Pontifícia Universidade Católica
do Rio de Janeiro

Dania González Morales

Two topics in degenerate elliptic equations involving a gradient term: existence of solutions and a priori estimates

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Boyan Slavchev Sirakov

Rio de Janeiro
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This thesis is lovingly dedicated to my mom.

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Abstract

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This thesis concerns the study of existence, nonexistence and *a priori* estimates of nonnegative solutions of some types of degenerate coercive and non coercive elliptic problems involving an additional term which depends on the gradient. Among other things, we obtain generalized integral conditions of Keller-Osserman type for the existence and nonexistence of solutions. Also, we show that different conditions are needed when $p \geq 2$ or $p \leq 2$, due to the degeneracy of the operator. The uniform a priori estimates are obtained for supersolutions and solutions of superlinear elliptic PDE or systems of such PDE in divergence form that can contain different operators and nonlinearities. We also give full boundary extensions to some "half Harnack" inequalities and quantitative Hopf lemmas, for degenerate elliptic operators like the p-Laplacian.

Keywords

Existence; Non existence; Degenerate PDEs; p-Laplacian; Dependence on the gradient; Keller-Osserman integral conditions; Boundary estimates; A priori estimates.

Resumo

Morales, Dania González; Sirakov, Boyan Slavchev. **Dois tópicos em equações elípticas degeneradas com dependência do gradiente: existência de soluções e estimativas a priori** . Rio de Janeiro, 2018. 85p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Esta tese tem o intuito do estudo da existência, não existência e estimativas *a priori* de soluções não negativas de alguns tipos de problemas elípticos degenerados coercivos e não coercivos com um termo adicional dependendo do gradiente. Dentre outras coisas, obtemos condições integrais generalizadas tipo Keller-Osserman para a existência e não existência de soluções. Também mostramos que condições adicionais e diferentes são necessárias quando $p \geq 2$ ou $p \leq 2$, devido ao caráter degenerado do operador. As estimativas a priori são obtidas para super-soluções e soluções de EDPs elípticas superlineares o sistemas de tais tipos de equações em forma divergente com diferentes operadores e não linearidades. Além do mais, obtemos extensões até a fronteira de algumas desigualdades de Harnack fracas e lemas quantitativos de Hopf para operadores elípticos como o p-Laplaciano.

Palavras-chave

Existência; Não existência; EDPs degeneradas; p-Laplaciano; Dependência do gradiente; Condições integrais Keller-Osserman; Estimativas até a fronteira; Estimativas a priori.

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1
Introduction

Fundamental topics in the theory of partial differential equations are that the study of the existence and the *a priori* estimates of solutions.

When a nonlinear mathematical model is represented by a degenerate elliptic partial differential equation, that is, when the uniform ellipticity of operator is lost, these two topics become even more interesting and challenging. This is because, on the one hand, several problems arising in natural processes are described by this type of equations, and, on the other hand, the degenerate structure of the equations requires different techniques from those used for equations involving uniformly elliptic operators.

This thesis is inserted in the effort to study these subjects in particular when the equations involve the degenerate p-Laplacian operator. More precisely, we study the following two problems.

($P1$) Existence and non existence of nonnegative entire solutions of coercive quasilinear elliptic equations with an additional term which depends on the gradient, like

$$\Delta_p u = f(u) \pm g(|\nabla u|) \qquad \text{in } \mathbb{R}^n,$$

where f and g satisfy some monotonicity and continuity hypotheses to be made precise in chapter 3.

($P2$) Uniform a priori L^∞–estimates for nonnegative weak, in the Sobolev sense, solutions of non coercive quasilinear elliptic equations in a domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$, such as

$$-Qu = H(x, u),$$

where Q is an operator whose the principal part corresponds to the p-Laplacian operator and $H : \Omega \times [0, \infty) \to [0, \infty)$ is a continuous bounded function that is superlinear at infinity (for super-solutions) and satisfies certain growth condition (in the case of sub-solutions), to be described in chapter 4.

This thesis is organized as follows. In the second chapter we give a brief introduction to the p-Laplacian operator which drives the equations in the two problems we study. The third chapter is devoted to the first problem $(P1)$ and is based on the paper "Existence and nonexistence of positive solutions of quasi-linear elliptic equations with gradient terms"([1], submitted). Sections 3.1 and 3.2 give some preliminaries on the subject; the main results are also stated. Section 3.3 contains results of local existence of a radial solution of the equation in $(P1)$. Section 3.4 continues the study of existence in the whole Euclidean space $\mathbb{R}^n$. We proceed by combining some integral conditions of generalized Keller–Osserman (KO) type which we obtain, with comparison principles due to Pucci and Serrin [2, Chapter 3]. In the last chapter we prove uniform a priori estimates to the equations in $(P2)$. These are based on the paper "Uniform a priori L^∞–estimates for the p-Laplacian", under preparation. Sections 4.1 and 4.2 are devoted to the preliminaries and the presentation of the main results on this subject. The third section contains results on boundary "half"-Harnack inequalities and quantitative Hopf lemmas. We finish by presenting the proof of Lebesgue and uniform L^∞–estimates.

2
Preliminaries

In this chapter we are going to give a short introduction to the p-Laplacian operator. To this goal we begin with general notations which we are going to use throughout the text. Other additional notations specific to each topic will be given at the beginning of each respective chapters, in order to facilitate the understanding of the material.

2.1
General Notations

We use the standard notations of the classical literature.

Simbols

$\mathbb{R}^n$: The n-dimensional Euclidean space

$x = (x_1, \ldots, x_n)$: A point $x \in \mathbb{R}^n$

$B_r(x_0)$: The euclidean ball centered at the point $x_0 \in \mathbb{R}^n$ with radius $r > 0$. If we write B_r, it is understand that the ball is centered at the origin of the Euclidean space.

