Since $\omega(\bar{z}) = \omega(z)$ and $\omega(z)$ is F-invariant, Lemma 7.2.2 implies that $\omega(z) \subset \overline{\text{graph } \varrho}$.

Suppose now that $\overline{A_t}$ is Lyapunov stable. Take any point $z = (\xi, x) \in \mathcal{D} \times X$. To prove the inclusion $\omega(z) \subset \overline{\text{graph } \varrho}$ take any open neighbourhood U of $\overline{A_t}$. Since $\overline{A_t}$ is Lyapunov stable there is a neighbourhood V of $\overline{A_t}$ such that $\mathcal{B}^n(V) \subset U$ for every $n \geq 0$. The definition of $\overline{A_t}$ and the fact the ξ is a disjunctive sequence imply that there is n_0 such $F^{n_0}(\xi, x) \in \Sigma_k \times V$. Hence $F^n(\xi, x) \in \Sigma_k \times U$ for every $n \geq n_0$, which implies that $\omega(z) \subset \Sigma_k \times U$. Since U is an arbitrary neighbourhood we have that $\omega(z) \subset \Sigma_k \times \overline{A_t}$. It follows from Lemma 7.2.2 that $\omega(z) \subset \overline{\text{graph } \varrho}$. ■

8
Global attracting measures for skew products

In this chapter we prove Theorem 8 and Corollary 9.

8.1
Preliminaries

We now introduce the main ingredients in the proof of Theorem 8. Recall the definition of the projection $\pi_1(\omega, x) \stackrel{\text{def}}{=} \omega$ in (2.3.1). Given a σ-invariant measure λ in Σ_k recall the definition of the set $\mathcal{M}_\lambda$ of measures of $\Sigma_k \times X$ with marginal λ in (2.3.2).

Proposition 8.1.1 *The set $\mathcal{M}_\lambda$ is F-invariant and compact.*

Proof. As $\mathcal{M}_\lambda$ is a subset of the compact set $\mathcal{M}_1(\Sigma_k \times X)$, to prove that it is compact it is enough to see that it is closed. Let (μ_n) be a sequence in $\mathcal{M}_\lambda$ such that $\mu_n \to \mu$. We need to see that $\mu \in \mathcal{M}_\lambda$. Since π_1 is continuous,

$$\lambda = (\pi_1)_* \mu_n \to (\pi_1)_* \mu,$$

proving that $\mu \in \mathcal{M}_\lambda$.

To prove the invariance $F_*(\mathcal{M}_\lambda) \subset \mathcal{M}_\lambda$ first note that $\pi_1 \circ F = \sigma \circ \pi_1$. Thus for $\mu \in \mathcal{M}_\lambda$ we have

$$(\pi_1)_* \, F_* \mu = (\pi_1 \circ F)_* \mu = (\sigma \circ \pi_1)_* \mu = \sigma_* \, (\pi_1)_* \mu = \sigma_* \lambda = \lambda,$$

which implies the F_*-invariance of $\mathcal{M}_\lambda$.

■

Recall the definitions of the subset of trivial spines S_{t}^- of Σ_k in (1.2.4) and of the generalised coding map ϱ defined on S_{t}^- in (1.2.5).

Remark 8.1.2 By the continuity of the maps T_i and recalling the definition of ϱ it follows that $T_{\xi_0} \circ \varrho(\xi) = \varrho \circ \sigma(\xi)$ for every $\xi \in S_{\mathrm{t}}^-$.

Given a compact metric space Z, we denote by $\mathbb{B}(Z)$ its Borel σ-algebra.

Definition 8.1.3 (Disintegration of a measure) Let λ be a probability measure defined on Σ_k and $\mu \in \mathcal{M}_\lambda$. A function

$$(\omega, B) \in \Sigma_k \times \mathbb{B}(X) \mapsto \mu_\omega(B) \in [0,1]$$

is a *disintegration of* μ *with respect to* λ if

1. for every $B \in \mathbb{B}(X)$ the map defined on Σ_k by $\omega \mapsto \mu_\omega(B)$ is $\mathbb{B}(\Sigma_k)$-measurable,
2. for λ-a.e. $\omega \in \Sigma_k$, the map defined on $\mathbb{B}(X)$ by $B \mapsto \mu_\omega(B)$ is a probability measure on $(X, \mathbb{B}(X))$,
3. for every $A \in \mathbb{B}(\Sigma_k \times X)$

$$\mu(A) = \int \mu_\omega(A^\omega)\, d\lambda(\omega), \quad \text{where } A^\omega \text{ is the } \omega\text{-section of } A.$$

There is the following result about existence and uniqueness of disintegrations.

Proposition 8.1.4 (Proposition 1.4.3 in (1)) *For every* $\mu \in \mathcal{M}_\lambda$ *its disintegration with respect to* λ *exists and is* λ*-a.e. unique.*

We need the following definition.

Definition 8.1.5 (Isomorphic transformations) Let $(X_1, \mathbb{B}_1, \mu_1)$ and $(X_2, \mathbb{B}_2, \mu_2)$ be a pair of probability spaces and consider measure preserving transformations $T_1 \colon X_1 \to X_1$ and $T_2 \colon X_2 \to X_2$. We say that the system (T_1, μ_1) is *isomorphic* to (T_2, μ_2) if there exist $M_1 \in \mathbb{B}_1$, $M_2 \in \mathbb{B}_2$ with $\mu_1(M_1) = \mu_2(M_2) = 1$ such that

1. $T_i(M_i) \subset M_i$ for every $i = 1, 2$.
2. There is an invertible measurable transformation $\phi \colon M_1 \to M_2$ whose inverse is also measurable, such that $\phi_* \mu_1 = \mu_2$ and $T_2 \circ \phi(x) = \phi \circ T_1(x)$ for every $x \in M_1$.