Ω: An open connected subset of $\mathbb{R}^n$, not necessarily bounded

$\partial\Omega$: The boundary of the set Ω

$|\xi|$: The Euclidean norm of a vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$\mathrm{div}\,(\xi) = \frac{\partial \xi_1}{\partial x_1} + \ldots + \frac{\partial \xi_n}{\partial x_n}$: The divergence of the vector $\xi \in \mathbb{R}^n$

$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$: The gradient of the function u

$\Delta_p u$: The p-Laplacian of the function u

Space of functions

$C^k(\Omega)$: The class of k-times continuously differentiable functions in Ω

$C_c^\infty(\Omega)$: The class of C^∞ functions with compact support in Ω

$L^p(\Omega)$: The class of Lebesgue integrable functions whose absolute value raised to the p-th power has finite integral

$L^p_{\mathrm{loc}}(\Omega)$: The class of functions whose absolute value raised to the p-th power has finite integral on compact subsets of Ω

$L^\infty(\Omega)$: The class of bounded functions

$W^{1,p}(\Omega)$: The first order Sobolev space (functions in $L^p(\Omega)$ such that also $|\nabla u| \in L^p(\Omega)$)

$W^{1,p}_{\mathrm{loc}}(\Omega)$: Functions in $L^p_{\mathrm{loc}}(\Omega)$ such that $|\nabla u| \in L^p_{\mathrm{loc}}(\Omega)$

2.2 Some features of the p-Laplacian operator

The p-Laplacian operator has been extensively studied during the past fifty years by nonlinear analysts. It appeared for the first time in the study of nonlinear flows in channels and ditches as a result of a nonlinear power law suggested as alternative to Darcy's law (the equation that describes the flow of a fluid through a porous media), but it also arises in many other contexts such as radiation of heat, non local diffusion, image and data processing, just to name a few. Problems involving the p-Laplacian operator and depending on the gradient arise naturally as stationary states of various other models in fluid mechanics. For a complete description of the origins of the operator and various applications the interested readers can consult [3] and [4, 5], respectively.

This nonlinear operator in divergence form is defined by

$$\Delta_p \cdot := \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot), \quad 1 < p < \infty, \tag{2.1}$$

and reduces to the Laplacian when p=2. However, the p-Laplacian has different structural properties from the Laplacian as $p \neq 2$.

Let us highlight the main properties that make the quasilinear case ($p \neq 2$) different from the semilinear one (p=2), and forces us to develop different techniques in a number of situations.

For $p \neq 2$ the p-Laplacian operator (2.1) is ($p-1$)-homogeneous but not additive.

In particular, if $u : \Omega \to \mathbb{R}$ is such that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $t \in \mathbb{R}$, $t \geq 0$, we have

$$\Delta_p(tu) = \operatorname{div}(|\nabla(tu)|^{p-2}\,\nabla(tu)) = \operatorname{div}(t^{p-1}\,|\nabla(u)|^{p-2}\,\nabla(u)) = t^{p-1}\Delta_p u,$$

but obviously if $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $p \neq 2$, $\Delta_p(u+v) \neq \Delta_p u + \Delta_p v$.

For $p \neq 2$ the character of the operator changes and the operator is not always uniformly elliptic.

To analyze the ellipticity of (2.1) we set

$$\begin{array}{rccl} A: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \xi & \longmapsto & A(\xi) := |\xi|^{p-2}\,\xi. \end{array}$$

By computing the Jacobian matrix associated to A we obtain

$$\partial_\xi A(\xi) = |\xi|^{p-2}\left(I_n + \frac{p-2}{|\xi|^2}\xi\otimes\xi\right). \tag{2.2}$$

So, for $\xi \neq 0$ the eigenvalues of this matrix are $|\xi|^{p-2}$ with multiplicity $n-1$ and $(p-1)\,|\xi|^{p-2}$ with multiplicity 1. Thus the operator it singular for $1 < p < 2$ and degenerate for $p > 2$. The values of $\xi = 0$ and $\xi = \infty$ correspond to the singularities of the p-Laplacian – in a neighborhood of these points we lose the uniform ellipticity.

Besides the mentioned differences due to the degeneracy of the operator there is another important difference when we deal with the p-Laplacian operator if there exists an additional term which depends on the gradient and it is the validity of a comparison principle. For instance, as is known by [2], the problem

$$\begin{cases} \Delta_4 u + |Du|^2 & = 0 \quad \text{in } \ B_R \subset \mathbb{R}^2 \\ \qquad\quad u & = 0 \quad \text{on } \ \partial B_R \end{cases} \tag{2.3}$$

admits the solutions $u(x) = 0$ and $v(x) = \frac{1}{8}(R^2 - |x|^2)$ in B_R which implies that the comparison principle cannot be satisfied in general. However, the comparison principle holds under additional hypotheses. Various results of this type are proven in [2]. In the next chapters we use some of these comparison principles, which will be stated in relation to each specific problem.

Unless otherwise stated, we consider weak solutions in the Sobolev sense according to the following definition.

Definition 2.1 (Weak Solution)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$; we say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak sub-solution of

$$\Delta_p u = f(x, u, |\nabla u|) \qquad \text{in } \ \Omega$$

if

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u \nabla\phi \,\mathrm{d}x \leq \int_\Omega -f(x, u, |\nabla u|)\phi \,\mathrm{d}x \tag{2.4}$$

for each $\phi \in C_c^\infty(\Omega)$, $\phi \geq 0$. Analogously, we can define a weak super-solution by presuming the opposite inequality. Finally we say that u is a weak solution, in the Sobolev sense, if it is sub-solution and super-solution at the same time.

It is well known that weak solution in Sobolev spaces are well suited to equations in divergence form.

3
Existence and nonexistence results for quasilinear equations

An important and very well studied type of second order PDEs are the so-called “coercive” ones. In particular, the equation

$$Lu = f(u), \quad f \geq 0,$$

where L an elliptic operator and f is increasing is of this type. The word coercive refers to the fact that growth of u (at infinity) causes growth in Lu and hence even more rapid growth of u. It becomes then a compelling question to understand under what hypotheses this process causes or does not cause explosion of u at a finite point, or in other words, under what hypotheses the equation has a solution in the whole space.

In this chapter we present our results on the first of the two problems stated in the introduction. We recall that the problem concerns the study of the existence and non existence of nonnegative entire solutions of coercive quasilinear elliptic equations with an additional term with depends on the gradient in the Euclidean space $\mathbb{R}^n$ such as

$$\Delta_p u = f(u) \pm g(|\nabla u|). \qquad (P_\pm)$$

We are going to assume that f and g satisfy the following conditions,

$$f \in C([0,\infty)), g \in C^{0,1}([0,\infty)) \text{ are strictly increasing with } f(0) = g(0) = 0. \qquad (3.1)$$

Throughout this chapter we also use also the following notations.