8.2
Proof of Theorem 8

We consider the following "candidate" attracting measure v_λ,

$$v_\lambda(A \times B) \stackrel{\text{def}}{=} \lambda(A \cap \varrho^{-1}(B)), \quad \text{for every } A \times B \subset \Sigma_k \times X. \tag{8.2.1}$$

Note that since $\lambda(S_{\mathrm{t}}^-) = 1$ this is a probability measure. This definition can be read as follows

$$v_\lambda(A \times B) = \int_A \chi_{\varrho^{-1}(B)} d\lambda = \int_A \delta_{\varrho(\omega)}(B)\, d\lambda(\omega),$$

where $\chi_{\varrho^{-1}(B)}$ is the characteristic function on $\varrho^{-1}(B)$. As $\lambda(S_{\mathrm{t}}^-) = 1$ this implies that $\delta_{\varrho(\omega)}$ is the disintegration of v_λ with respect to λ.

It remains to see that v_λ is the global attractor for the restriction of F_* to $\mathcal{M}_\lambda$. This follows from the next lemma.

Lemma 8.2.1 *Consider $\mu \in \mathcal{M}_\lambda$ and its disintegration $(\mu_\omega)_{\omega \in \Sigma_k}$ with respect to λ. Then the disintegration of $F_*^n \mu$ with respect to λ is given by the family of measures*

$$(T_{\omega_{-1}*} \dots T_{\omega_{-n}*} \mu_{\sigma^{-n}(\omega)})_{\omega \in \Sigma_k}.$$

Proof.[Proof of Lemma 8.2.1] Consider any rectangle $A \times B$ in $\Sigma_k \times X$. Then, by definition of a disintegration,

$$F_*^n \mu(A \times B) = \mu\Big(F^{-n}(A \times B)\Big) = \int \mu_\omega\Big((F^{-n}(A \times B))^\omega\Big)\, d\lambda(\omega).$$

Note that if $\omega \in \sigma^{-n}(A)$ then

$$(F^{-n}(A \times B))^\omega = (T_{\omega_{n-1}} \circ \dots \circ T_{\omega_0})^{-1}(B).$$

Otherwise, $(F^{-n}(A \times B))^\omega = \emptyset$. Thus

$$\begin{aligned} F_*^n \mu(A \times B) &= \int_{\sigma^{-n}(A)} \mu_\omega\Big((T_{\omega_{n-1}} \circ \cdots T_{\omega_0})^{-1}(B)\Big)\, d\lambda(\omega) \\ &= \int (\chi_A \circ \sigma^n)(\omega)\, \mu_\omega\Big((T_{\omega_{n-1}} \circ \dots \circ T_{\omega_0})^{-1}(B)\Big)\, d\lambda(\omega). \end{aligned}$$

Defining

$$g = g_{B,n} \colon \Sigma_k \to \mathbb{R}, \quad g(\omega) \stackrel{\text{def}}{=} \mu_{\sigma^{-n}(\omega)}\Big((T_{\omega_{-1}} \circ \dots \circ T_{\omega_{-n}})^{-1}(B)\Big)$$

and recalling that λ is σ-invariant we get

$$\begin{aligned} F_*^n\mu(A\times B) &= \int (\chi_A\circ\sigma^n)(\omega)\,(g\circ\sigma^n)(\omega)\,d\lambda(\omega) = \int (\chi_A\, g)\circ\sigma^n\,d\lambda(\omega) \\ &= \int \chi_A\, g\,d\lambda(\omega) = \int_A \mu_{\sigma^{-n}(\omega)}((T_{\omega_{-1}}\circ\cdots\circ T_{\omega_{-n}})^{-1}(B))\,d\lambda(\omega) \\ &= \int_A T_{\omega_{-1}*}\cdots_* T_{\omega_{-n}*}\mu_{\sigma^{-n}(\omega)}(B)\,d\lambda(\omega). \end{aligned}$$

Since this identity holds for every rectangle, we get that the family of measures $T_{\omega_{-1}*}\cdots_* T_{\omega_{-n}*}\mu_{\sigma^{-n}(\omega)}$ is the disintegration of $F_*^n\mu$ with respect to λ. ■

Observe that if $\omega = (\omega_i)_{i\in\mathbb{Z}} \in S_{\mathrm{t}}^-$ then $\omega^- = (\omega_{-i})_{i\geq 1} \in S_{\mathrm{t}}$ and $\varrho(\omega) = \pi(\omega^-)$. Now from Lemmas 8.2.1 and 6.1.3 we get that

$$\lim_{n\to\infty}(F_*^n\mu)_\omega = \delta_{\pi(\omega^-)} = \delta_{\varrho(\omega)}, \quad \text{for } \lambda\text{-almost every } \omega. \tag{8.2.2}$$

Consider now any continuous real map f of $\Sigma_k\times X$. We get the following

$$\begin{aligned} \lim_{n\to\infty}\int f(\omega,x)\,dF_*^n\mu(\omega,x) &\underset{(a)}{=} \lim_{n\to\infty}\int_{\Sigma_k}\int_X f(\omega,x)\,d(F_*^n\mu)_\omega(x)\,d\lambda(\omega) \\ &\underset{(b)}{=} \int_{\Sigma_k}\left(\lim_{n\to\infty}\int_X f(\omega,x)d(F_*^n\mu)_\omega(x)\right)d\lambda(\omega) \\ &\underset{(c)}{=} \int_{\Sigma_k}\int_X f(\omega,x)\,d\delta_{\varrho(\omega)}(x)\,d\lambda(\omega) \underset{(d)}{=} \int f(\omega,x)\,dv_\lambda(\omega,x), \end{aligned}$$

where (a) uses the definition of a disintegration, (b) the dominated convergence, (c) (8.2.2) and the definition of the weak$*$, and (d) the definition of v_λ. Since f is an arbitrary continuous function, we conclude that v_λ is a global attracting measure in $\mathcal{M}_\lambda$, ending the proof of item (1).

To prove the second item, we need to construct an isomorphism between (F, v_λ) and (σ,λ). We first observe that the map $\phi\colon S_{\mathrm{t}}^-\to \operatorname{graph}\varrho$ defined by $\phi(\xi) = (\xi,\varrho(\xi))$ is an invertible measurable transformation whose inverse is also measurable and is given by $\phi^{-1}(\xi,x) = \xi$. Note that $\sigma(S_{\mathrm{t}}^-)\subset S_{\mathrm{t}}^-$ and $F(\operatorname{graph}\varrho)\subset\operatorname{graph}\varrho$. We will see that, $\phi_*\lambda = v_\lambda$ and $\phi\circ\sigma(\xi) = F\circ\phi(\xi)$ for every $\xi\in S_{\mathrm{t}}$.

For the first assertion take any measurable rectangle $A\times B\subset\Sigma_k\times X$. Recalling the definition of v_λ we have that

$$\phi_*\lambda(A\times B) = \lambda(\phi^{-1}(A\times B)) = \lambda(A\cap\varrho^{-1}(B)) = v_\lambda(A\times B).$$

To get the conjugacy between σ and F we recall that $T_{\xi_0}\circ\varrho(\xi) = \varrho\circ\sigma(\xi)$, see Remark 8.1.2. Hence for every $\xi\in S_{\mathrm{t}}$ we have that

$$F\circ\phi(\xi) = F(\xi,\varrho(\xi)) = (\sigma(\xi), T_{\xi_0}\circ\varrho(\xi)) = (\sigma(\xi),\varrho\circ\sigma(\xi)) = \phi\circ\sigma(\xi).$$

By hypothesis $\lambda(S_{\mathrm{t}}^-) = 1$, and thus $\upsilon_\lambda(\operatorname{graph}\varrho) = \lambda(\phi^{-1}(\operatorname{graph}\varrho)) = 1$. Therefore (F, υ_λ) and (σ, λ) are isomorphic.

We now prove item (3): $\operatorname{supp}\upsilon_\lambda = \overline{\operatorname{graph}(\varrho_{|\operatorname{supp}\lambda})}$. Since $\upsilon_\lambda = \phi_*\lambda$, we immediately get that $\upsilon_\lambda(\operatorname{graph}(\varrho_{|\operatorname{supp}\lambda})) = 1$ and hence

$$\operatorname{supp}\upsilon_\lambda \subset \overline{\operatorname{graph}(\varrho_{|\operatorname{supp}\lambda})}.$$

The next claim implies the inclusion $\subset$ and hence the equality in item (3).

Claim 8.2.2 $\operatorname{graph}(\varrho_{|\operatorname{supp}\lambda}) \subset \operatorname{supp}\upsilon_\lambda$.

Proof. Take any point $(\xi, x) \in \operatorname{graph}(\varrho_{|\operatorname{supp}\lambda})$, to prove the claim it is enough to see that for every $\ell > 0$ and every neighborhood V of x it holds

$$\upsilon_\lambda\Big([\xi_{-\ell}\dots\xi_\ell]\times V\Big) > 0.$$

Since $x \in V$ and $x = \varrho(\xi)$ there is $m \geq 0$ such that

$$\varrho([\xi_{-(\ell+m)}\dots\xi_\ell]\cap S_{\mathrm{t}}^-) \subset V. \tag{8.2.3}$$

Thus by the definition of υ_λ in (8.2.1) we have

$$\upsilon_\lambda\Big([\xi_{-\ell}\dots\xi_\ell]\times V\Big) \geq \upsilon_\lambda\Big([\xi_{-(\ell+m)}\dots\xi_\ell]\times V\Big) = \lambda\Big([\xi_{-(\ell+m)}\dots\xi_\ell]\cap\varrho^{-1}(V)\Big).$$

Note that by (8.2.3)

$$\lambda\Big([\xi_{-(\ell+m)}\dots\xi_\ell]\cap\varrho^{-1}(V)\Big) = \lambda([\xi_{-(\ell+m)}\dots\xi_\ell]\cap S_{\mathrm{t}}^-).$$

Since that, by hypothesis, $\xi \in \operatorname{supp}\lambda \cap [\xi_{-(\ell+m)}\dots\xi_\ell]$ and $\lambda(S_{\mathrm{t}}^-) = 1$ we get that

$$\upsilon_\lambda\Big([\xi_{-\ell}\dots\xi_\ell]\times V\Big) \geq \lambda\Big([\xi_{-(\ell+m)}\dots\xi_\ell]\cap S_{\mathrm{t}}^-\Big) > 0,$$

which implies the claim. ■

We now prove item (4) of the theorem. The argument follow closely the ideas in the proof of (19, page 21). Since (λ, σ) is ergodic we have that for λ-almost every ω it holds

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{\sigma^i(\omega)} \to \lambda. \tag{8.2.4}$$

By definition of π_1 for every (ξ, x) it holds

$$\pi_{1*}\left(\frac{1}{n}\sum_{i=0}^{n-1}\delta_{F^i(\omega,x)}\right) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{\sigma^i(\omega)}. \tag{8.2.5}$$

Therefore if (ω, x) is any point with ω satisfying (8.2.4), then every accumulation point of $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{F^i(\omega,x)}$ is an F-invariant measure (this follows from Krilov-Bogolyubov Theorem, see (26)) with marginal λ (this follows from equations (8.2.4) and (8.2.5)). Since by item (1) of the theorem the attracting measure v_λ is the unique F-invariant measure with marginal λ we get

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{F^i(\omega,x)} \to v_\lambda,$$

ending the proof of item (4) and of the theorem. ■

8.3 Proof of Corollary 9

Consider a recurrent IFS$(T_1, \dots T_k; \mathbb{P}^+)$ and the corresponding step skew product F. Suppose that $\mathbb{P}^-(S_{\mathrm{t}}) = 1$. Consider the Markov measure $\mathbb{P}$ on Σ_k determined by the transition matrix associated to $\mathbb{P}^+$.

Lemma 8.3.1 *Consider the map* $\Pi : \Sigma_k \to \Sigma_k^+$ *defined by*

$$\Pi(\xi) = \xi_{-1}\xi_{-2}\dots.$$

Then $S_{\mathrm{t}}^- = \Pi^{-1}(S_{\mathrm{t}})$ *and* $\Pi_*\mathbb{P} = \mathbb{P}^-$.