$C^{0,1}$: The class of the Lipschitz continuous functions

$v(r)$: If the solution u is a radial function we set $v(r) = v(|x|) = u(x)$

$'$: The prime means the derivative of the function with respect to the variable r

$\|\cdot\|_\star$: The norm of the function $\cdot$ in the class of functions $\star$

3.1 Introduction

The study of the equations $(P_\pm)$ goes back to the famous work [6] of Lasry and Lions, who studied explosive solutions defined in a bounded domain in the particular case p=2. Concerning unbounded domains, many novel results appeared later in the paper of Farina and Serrin [7]. They studied positive solutions in the whole Euclidean space when the term g which depends on the gradient is a power of the gradient, and the sum in the right-hand side of our equation is replaced by a product of f and g. In a more recent work, Felmer, Quaas and Sirakov [8] showed results of existence and non existence of $(P_\pm)$, when the operator is uniformly elliptic, such as the Laplacian, for viscosity solutions in the whole Euclidean space. They gave a rather precise description of the way that the interaction between the terms f and g in the nonlinearity influences the solvability of these problems.

As far as the general case, when p is not necessarily equal to two, is concerned, we first quote the pioneering work [9] of Mitidieri and Pohozaev. They studied nonexistence results for entire weak solutions, in the Sobolev sense, of equations without dependence on the gradient (g=0). Mitidieri and Pohozaev used a priori estimates obtained with a careful choice of test functions. Other important existence and nonexistences results for equations like $(P_\pm)$ when the gradient term has a particular (power) form are due to Filipucci, Pucci and Rigoli [10, 11, 12]. They used refined techniques based on comparison principles and the weak solutions they considered were assumed to be in C^1. Furthermore, the above quoted work by Farina and Serrin [7] contains important contributions also for equations involving the p-Laplacian, with nonlinearities which behave like products (as opposed to sums and differences) of u and its gradient, possibly dependent on x with conditions required only for large radii.

However, to our knowledge, if the problem involves a term which depends on the gradient, in all references, it is a power or the nonlinearity behaves exactly as a product of a term in u and a term which depends on the gradient.

Motivated by all these results, our goal is to get a generalization of the results in [8] to the case of the p-Laplacian.

The main novelty of our work resides in the concurrent resolution of difficulties which arise, on one hand, from the singular or degenerate nature of the operator, and on the other hand, from the general form of the right-hand side and the need to understand how the interaction between f and g affects the solvability of the problem. The presence of the gradient term combined with the intrinsic properties of the p-Laplacian operator makes our study of this type of problems different from previous works, to our knowledge. In addition, for the case of (P_+), we manage to prove the nonexistence of a positive solution in the weak Sobolev sense, according to Definition 2.1. For $(P_\pm)$, this definition states that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ is an entire weak sub-solution of $(P_\pm)$ if, for each $\phi \in C_c^\infty(\mathbb{R}^n)$, $\phi \geq 0$,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \phi \, \mathrm{d}x \leq \int_{\mathbb{R}^n} -(f(u) \pm g(|\nabla u|))\phi \, \mathrm{d}x. \tag{3.2}$$

Remark 3.1 *Actually, most of our nonexistence results are valid for weak solutions which belong to the smaller class $W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. However, as we explain later in Section 3.4, we can still obtain a nonexistence result for sub-solutions in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, if a comparison principle were proven for solutions in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ of (P_+).*

A discussion of the results is in order. The first theorem below contains our existence and non-existence statements for the problem (P_+). The non-existence result says that if a condition on f, resp. g, which guarantees lack of solutions for the problem with $g = 0$, resp. $f = 0$, is valid, then there are no solutions for the full problem. In the existence result we establish how the value of p influences the solvability of the problem. For p larger or smaller than 2 we obtain an explicit relation between the functions f and g, which permits us to conclude about the existence.

Theorem 1 (Existence theorem for (P_+)) *Let f, g be functions satisfying (3.1) and $F(s) = \int_0^s f(t)\,\mathrm{d}t$.*

(i) If either

$$\int_1^\infty \frac{1}{(F(s))^{\frac{1}{p}}} \mathrm{d}s < +\infty \qquad \text{or} \qquad \int_1^\infty \frac{s^{p-2}}{g(s)} \mathrm{d}s < +\infty \tag{3.3}$$

then any nonnegative weak sub-solution in the space $W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ of

$$\Delta_p u = f(u) + g(|\nabla u|) \ \text{ in } \ \mathbb{R}^n \tag{P_+}$$

is identically zero.

(ii) If

$$\int_1^\infty \frac{1}{(F(s))^{\frac{1}{p}}} \mathrm{d}s = +\infty \quad (3.4) \qquad \text{and} \qquad \int_1^\infty \frac{s^{p-2}}{g(s)} \mathrm{d}s = +\infty \quad (3.5)$$

and there exist constants $A_0, \epsilon_0 > 0$ such that for all $A \geq A_0$, either

$$\textit{if} \quad p \leq 2, \qquad \liminf_{s\to\infty} \frac{g(AF(s)^{\frac{1}{p}})}{A^p f(s)} > \frac{1}{p} + \epsilon_0 \tag{3.6}$$

or

$$\textit{if} \quad p \geq 2, \qquad \limsup_{s\to\infty} \frac{g(AF(s)^{\frac{1}{p}})}{A^p f(s)} < \frac{1}{p} - \epsilon_0 \tag{3.7}$$

then (P_+) *has at least one positive solution.*

Observe that in the problem (P_+) a growth of f and g leads to a growth in the p-Laplacian, hence, at least morally, to growth of u. In this way, (3.4) and (3.5) describe how quickly $f(s)$ and $g(s)$ can grow as s tend to infinity, so that the positive solution does not blow up at some finite point.

We recall that the condition on F in (3.3) with p=2 is precisely the one obtained by Keller [13] and Osserman [14] for the semilinear equation $\Delta u = f(u)$, this is, when $g = 0$.

To state a few examples of f and g in (P_+), the generalized (KO) integral condition associated to f, (3.4), is satisfied by standard functions whose growth at infinity does not exceed that of t^q , $q \leq p-1$ or $t^{p-1}(\log t)^q$, $0 < q \leq p$. The (KO) integral condition (3.5) associated to g is valid, for example, by functions whose growth at infinity does not exceed that of t^q , $q \leq p-1$ or $t^{p-1}(\log t)^q$, $0 < q \leq 1$.