Proof. The assertion $S_{\mathrm{t}}^- = \Pi^{-1}(S_{\mathrm{t}})$ follows immediately from definition of Π. To see that $\Pi_*\mathbb{P} = \mathbb{P}^-$ take any cylinder $[a_0 \dots a_\ell]$ in Σ_k^+. Then, by definition (recall the notation in (3.2.1))

$$\Pi_*\mathbb{P}([a_0 \dots a_\ell]) = \mathbb{P}([-(\ell+1); a_\ell \dots a_0]) = \mathbb{P}^+([a_\ell \dots a_0]) = \mathbb{P}^-([a_0 \dots a_\ell]).$$

Since this holds for every cylinder we get that $\Pi_*\mathbb{P} = \mathbb{P}^-$. ■

The previous lemma implies that $\mathbb{P}(S_{\mathrm{t}}^-) = \mathbb{P}^-(S_{\mathrm{t}}) = 1$. We now can apply Theorem 8. Consider a continuous function $f\colon X \to \mathbb{R}$. Define $\bar{f}\colon \Sigma_k \times X \to \Sigma_k \times X$ by $\bar{f}(\omega, x) = f(x)$. Let $v_{\mathbb{P}}$ be the attracting measure in Theorem 8. Since $\bar{f}$ is continuous and $F^i(\xi, x) = (\sigma^i(\xi), T_{\xi_{i-1}} \circ \dots \circ T_{\xi_0}(x))$, it follows from item (4) of Theorem 8 that for $\mathbb{P}$-almost every sequence ξ and all point $x \in X$ it holds

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T_{\xi_{i-1}} \circ \dots \circ T_{\xi_0}(x)) = \int \bar{f}\, dv_{\mathbb{P}}.$$

Claim 8.3.2 $\displaystyle\int \bar{f}\, dv_{\mathbb{P}} = \int f\, d(\varrho_{*\mathbb{P}})$.

Proof. Let $\bar{\varrho}(\xi) = (\xi, \varrho(\xi))$. It follows from the definition of $\upsilon_{\mathbb{P}}$ that $\bar{\varrho}_*\mathbb{P} = \upsilon_{\mathbb{P}}$, just write

$$\bar{\varrho}_*\mathbb{P}(A \times B) = \mathbb{P}\Big(\bar{\varrho}^{-1}(A \times B)\Big) = \mathbb{P}(A \cap \varrho^{-1}(B) = \upsilon_{\mathbb{P}}(A \times B).$$

Noting that $\bar{f} \circ \bar{\varrho}(\xi) = f \circ \varrho$ we get

$$\int \bar{f}\, d\upsilon_{\mathbb{P}} = \int \bar{f}\, d(\bar{\varrho}_*\mathbb{P}) = \int \bar{f} \circ \bar{\varrho}\, d\mathbb{P} = \int f \circ \varrho\, d\mathbb{P} = \int f\, d(\varrho_*\mathbb{P}),$$

proving the claim. ■

Since $\varrho = \pi \circ \varpi$ and $\mathbb{P}^- = \varpi_*\mathbb{P}$ it follows that $\varrho_*\mathbb{P} = \pi_*\mathbb{P}^-$ and we conclude that for $\mathbb{P}$-almost every sequence ξ and every point x it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T_{\xi_{i-1}} \circ \cdots \circ T_{\xi_0}(x)) = \int f\, d(\pi_*\mathbb{P}^-),$$

proving the corollary. ■

9
Milnor attractors. Proof of Theorem 10

Let A be a Milnor attractor of F. By definition, the realm of attraction $\delta(A)$ of A has positive measure. Since the set of disjunctive sequences has measure 1 the set $\delta(A)$ contains a disjunctive sequence ξ. Therefore, from definition of $\delta(A)$ and the first item of Theorem 7, it follows that

$$\overline{\operatorname{graph}\varrho} \subset \omega(z) \subset A.$$

We now prove that $\overline{\operatorname{graph}\varrho} \subset A_{\text{stat}}$. Let $\mathcal{C}$ denote the set of all cylinders of Σ_k. Since $(\sigma, \mathfrak{b})$ is ergodic there is a set E with $\mathfrak{b}(E) = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n}\#\{i; \sigma^i(\xi) \in C, 0 \le i < n\} = \mathfrak{b}(C) \quad \text{for every } \xi \in E \text{ and every } C \in \mathcal{C}.$$

Claim 9.0.3 *For every $z = (\xi, x) \in E \times [0,1]$ it holds $\overline{\operatorname{graph}\varrho} \subset \omega_{\text{stat}}(z)$.*

Proof. Fix $z = (\xi, x) \in E \times [0,1]$. Take a point $(\beta, \varrho(\beta)) \in \operatorname{graph}\varrho$ and an open neighbourhood $U = [-\ell; \beta_{-\ell} \dots \beta_\ell] \times V$ of $(\beta, \varrho(\beta))$. By definition of S_{t}^-, there is m such that

$$T_{\beta_{-1}} \circ \dots T_{\beta_{-(\ell+m)}}(X) \subset V.$$

Note that if $\sigma^i(\xi) \in [0; \beta_{-(\ell+m)} \dots \beta_\ell] \overset{\text{def}}{=} D$ then $F^i(F^{\ell+m}(\xi, x)) \in [-\ell; \beta_{-\ell} \dots \beta_\ell] \times V$. Therefore we have the following

$$\#\{i \colon F^i(F^{\ell+m}((\xi, x)) \in U, 0 \le i < n\} \ge \#\{i \colon \sigma^i(\xi) \in D,\ 0 \le i < n\}.$$

Since $\xi \in E$ it follows that

$$\lim_{n\to\infty} \frac{1}{n}\#\{i \colon \sigma^i(\xi) \in D,\ 0 \le i < n\} = \mathfrak{b}(D) > 0.$$

Therefore for every $(\xi, x) \in E \times [0,1]$ it holds

$$\limsup_{n} \frac{1}{n}\#\{i : F^i(\xi, x) \in U,\ 0 \le i < n\} \ge \mathfrak{b}(D) > 0.$$

Hence $\operatorname{graph}\varrho \subset \omega_{\text{stat}}(z)$, which implies that $\overline{\operatorname{graph}\varrho} \subset \omega_{\text{stat}}(z)$, proving the claim. ■

On the one hand, the claim implies that for $(\mathfrak{b} \times m)$-almost every point z it holds $\overline{\text{graph}\,\varrho} \subset \omega_{\text{stat}}(z)$. On the other hand, by definition $(\mathfrak{b} \times m)$-almost every point z we have $\omega_{\text{stat}}(z) \subset A_{\text{stat}}$. The proof of the first item of the theorem follows intersecting these two sets with full $(\mathfrak{b} \times m)$-measure.

We now prove item (2). Suppose that either $\overline{A_{\mathrm{t}}}$ is a Lyapunov stable fixed point of the Barnsley-Hutchinson operator or it has non-empty interior. It follows from item (4) of Theorem 7 that for every disjunctive sequence ξ it holds $\overline{\text{graph}\,\varrho} = \omega(\xi, x)$ for every $x \in [0,1]$. Hence, by the definition of A_M, we have that $A_M = \overline{\text{graph}\,\varrho}$. Since every Milnor attractor contains $\overline{\text{graph}\,\varrho}$ and is contained in A_M it follows that A_M is the unique Milnor attractor of F. Finally, since $A_{\text{stat}} \subset A_M$ it follows from item (1) that

$$\overline{\text{graph}\,\varrho} \subset A_{\text{stat}} \subset A_M = \overline{\text{graph}\,\varrho},$$

Item (3) follows from Theorem 8 and Theorem5.1.2. Item (4) follows from item (2), (3) and Theorem 8. The proof of the Theorem is now complete. ■

10
Milnor attractors with disconnected bones

In this chapter we prove Theorem 11. To construct the open set $\mathcal{S}$ in the theorem we need to recall the construction by Kudryashov in (20), see also Example 11.0.10, claiming that for every non-trivial subinterval $[a,b]$ of $[0,1]$ there are C^1 increasing maps $K_1, K_2\colon [a,b] \to [a,b]$ such that:

1. the $\mathrm{IFS}(K_1, K_2)$ is minimal, and

2. $K_1 \circ K_2$ has a repelling fixed point.

We call a pair of maps K_1, K_2 as above a *$\mathcal{K}$-pair in* $[a,b]$. Moreover, according to (20) the K-pairs contains an open subset of the C^1 step skew products over Σ_2 with fiber $[a,b]$.

We now consider k disjoint subintervals of $[0,1]$, namely $I_1 = [a_1,b_1],\dots,I_k=[a_k,b_k]$, with $a_1 = 0 < b_1 < b_2 < \cdots < b_k = 1$ and two C^1-strictly increasing maps $T_1, T_2\colon [0,1]\to[0,1]$ such that for each $i \in \{1,\dots,k\}$ we have that $T_1(I_i) \subset I_i$ and $T_2(I_i) \subset I_i$ and $T_{1|I_i}, T_{2|I_i}$ is a $\mathcal{K}$-pair in I_i.

We let T_3 an increasing contraction with $T_3([0,1]) \subset I_1$ and T_4 an increasing map such that $T_4(I_i) \subset \mathrm{int}(I_{i+1})$ for $i \in \{1,\dots,k-1\}$ and $T_4(I_k) \subset \mathrm{int}(I_k)$, see Figure 10.1.

This construction can be done using C^1 maps and in a robust way. The maps outside of $I_1 \cup \cdots \cup I_k$ are defined so that T_1 and T_2 are increasing and C^1.

We will see that the skew product $F = F_{\mathfrak{F}}\colon \Sigma_4 \times [0,1] \to \Sigma_4 \times [0,1]$ associated to $\mathfrak{F} = \mathrm{IFS}(T_1,T_2,T_3,T_4)$ satisfies all the properties in the statement of the theorem.

Lemma 10.0.4 $\overline{A_{\mathrm{t}}} = \bigcup_{i=1}^k I_i$.

Proof. Note that, by construction, the set $\cup_{i=1}^k I_i$ is invariant under the Barnsley-Hutchinson operator of $\mathfrak{F}$. Therefore, from the minimality of $\overline{A_{\mathrm{t}}}$, it follows

$$\overline{A_{\mathrm{t}}} \subset \bigcup_{i=1}^k I_i.$$

Fix any $i \in \{1,\dots,k\}$. We now see that $I_i \subset \overline{A_{\mathrm{t}}}$. First observe that $I_i \cap \overline{A_{\mathrm{t}}} \neq \emptyset$: just note that the fixed point $p \in I_1$ of T_3 belongs to $p \in \overline{A_{\mathrm{t}}}$. By construction, for $i \in \{2, \cdots, k\}$, $T_4^{i-1}(p) \in I_i$. Since $\overline{A_\mathrm{t}}$ is $\mathfrak{F}$-invariant it follows that $I_i \cap \overline{A_\mathrm{t}} \neq \emptyset$ for every i.

Let $p_1 = p$ and $p_i \stackrel{\text{def}}{=} T_4^{i-1}(p) \in I_i$ for $i \in \{2, \ldots, k\}$. The minimality of $\mathfrak{G}_\mathrm{i} = \mathrm{IFS}(T_{1|I_i}, T_{2|I_i})$ implies that the $\mathfrak{G}$-orbit of each p_i is dense I_i. Since $\overline{A_\mathrm{t}}$ is $\mathfrak{F}$-invariant it follows that $I_i \subset \overline{A_\mathrm{t}}$. The proof of the lemma is now complete. ■

We are now ready to prove itens (1)-(3) of the theorem about topological properties of the Milnor attractor. For that recall the definitions of the set S_t^- in (1.2.4) and of the projection ϱ defined on such a set in (1.2.5). Since $\overline{A_\mathrm{t}}$ has non-empty interior, Theorem 7 implies that $\omega(z) = \overline{\mathrm{graph}\,\varrho}$ for $\mathfrak{s}$-almost every z. Hence, by definition of likely limit set, $A_M = \overline{\mathrm{graph}\,\varrho}$. Theorem 10 implies that A_M is a Milnor attractor and that the Milnor attractor is unique. Finally, by Theorem 7 we get that $F_{|A_M}$ is topologically mixing. This ends the proof of item (1).