The hypotheses (3.6) and (3.7) set a comparison between $g \circ F^{\frac{1}{p}}(s)$ and $f(s)$ for large values of s which depends on the values of p. We note that for most standard functions the limits in (3.6) and (3.7) are zero or infinity, so these hypotheses are easy to verify. We also note that if $g \circ F^{\frac{1}{p}}$ grows no faster than f at infinity we cannot conclude anything about the existence of positive solutions if $p < 2$ even if (3.4) and (3.5) are satisfied. Analogously, if $g \circ F^{\frac{1}{p}}$ grows strictly faster than f, and $p > 2$.

Remark 3.2 *Assumptions* (3.6) *and* (3.7) *can be avoided in the particular case where* $g(s)$ *has at most* s^{p-1} *growth as* s *tends to infinity. This will be explained later.*

The treatment for (P_-) is somewhat different and in some way simpler, because we don't need additional conditions like (3.6) and (3.7) to conclude about the existence.

We summarize our results on (P_-) in the next theorem.

Theorem 2 (Existence theorem for (P_-)**)** *Let* f *and* g *be functions that satisfy* (3.1).

(i) If

$$\int_1^\infty \frac{1}{\Gamma^{-1}(F(s))}\,\mathrm{d}s < \infty \tag{3.8}$$

with Γ *defined by*

$$\Gamma(s) = \int_0^{2s} g(t)\,\mathrm{d}t + \frac{p-1}{p} c s^p, \tag{3.9}$$

then any sub-solution of (P_-) *vanishes identically.*

(ii) If

$$\int_1^\infty \frac{1}{(F(s))^{\frac{1}{p}}}\,\mathrm{d}s = \infty \quad \textit{or} \quad \int_1^\infty \frac{1}{g^{-1}(f(s))}\,\mathrm{d}s = \infty, \tag{3.10}$$

then (P_-) *admits at least one positive solution.*

Observe that, by the definition of Γ, if (3.8) is satisfied then none of the assumptions in (3.10) are possible. Analogously if at least one of the integral condition in (3.10) is valid, then (3.8) is not possible. On the other hand, nothing can be said about (3.8) if (3.10) is not satisfied.

Finally, it is worth noting that it is only technical to extend all the above results to non-autonomous equations, when positive continuous weight functions (depending on $|x|$) multiply f and g. This naturally leads to modifications in the integral conditions. We also observe that, by applying for instance Young's inequality, the case of a right-hand side which behaves as a product of functions of u and its gradient can be reduced to the problem (P_+).

3.2 Main tools to deal with the existence results

The comparison principle plays a paramount role in the theory of elliptic PDE. To prove our non-existence results we will make use of a version of this principle adapted to the equations which we study. In the following subsection we give the detailed statement.

Then, in section 3.2.2 we will introduce a initial value problem for a singular one-dimensional equation, which corresponds to the radial version of $(P_\pm)$. We will give a description of some properties of the solutions of that problem.

3.2.1 The comparison principle

The singular/degenerate character of the p-Laplacian causes difficulties in establishing comparison principles as was mentioned in the previous chapter. We will use the following version of this principle given by Pucci and Serrin in [2, Corollary 3.6.3].

Theorem 3 (Comparison principle) *Assume that $B = B(x, z, \xi)$ is locally Lipschitz continuous with respect to ξ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and is non-increasing in the variable z. Let u and v be solutions of class $W^{1,\infty}_{\text{loc}}(\Omega)$ of*

$$\Delta_p u + B(x, u, Du) \geq 0 \text{ in } \Omega \quad , \quad \Delta_p v + B(x, v, Dv) \leq 0 \text{ in } \Omega,$$

where $p > 1$. Suppose that

$$\operatorname*{ess\,inf}_{\Omega} \{|Du| + |Dv|\} > 0.$$

If $u \leq v + M$ in $\partial\Omega$ where $M \geq 0$ is constant, then $u \leq v + M$ in Ω.

A complete proof can be found in the cited reference.

Remark 3.3 *The condition $\operatorname{ess\,inf}_{\Omega} \{|Du| + |Dv|\} > 0$ is vital for the comparison principle to be satisfied. In fact, for the solutions of (2.3) we have that $|Du| + |Dv| = 0$ at zero, and consequently $\operatorname{ess\,inf}_{\Omega} \{|Du| + |Dv|\} = 0$. We will be able to apply the comparison principle in our equations thanks to the hypotheses (3.1) for f and g.*

3.2.2 Associated ODE problem

In this section we focus our attention on the associated ODE of the problems $(P_{\pm})$.

Take $u(x) = v(|x|) = v(r)$ in $(P_{\pm})$ and add the initial conditions $u(0) = v(0) = v_0 > 0$ and $v'(0) = 0$, which are consistent with our goals. Thus, the problem

$$\begin{cases} (r^{n-1}(v')^{p-1})' & = r^{n-1}\left(f(v) \pm g(v')\right) \quad \text{in} \quad (0, \infty) \\ \qquad v(0) & = v_0 > 0 \\ \qquad v'(0) & = 0 \end{cases} \tag{3.11}$$

corresponds to the radial version of $(P_{\pm})$ with fixed initial data.

More precisely, if we compute the p-Laplacian of a radial function $u(x) = v(r)$ we obtain the one-dimensional operator

$$\operatorname{sign}(v'(r)) \left(r^{n-1} |v'(r)|^{p-1}\right)', \tag{3.12}$$

and of course $g(|\nabla u|) = g(|v'|)$.

In (3.11) the equation is written in a simplified form because, as we will see next, v is positive and v' is such that $v'(0) = 0$ and $v'(r) > 0$ for all $r > 0$.

It is essentially known that for an ODE problem like (3.11) the non-negativity of the solutions and its derivatives can be deduced a priori. We give a full proof of this fact, for the reader's convenience.

Lemma 3.4 (A priori properties of the radial solutions) *Let $v = v(r)$ be a solution of*

$$\begin{cases} \operatorname{sign}(v')\left(r^{n-1}\,|v'|^{p-1}\right)' & = r^{n-1}\left(f(v) \pm g(|v'|)\right) \quad in \quad (0,\infty) \\ v(0) & = v_0 > 0 \\ v'(0) & = 0 \end{cases} \tag{3.13}$$

in some interval $[0,R]$ with $0 < R < \infty$. Then $v > 0$, $v' > 0$, $v'' \geq 0$ and $v(r) \leq v_0 + Rv'(r)$, for all $r \in (0,R)$.