To prove the second item note that A_t has uncountable many elements. Hence by Theorem 5 we have that $\mathfrak{b}(S_\mathrm{t}^-) = 1$. Therefore ϱ is defined almost everywhere, proving item (2).

The prove of the third item has two parts. We first see that there is a sequence ϑ such that $(\{\vartheta\} \times [0,1]) \cap A_M$ is the union of k disjoint non-trivial intervals. Thereafter, using such a sequence,

Lemma 10.0.5 *There is a sequence ϑ such that $(\{\vartheta\} \times [0,1]) \cap A_M$ is the union of k disjoint non-trivial intervals.*

Proof. Since $A_M = \overline{\mathrm{graph}\,\varrho}$ it follows from Theorem 7 that A_M is the maximal attractor of the map $F_\mathrm{t} \stackrel{\text{def}}{=} F_{|\Sigma_k \times \overline{A_\mathrm{t}}}$. By the characterisation of the maximal attractor in Proposition 7.1.1, we have that

$$A_M = \bigcup_{\xi} \{\xi\} \times I_\xi^\mathrm{t}, \quad \text{where} \quad I_\xi^\mathrm{t} = \bigcap T_{\xi_{-1}} \circ \cdots \circ T_{\xi_{-n}}(\overline{A_\mathrm{t}}).$$

Therefore

$$(\{\xi\} \times [0,1]) \cap A_M = \{\xi\} \times I_\xi^\mathrm{t} \quad \text{for every } \xi \in \Sigma_k.$$

Hence to prove the lemma construct a sequence ϑ such that I_ϑ^t is a union of k disjoint intervals. For that recall that $\overline{A_\mathrm{t}} = \bigcup_{i=1}^k I_i$. Thus

$$I_\vartheta^\mathrm{t} = \bigcap_{n \geq 1} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(\overline{A_\mathrm{t}}) = \bigcap_{n \geq 1} \bigcup_{i=1}^{k} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_i).$$

Claim 10.0.6 *Consider any $i \in \{1, \dots, k\}$ and let $I^{\mathrm{t}}_{\vartheta}(i)$ be the spine of the sequence ϑ with respect to the* IFS$(T_{1|I_i}, T_{2|I_i})$. *Then*

$$I^{\mathrm{t}}_{\vartheta} = \bigcup_{i=1}^{k} I^{\mathrm{t}}_{\vartheta}(i)$$

for every $\vartheta \in \Sigma_{k+2}$ such that $\vartheta_{-n} \in \{1, 2\}$ for every $n \geq 1$.

Proof. Recalling the definition of $I^{\mathrm{t}}_{\vartheta}(i)$ we have that it is enough to see that

$$\bigcap_{n\geq 1} \bigcup_{i=1}^{k} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_i) = \bigcup_{i=1}^{k} \bigcap_{n\geq 1} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_i).$$

The inclusion " $\supset$ " it is straightforward. To see the inclusion " $\subset$ " take a point $p \in \bigcap_{n\geq 1} \bigcup_{i=1}^{k} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_i)$. By definition, for every $n \geq 1$ there is i_n such that $p \in T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_{i_n})$. Since $T_j(I_i) \subset I_i$ for every $j = 1, 2$ and every $i \in \{1, \dots, k\}$, recalling that the intervals I_i's are pairwise disjoint we conclude that i_n is independent of n, say $i_n = s$ for all n. Therefore

$$p \in T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_s) \quad \text{for every} \quad n \geq 1$$

and hence

$$p \in \bigcup_{i=1}^{k} \bigcap_{n\geq 1} T_{\vartheta_{-1}} \circ \cdots \circ T_{\vartheta_{-n}}(I_i),$$

proving the claim. ∎

Fix any sequence ϑ with $\vartheta_{-1} \dots \vartheta_{-n} \dots = 12121212\dots$. By Claim 10.0.6 to conclude that $I^{\mathrm{t}}_{\vartheta}$ is the union of k disjoint non-trivial intervals it is sufficient to see that $I^{\mathrm{t}}_{\vartheta}(i)$ is a non-trivial interval for every $i \in \{1, \dots, k\}$. This follows from the fact that $T_{1|I_i} \circ T_{2|I_i}$ has a repelling fixed point. ∎

To get a dense subset of Σ_4 consisting of sequences ω such that the spine I^{t}_{ω} is the union of k disjoint non-trivial intervals fix a sequence ϑ with $\vartheta_{-1} \dots \vartheta_{-n} \dots = 12121212\dots$. Note that if $\omega \in \Sigma_4$ is such that $\omega_{-(n+\ell)} = \vartheta_{-\ell}$ for every $\ell \geq 1$ then

$$I^{\mathrm{t}}_{\omega} = T_{\omega_{-1}} \circ \cdots \circ T_{\omega_{-n}}(I^{\mathrm{t}}_{\vartheta}).$$

Since the maps T_i's are monotone and $I^{\mathrm{t}}_{\vartheta}$ is a union of k non-trivial disjoint intervals, we conclude that I^{t}_{ω} is also a disjoint union of k non-trivial intervals. ∎

Remark 10.0.7 Given any $k \geq 1$, we have exhibited IFSs with 4 increasing maps such that the closure of its target set $\overline{A_{\mathrm{t}}}$ consists of k disjoint intervals. We do not know of number 4 is not sharp.

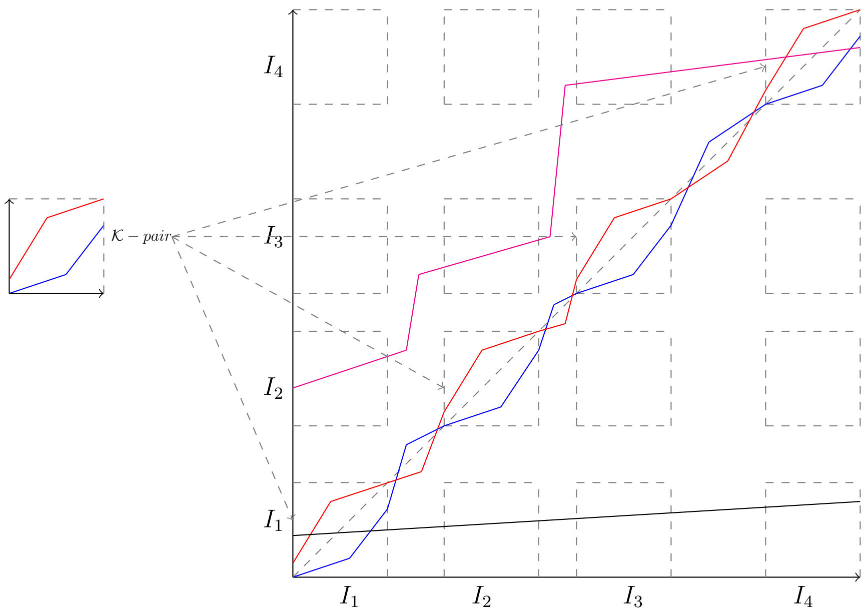

Figure 10.1: A $\mathcal{K}$-pair and a piecewise linear model

11
Examples

Example 11.0.8 (A non-regular IFS with $S_{\mathrm{t}} \neq \emptyset$) Consider an IFS defined on $[0,1]$ consisting of two injective continuous maps T_1 and T_2 as in Figure 11.1.

- The map T_1 has exactly two fixed points $0, 1$, where 0 is a repeller and 1 is an attractor.
- The map T_2 has (exactly three) fixed points $p_1 < p_2 < p_3$, where p_1 and p_3 are attractors and p_2 is a repeller, $T_2([0,1]) = [\alpha, \beta] \subset (0,1)$, and $T_1(p_1) < \beta$.

Obviously, IFS(T_1) and IFS(T_2) are not asymptotically stable. To see that IFS(T_1, T_2) is not asymptotically stable just note that $[0,1]$ and $[p_1, 1]$ are fixed points of the Barnsley-Hutchinson operator. For the last assertion we use that $T_1(p_1) < \beta$. This implies that IFS(T_1, T_2) is non-regular.

Finally, to see that $S_{\mathrm{t}} \neq \emptyset$ note that since 1 is an attracting fixed point of T_1 and $T_2([0,1]) \subset (0,1)$ we have that $T_1^n \circ T_2([0,1]) \cap T_2([0,1]) = \emptyset$ for every n sufficiently large. Now Theorem 5.3.1 implies that $S_{\mathrm{t}} \neq \emptyset$. To see that $\#(A_{\mathrm{t}}) \geq 2$ just note that given any $x \in A_{\mathrm{t}}$ then $T_i(x) \in A_{\mathrm{t}}$ and that $T_1(x) \neq T_2(x)$.

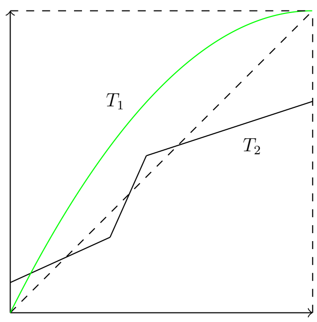

Figure 11.1: A non-regular IFS with a weakly hyperbolic sequence

Example 11.0.9 ($A_{\mathrm{t}} \subsetneq \overline{A_{\mathrm{t}}} = [0,1]$) In this example we consider the underlying IFS of the porcupine-like horseshoes in (12). We translate the construction in (13, page 12) to our context.

Consider an injective IFS(T_1, T_2) defined on $[0,1]$ such that $T_1(x) = \lambda(1-x)$, $\lambda \in (0,1)$, and T_2 is a continuous function with exactly two fixed points, the repelling fixed point 0 and the attracting fixed point 1, see Figure 11.2. We assume that T_2 is a uniform contraction on $[T_2^{-1}(\lambda), 1]$. Then $\overline{A_t} = [0,1]$ and $1 \notin A_t$.

To prove the first assertion note that $\lambda \in \overline{A_t}$. For that take an open neighbourhood $V \subset (0,1)$ of λ. Note that $T_1^{-1}(V)$ is a neighbourhood of 0. Consider the fixed point $p = \frac{\lambda}{1+\lambda} \in (0,1)$ of T_1 and note that $p \in A_t$. Since $T_2^n(p) \to 1$ as $n \to \infty$ and $T_1(1) = 0$, there is ℓ such that $T_1 \circ T_2^\ell(p) \in T_1^{-1}(V)$. Hence $T_1^2 \circ T_2^\ell(p) \in V$. By the invariance of A_t we have that $A_t \cap V \neq \emptyset$. Since this holds for every neighbourhood V of λ we get $\lambda \in \overline{A_t}$.

We now prove that A_t is dense in $[0,1]$. Take any open interval $J \subset (0,1)$. We need to see that $J \cap A_t \neq \emptyset$. If $\lambda \in J$ we are done. Otherwise $\lambda \notin J$ and either $J \subset (\lambda, 1] = I_2$ or $J \subset [0, \lambda) = I_1$. We now construct a finite sequence $\xi_0 \dots \xi_m$ such that

$$\lambda \in T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J).$$

For that let $\xi_0 = i$ if $J \subset I_i$ and define recursively $\xi_{\ell+1} = i$ if $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \subset I_i$. Note that if $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \cap I_i \neq \emptyset$ and $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \cap I_i \not\subseteq I_i$ some $i = 1, 2$, we are done. Since T_2^{-1} is a uniform expansion on $(\lambda, 1]$ and T_1^{-1} is a uniform expansion on $[0, \lambda]$ the recursion stops after a finitely many steps: there is m such that $\lambda \in T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J)$. Since $\lambda \in \overline{A_t} \cap T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J)$, the invariance of A_t implies that $J \cap A_t \neq \emptyset$.