Proof of Lemma 3.4. By simple computation it is easy to see that the problem (3.13) can be rewritten as

$$\begin{cases} \left(|v'|^{p-2}\,v'\right)' + \frac{n-1}{r}\,|v'|^{p-2}\,v' & = f(v) \pm g(|v'|) \text{ in } \quad (0,\infty) \\ v(0) & = v_0 > 0 \\ v'(0) & = 0. \end{cases} \tag{3.14}$$

First we deal with the signs of v' and v. By letting $r \to 0$ we obtain that

$$\lim_{r\to 0}\left(|v'|^{p-2}\,v'\right)'(r) = \frac{1}{n}f(v_0) > 0,$$

which implies that $\left(|v'|^{p-2}\,v'\right)'(r) > 0$ for all $r > 0$ close enough to zero. Then obviously $|v'|^{p-2}\,v'(r) > 0$ for all $r > 0$ close enough to zero as well. Consequently $v'(r) > 0$ for all $r > 0$ close to zero. Now we need to study if this behavior is the same for all $r > 0$.

Suppose that there exists $r_1 > 0$ such that $|v'|^{p-2}\,v'(r) > 0$ in $(0,r_1)$ and $|v'|^{p-2}\,v'(r_1) = 0$. Then we would have $\left(|v'|^{p-2}\,v'\right)'(r_1) \leq 0$. On the other hand, by using the equation in (3.14) we obtain that $\left(|v'|^{p-2}\,v'\right)'(r_1) > 0$, a contradiction. Hence $v'(r) > 0$ for all $r > 0$. Since $v(0) = v_0 > 0$ we also conclude that $v(r) > 0$ for all $r > 0$.

Knowing the signs of v and v' we rewrite our equation as

$$\left((v')^{p-1}\right)' + \frac{n-1}{r}(v')^{p-1} = f(v) \pm g(v').$$

In order to deal with the sign of v'' we separate the argument in two cases, according to the signs $+$ or $-$ in the nonlinearity.

Case I: Positive sign in the nonlinearity.

$$\begin{cases} ((v')^{p-1})' + \frac{n-1}{r}(v')^{p-1} & = f(v) + g(v') \ \text{ in } \ (0,\infty) \\ v(0) & = v_0 > 0 \\ v'(0) & = 0 \end{cases} \tag{3.15}$$

Suppose that there exist ϵ, $r_1 > 0$ such that $((v')^{p-1})'(r) > 0$ in $(r_1 - \epsilon, r_1)$ and $((v')^{p-1})'(r_1) = 0$. Taking $h > 0$ sufficiently small we have

$$((v')^{p-1})'(r_1 - h) + (n-1)\frac{(v')^{p-1}(r_1 - h)}{r_1 - h} \leq ((v')^{p-1})'(r_1) + (n-1)\frac{(v')^{p-1}(r_1)}{r_1},$$

since $f(v(r))$ and $g(v'(r))$ are strictly increasing for all $r > 0$. Then,

$$\begin{aligned} (n-1)\left(\frac{(v')^{p-1}(r_1)}{r_1} - \frac{(v')^{p-1}(r_1 - h)}{r_1 - h}\right) & \geq ((v')^{p-1})'(r_1 - h) - ((v')^{p-1})'(r_1) \\ & = ((v')^{p-1})'(r_1 - h) > 0. \end{aligned}$$

Now dividing by h and letting it tend to zero

$$\begin{aligned} 0 & < \lim_{h\to 0}(n-1)\frac{1}{h}\left[\frac{(v')^{p-1}(r_1)}{r_1} - \frac{(v')^{p-1}(r_1 - h)}{r_1 - h}\right] \\ & = (n-1)\left[\frac{(v')^{p-1}}{r}\right]'\Bigg|_{r=r_1} \\ & = (n-1)\left[\frac{((v')^{p-1})'(r_1)r_1 - (v')^{p-1}(r_1)}{r_1^2}\right] \\ & = -(n-1)\frac{(v')^{p-1}(r_1)}{r_1^2}. \end{aligned}$$

Thus $(v')^{p-1}(r_1) < 0$, and since r_1 is arbitrary we obtain a contradiction with the fact that $v'(r) > 0$ for all $r > 0$.

Case II: Negative sign in the nonlinearity.

$$\begin{cases} ((v')^{p-1})' + \frac{n-1}{r}(v')^{p-1} & = f(v) - g(v') \ \text{ in } \ (0,\infty) \\ v(0) & = v_0 > 0 \\ v'(0) & = 0. \end{cases} \tag{3.16}$$

Suppose that $((v')^{p-1})'(r_1) < 0$ for some $r_1 > 0$. Let

$$r_2 = \inf\left\{\tilde{r} \ : \ ((v')^{p-1})'(r) < 0 \text{ in } (\tilde{r}, r_1)\right\}.$$

Since $\lim_{r\to 0}((v')^{p-1})'(r) > 0$ we have that $r_2 > 0$ and $((v')^{p-1})'(r_2) = 0$. Moreover $((v')^{p-1})'(r) < 0$ for $r > r_2$ sufficiently close to r_2. This implies that $(v')^{p-1}(r)$ is decreasing for $r > r_2$ sufficiently close to r_2. On the other hand, since v is increasing

$$((v')^{p-1})'(r) = f(v(r)) - \frac{n-1}{r}(v')^{p-1} - g(v')$$

is increasing. Thus $((v')^{p-1})'(r_2) = 0$ implies that $((v')^{p-1})'(r) > 0$ if $r > r_2$ close to r_2, a contradiction, and we are done.

Also, since $v''(r) > 0$ for all $r > 0$ we have that $v'(r)$ is non-decreasing for $0 < r < R$, then

$$v(r) = v_0 + \int_0^r v'(s)\,\mathrm{d}s \leq v_0 + Rv'(r).$$

$\blacksquare$

3.3 Local existence for the associated radial problem

We will now study the existence of solution of (3.11) in a right neighborhood of zero. Because of the presence of the gradient in the right-hand side we will need to apply topological tools, in particular, the Leray-Schauder theorem. This dependence in the gradient makes it necessary to bound not only the function but also its derivative. Consequently, in this way we can only show the existence of solution in a neighborhood of zero. Furthermore, even this is not immediate when $p \neq 2$ because of the expression involving the exponent $p-1$ inside the derivative on the left-hand side.