The fact that $1 \notin A_t$ follows observing that $\bar{2} \notin S_t$ and that every finite sequence $\xi_0 \dots \xi_n$ such that $\xi_i = 1$ for some i satisfies $1 \notin T_{\xi_0} \circ \dots \circ T_{\xi_n}([0,1])$.

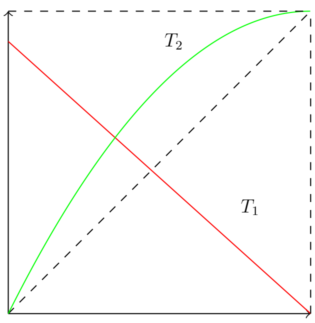

Figure 11.2: The underlying IFS of a porcupine-like horseshoe

Example 11.0.10 (A non-weakly hyperbolic IFS in $[0,1]$ **with** $A_t = [0,1]$**)** We consider the underlying IFS of the bony attractors in (20).

Consider the IFS(T_1, T_2) defined on $[0,1]$ as follows, T_1 is the piecewise-linear map with "vertices" $(0,0)$, $(0.6, 0.2)$, and $(1, 0.8)$ and T_2 is the piecewise-linear map with "vertices" $(0, 0.15)$, $(0.4, 0.8)$, and $(1,1)$, see Figure 11.3. We claim that the IFS(T_1, T_2) is not weakly hyperbolic and $A_t = [0,1]$.

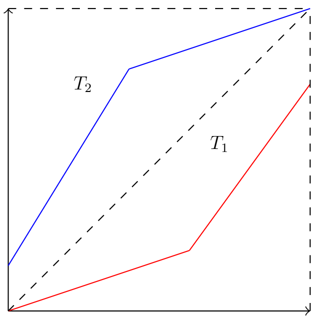

Figure 11.3: A $\mathcal{K}$-pair

To prove the first assertion note that $T_1 \circ T_2$ has a repelling fixed point, see (20). Therefore the periodic sequence $\overline{12}$ does not belong to S_{t}, hence the IFS is not weakly hyperbolic.

To see the second assertion, note that the compositions T_1^3, $T_1^2 \circ T_2$, $T_2^2 \circ T_1$ and T_2^5 are uniform contractions and that the union of their images is $[0,1]$, see (20). In other words, the IFS$(T_1^3, T_1^2 \circ T_2, T_2^2 \circ T_1, T_2^5)$ is hyperbolic and $[0,1]$ is the unique fixed point of its Barnsley-Hutchinson operator. Consider the finite set of words

$$W = \{111, 112, 221, 22222\}$$

and let E_W be the subset of Σ_k^+ consisting of sequences ξ that are a concatenation of words of W[1]. Let S_{t} be the set of weakly hyperbolic sequences corresponding to the IFS(T_1, T_2) and π the associated coding map. By construction we have that $E_W \subset S_t$ and $\pi(E_W) = [0,1]$. Since $A_t = \pi(S_{\mathrm{t}})$ we have that $A_{\mathrm{t}} = [0,1]$.

Example 11.0.11 In this example, we see an IFS defined on the square $[0,1]^2$ such that the closure of its target set is Lyapunov stable, however this set is not a Conley attractor and the maps of the IFS are not Lipschitz with constant ≤ 1 (indeed one of the maps have Lipschitz constant greater than one 1).

Let $T_1, T_2 \colon [0,1] \to [0,1]$ be the maps in Example 11.0.9. Consider the IFS(f_1, f_2, f_3) defined in the square $[0,1]^2$, where $f_3(x,y) = \frac{1}{2}(x,y)$ and $f_1(x,y) = (T_1(x), y)$, and $f_2(x,y) = (T_2(x), y)$.

We first claim that $\overline{A_{\mathrm{t}}} = [0,1] \times \{0\}$. Since $(0,0)$ is the fixed point of the contraction f_3 we have that $(0,0) \in A_{\mathrm{t}}$. The minimality of the IFS(T_1, T_2) implies that IFS(f_1, f_2) acts transitively in each fibre $[0,1] \times \{y\}$. Therefore the invariance of A_{t} implies that $[0,1] \times \{0\} \subset \overline{A_{\mathrm{t}}}$. Since the set $[0,1] \times \{0\}$ is

[1]There is an increasing sequence $(i_\ell)_{\ell \in \mathbb{N}}$ with $\xi_0 = 0$ such that $\xi_{i_\ell} \dots \xi_{i_{\ell+1}-1} \in W$ for every $\ell \in \mathbb{N}$.

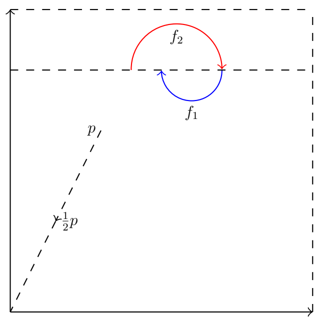

Figure 11.4: Actions of the maps of the IFS

invariant under the Barnsley-Hutchinson operator, the minimality of $\overline{A_t}$ implies that $\overline{A_t} \subset [0,1] \times \{0\}$.

To see that $\overline{A_t}$ is Lyapunov stable given an open neighbourhood U of it consider ϵ such that $[0,1] \times [0,\epsilon) \subset U$. By definition of the IFS $\mathcal{B}([0,1] \times [0,\epsilon)) = [0,1] \times [0,\epsilon) \subset U$. This also prevents the set $\overline{A_t}$ to be a Conley attractor.

For the assertion about the Lipschitz constant, just observe that T_2 is an expanding map, this implies that the Lipschitz constant of f_2 cannot be ≤ 1.

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