In the next lemma we prove the existence of solution in a neighborhood of zero. Similar results have appeared for instance in [4] and [15].

Lemma 3.5 (Existence of solution in a neighborhood of zero) *Let f and g be continuous increasing functions. Then there exists $0 < r_1 \leq 1$ such that the problem (3.11) has a positive solution $v(r) \in C^2[0, r_1]$.*

This lemma will be obtained by observing that finding a solution for the problem is equivalent to finding a fixed point of some well chosen integral operator. Then, using the Arzelá-Ascoli theorem and the dominated convergence theorem, we prove the continuity and compactness of this operator. Finally we apply the Leray-Schauder fixed point theorem (in [16, Corollary 11.2]), to this operator in a well chosen closed, convex and bounded set.

Proof. To simply the notation set

$$H(t) := t^{p-1}, \quad t > 0$$

and

$$F(v,v') := f(v) \pm g(v').$$

With this notation, we rewrite the radial version of our problem as

$$\begin{cases} (r^{n-1}H(v'))' & = F(v,v') \ \text{ in } \ (0,\infty) \\ \qquad v(0) & = v_0 > 0 \\ \qquad v'(0) & = 0 \end{cases} \tag{3.17}$$

We consider the Banach space $X = C^1([0,r_1])$, for some $r_1 > 0$, with the associated norm

$$\|u\|_{C^1([0,r_1])} = \|u\|_{L^\infty([0,r_1])} + \|u'\|_{L^\infty([0,r_1])}$$

and $0 < r_1 < 1$ that will be chosen posteriosly. Let the integral operator

$$T: \ C^1([0,r_1]) \to C^1([0,r_1])$$

be defined as

$$Tv(r) = v_0 + \int_0^r H^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{n-1} F(v(t),v'(t))\,\mathrm{d}t\right) \mathrm{d}s. \tag{3.18}$$

Observe that solving (3.17) is equivalent to finding a fixed point of this integral operator (3.18). From now on we focus on showing the existence of such a fixed point.

Let $\{v_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $C^1([0,r_1])$,

$$\|v_k\|_{C^1([0,r_1])} \le M, \quad \forall k \in \mathbb{Z}.$$

By the monotonicity properties of f and g, for all $r \in [0,r_1]$

$$f(v_k(r)) \le f(M) := k_1, \qquad g(v_k'(r)) \le g(M) := k_2,$$

and $F(v,v') \le k_1 + k_2$.

Then, using that $0 \le r < r_1 < 1$ and the fact that H^{-1} is strictly increasing we have

$$\begin{aligned} Tv_k(r) &= v_0 + \int_0^r H^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{n-1} F(v_k(t), v_k'(t))\,\mathrm{d}t\right)\mathrm{d}s \\ &\le v_0 + \int_0^{r_1} H^{-1}\left((k_1+k_2)\int_0^s \left(\frac{t}{s}\right)^{n-1}\mathrm{d}t\right)\mathrm{d}s \\ &\le v_0 + r_1 H^{-1}\left((k_1+k_2)\int_0^{r_1}\mathrm{d}t\right) \\ &\le v_0 + H^{-1}\left(k_1+k_2\right). \end{aligned}$$

Consequently $Tv_k(r)$ is uniformly bounded in the sup norm. We also have

$$\begin{aligned} (Tv_k)'(r) &= H^{-1}\left(\int_0^r \left(\frac{t}{r}\right)^{n-1} F(v_k(t), v_k'(t))\,\mathrm{d}t\right) \\ &\le H^{-1}\left((k_1+k_2)\int_0^r \left(\frac{t}{r}\right)^{n-1}\mathrm{d}t\right) \\ &\le H^{-1}\left((k_1+k_2)\int_0^r \mathrm{d}t\right) \\ &\le H^{-1}\left((k_1+k_2)\,r\right) \qquad (3.19) \\ &\le H^{-1}\left(k_1+k_2\right), \end{aligned}$$

showing that $(Tv_k)'(r)$ is uniformly bounded in the sup norm. Then $Tv_k(r)$ is equicontinuous.

Since we deal with the Banach space $(C^1, \|\cdot\|_{C^1})$ we need to prove the equicontinuity also for $(Tv_k)'(r)$. Thus, we obtain (by denoting $\widetilde{F} := F(v_k, v_k')$)

$$(Tv_k)''(r) = \frac{1}{p-1}\left(\int_0^r \left(\frac{t}{r}\right)^{n-1}\widetilde{F}\,\mathrm{d}t\right)^{\frac{2-p}{p-1}}\left(\widetilde{F} + \frac{1-n}{r}\int_0^r \left(\frac{t}{r}\right)^{n-1}\widetilde{F}\,\mathrm{d}t\right).$$

Letting $\phi_k = (Tv_k)'$

$$|(Tv_k)''(r)| = \frac{1}{p-1}\left[|\phi_k(r)|^{2-p} F(v_k(r), v_k'(r)) + \frac{n-1}{r}|\phi_k(r)|\right]. \qquad (3.20)$$

To study the uniform boundedness of this derivative we need to pay attention to each of the two terms in this sum. Observe that in the first term there is an exponent $2-p$, which leads to separation in two cases, according to the degeneracy character of the operator. As for the second term, note that although ϕ_k is uniformly bounded conclusion is not inmediate because of the term $1/r$. Besides, from (3.19) we know that

$$\phi_k(r) = (Tv_k)'(r) \le H^{-1}(k_1+k_2)r^{\frac{1}{p-1}}$$

which implies that

$$\frac{(n-1)\phi_k(r)}{r} \leq (n-1)H^{-1}(k_1+k_2)r^{\frac{2-p}{p-1}}. \tag{3.21}$$

Thus, we also need to consider the values of p for the second term of the sum.

Case I: $1 < p \leq 2$.

If $1 < p \leq 2$, then $2-p \geq 0$. Since ϕ_k is uniformly bounded as we saw earlier, the first term on the right-hand side of (3.20) is also uniformly bounded. On the other hand by (3.21) the second term is uniformly bounded too. Then $(Tv_k)'$ is Lipschitz continuous uniformly with respect to k, and consequently equicontinuous.

Case II: $p > 2$.

If $p > 2$, then $2-p < 0$. Hence the Lipschitz continuity is lost and we can not conclude about the equicontinuity of $(Tv_k)'$ as above. We will prove that $(Tv_k)'$ is α-Hölder continuous with $\alpha = \dfrac{1}{p-1} < 1$, to obtain the equicontinuity.

Let $\lambda_k(r) = \displaystyle\int_0^r \left(\frac{t}{r}\right)^{n-1} F(v_k(t), v_k'(t))\,\mathrm{d}t$, then

$$(Tv_k)'(r) = \left(H^{-1}\circ\lambda_k\right)(r),$$

where, $H^{-1}(t) = t^{\frac{1}{p-1}}$, $t > 0$, is α- Hölder continuous with $\alpha = \dfrac{1}{p-1}$. On the other hand, we observe that $\lambda_k(r) \in C^2([0,r_1])$ with

$$\lim_{r\to 0^+}\lambda_k(r) = 0 \quad\text{and}\quad \lim_{r\to 0^+}\lambda_k'(r) = \frac{F(v_0,0)}{n} = \frac{f(v_0)}{n}.$$

Thus, using the mean value theorem we obtain that there exists $L > 0$ such that

$$|\lambda_k(r) - \lambda_k(\ell)| \leq L\,|r-\ell|\,. \tag{3.22}$$

This means that λ_k is locally Lipschitz continuous uniformly in k. Then, using the Hölder continuity of $H^{-1}(t)$ and (3.22)

$$\begin{aligned} |(Tv_k)'(r) - (Tv_k)'(\ell)| &= \left|(H^{-1}\circ\lambda_k)(r) - (H^{-1}\circ\lambda_k)(\ell)\right| \\ &\leq |\lambda_k(r)-\lambda_k(\ell)|^{\frac{1}{p-1}} \\ &\leq L^{\frac{1}{p-1}}\,|r-\ell|^{\frac{1}{p-1}}\,. \end{aligned}$$

Consequently $(Tv_k)'(r)$ is Hölder continuous with exponent $\frac{1}{p-1}$ uniformly in k, which implies that $Tv_k(r)$ is equicontinuous also for $p > 2$.

To prove the continuity of T we observe that, if $v_k \to v$ uniformly in $C^1([0,r_1])$, then by Lebesgue's dominated convergence theorem, for any sub-sequence $\left\{v_{k_j}\right\}$ of $\{v_k\}$ there is another sub-sequence (still denoted by v_{k_j}), such that $Tv_{k_j} \to Tv$ in $C^1([0,r_1])$. The continuity of the integral operator T follows.

Consequently, by the the Arzelá-Ascoli theorem we conclude that T is a compact and continuous operator.

At last we need to choose an appropriate closed, convex and bounded subset such that the operator T maps this set into itself. Given $0 < m_2 < v_0$, $m_3 > 0$, by the continuity of v and v' we can choose a sufficiently small constant $r_0 > 0$ such that

$$\max_{0\leq r\leq r_0} \{|v(r) - v_0|\} \leq m_2, \qquad \text{and} \qquad \max_{0\leq r\leq r_0} \{|v'(r)|\} \leq m_3.$$

Denote, in the case of (P_+)

$$r_\alpha := \max\left\{ (\frac{p}{p-1})^{\frac{p-1}{p}} \frac{n^{\frac{1}{p}} m_2^{\frac{p-1}{p}}}{(F(v_0+m_2,m_3))^{\frac{1}{p}}} \ , \ \frac{n\ m_3^{p-1}}{F(v_0+m_2,m_3)} \right\},$$

and in the case of (P_-)

$$r_\alpha := \max\left\{ (\frac{p}{p-1})^{\frac{p-1}{p}} \frac{n^{\frac{1}{p}}\ m_2^{\frac{p-1}{p}}}{(f(v_0+m_2))^{\frac{1}{p}}} \ , \ \frac{n\ m_3^{p-1}}{f(v_0+m_2)} \right\}.$$

Also denote a closed convex and bounded subset of the space $C^1([0,r_\alpha])$ by

$$B_\alpha = \left\{ \phi \in C^1([0,r_\alpha]);\ \ |\phi(s) - v_0| \leq m_2\ ,\ |\phi'(s)| \leq m_3 \quad \text{for all } 0 \leq s \leq r_\alpha \right\}. \tag{3.23}$$

We need to show that T maps B_α into B_α. For this purpose we are going to show that

$$|T\phi(r) - v_0| \leq m_2, \qquad \text{and} \qquad |(T\phi)'(r)| \leq m_3.$$

Since f and g are increasing we have in the case of (P_+) that

$$F(\phi(t),\phi'(t)) \leq F(v_0+m_2,m_3)$$

and in the case of (P_-) that

$$F(\phi(t),\phi'(t)) \leq F(v_0+m_2,0) = f(v_0+m_2).$$

Also, using the monotonicity of H^{-1} we have in the case of (P_+)

$$
\begin{aligned}
|T\phi(r) - v_0| &\le \int_0^r \left| H^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{n-1} F(\phi(t), \phi'(t))\,\mathrm{d}t \right) \right| \mathrm{d}s \\
&\le \int_0^r H^{-1}\left(F(v_0 + m_2, m_3)\frac{s}{n} \right) \mathrm{d}s \\
&= \frac{p-1}{p} \left(\frac{F(v_0 + m_2, m_3)}{n} \right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \le m_2.
\end{aligned}
$$

Analogously for the derivative

$$
\begin{aligned}
|(T\phi)'(r)| &\le \left| H^{-1}\left(\int_0^s \left(\frac{t}{s}\right)^{n-1} F(\phi(t), \phi'(t))\,\mathrm{d}t \right) \right| \\
&\le F(v_0 + m_2, m_3)^{\frac{1}{p-1}} H^{-1}\left(\int_0^r \left(\frac{t}{r}\right)^{n-1} \mathrm{d}t \right) \\
&= \left(\frac{F(v_0 + m_2, m_3)}{n} \right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \le m_3.
\end{aligned}
$$

In the same way for (P_-)

$$
|T\phi(r) - v_0| \le \frac{p-1}{p} \left(\frac{f(v_0 + m_2)}{n} \right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} \le m_2
$$

and for the derivative

$$
|T'\phi(r)| \le \left(\frac{f(v_0 + m_2)}{n} \right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \le m_3.
$$

Thus $T(B_\alpha) \subset B_\alpha$, this means that T is a continuous, compact mapping from B_α into B_α. By the Leray-Schauder theorem, the integral operator T has a fixed point $v(r) = Tv(r)$ in B_α. Therefore the problem has a radial solution $v \in C^1([0, r_\alpha])$.

On the other hand, by the definition of the operator we have that

$$
Tw(r) = u(r) \in C^2((0, r_\alpha)) \cap C([0, r_\alpha])
$$

for each $w \in C^1([0, r_\alpha])$, in particular, for the fixed point v. Consequently the radial solution of the problem is such that $v(r) \in C^2((0, r_\alpha)) \cap C([0, r_\alpha])$. Taking $r_1 < r_\alpha$ we obtain the desired result. ■

Note that the above nonlinear technique gives existence for the ODE only for some finite interval of r, or equivalently, existence for $(P_\pm)$ in some (possibly small) ball. As we are interested in the existence or not of non-negative solutions in the whole $\mathbb{R}^n$, a first candidate for the existence is exactly this radial solution, provided we are able to check that it actually exists globally.

In the next lemma we show explicitly, in case of (P_+), that under some hypotheses any radial solution of the problem exists on a maximal *finite* interval. So, under these hypotheses, the candidate fails. In the next section we are going to see that the opposite hypotheses (with respect to the lemma) are necessary for the existence.

Lemma 3.6 *Let f and g satisfy* (3.1)*. Then, the following statements hold.*

(i) Under the assumptions in (3.3) *(Theorem 1 (i)) any solution of*

$$\begin{cases} \left(r^{n-1}(v')^{p-1}\right)' & = r^{n-1}\left(f(v)+g(v')\right) \quad in \quad (0,\infty) \\ \qquad v(0) & = v_0 > 0 \\ \qquad v'(0) & = 0 \end{cases} \tag{3.24}$$

exists on a maximal interval $(0,R)$*, where* $0<R<\infty$ *and it is such that*

$$v'(r) \to \infty \ \ as \ \ r \to R.$$

(ii) Furthermore,

$$v(r) \xrightarrow[r\to R]{} \infty \iff \int_1^\infty \frac{s^{p-1}}{g(s)}\,\mathrm{d}s = \infty. \tag{3.25}$$

Remark 3.7 *Note that the integral condition in* (3.25) *and* (3.5) *are not exhaustive. In fact for $g(t)=t^q$ where $q>p$, neither of them is satisfied.*

Proof of item (i). Integrating from 0 to r the equation in (3.24), by the monotonicity of f and g and the a priori properties in Lemma 3.4 we obtain that

$$\begin{aligned} (v')^{p-1} &= \frac{1}{r^{n-1}}\int_0^r s^{n-1}\left(f(v(s))+g(v'(s))\right)\mathrm{d}s \\ &\leq \frac{r}{n}\left(f(v(r))+g(v'(r))\right). \end{aligned}$$

Then using the rewritten version (3.15) of (3.24) results in

$$\begin{aligned} \left((v')^{p-1}\right)' &\geq f(v)+g(v') - \frac{n-1}{n}\left(f(v(r))+g(v'(r))\right) \\ &= \frac{1}{n}\left(f(v(r))+g(v'(r))\right). \end{aligned}$$

thus $\left((v')^{p-1}\right)' \geq \frac{1}{n}f(v(r))$. Multiplying the last inequality by $\frac{p}{p-1}v' \geq 0$ we have

$$\frac{p}{p-1}\left((v')^{p-1}\right)' v' \geq \frac{p}{p-1}\frac{1}{n}\left(F(v)\right)',$$

then integrating from 0 to r

$$\begin{aligned}\int_0^r p(v')^{p-1}v''\,\mathrm{d}r &\geq \frac{p}{n(p-1)}\int_0^r f(v(r))v'(r)\,\mathrm{d}r\\ \int_0^r ((v')^p)'\,\mathrm{d}r &\geq \frac{p}{n(p-1)}(F(v(r))-F(v_0)).\end{aligned}$$

Thus

$$v' \geq \left(\frac{p}{n(p-1)}\right)^{\frac{1}{p}}(F(v)-F(v_0))^{\frac{1}{p}}.$$

Dividing by $(F(v)-F(v_0))^{\frac{1}{p}}$ and taking the integral from 0 to r, we obtain

$$\int_0^r \frac{v'(s)}{(F(v(r))-F(v_0))^{\frac{1}{p}}}\,\mathrm{d}s \geq \left(\frac{p}{n(p-1)}\right)^{\frac{1}{p}} r,$$

i.e,

$$\int_{v_0}^{v(r)} \frac{1}{(F(s)-F(v_0))^{\frac{1}{p}}}\,\mathrm{d}s \geq \left(\frac{p}{n(p-1)}\right)^{\frac{1}{p}} r. \tag{3.26}$$

Analogously,

$$\left((v')^{p-1}\right)' \geq \frac{1}{n}g(v'(r)). \tag{3.27}$$

Dividing by $g(v')$ and integrating from some $r_0>0$ to $r>r_0$, we obtain

$$\begin{aligned}\frac{\left((v')^{p-1}\right)'}{g(v'(r))} &\geq \frac{1}{n}\\ \int_{r_0}^r \frac{\left((v')^{p-1}\right)'}{g(v'(r))}\,\mathrm{d}r &\geq \frac{1}{n}(r-r_0),\end{aligned}$$

then

$$\int_{v'(r_0)}^{v'(r)} \frac{s^{p-2}}{g(s)}\,\mathrm{d}s \geq \frac{1}{n}(r-r_0). \tag{3.28}$$

Since $F(v_0)$ is a constant it does not affect the convergence of the integral in the inequality (3.26). Using the assumptions (3.3) of the theorem we see that at least one of the integrals on the left hand side of (3.26) and (3.28) stays bounded independently of r. Thus r is bounded above, consequently R is finite.

Observe that the fact that the maximal interval is finite can be due to $v(r)\to\infty$ or $v'(r)\to\infty$ when r tends to R. Since v , v' are increasing and $v(r)\leq v_0+Rv'(r)$ (as we showed in Lemma 3.4) we conclude that $v'(r)\to\infty$ as r tends to R. ■

Proof of item (ii). Suppose that the integral in the right-hand side is finite. Then, multiplying (3.27) by v' and dividing by $g(v')$ we obtain

$$(p-1)(v')^{p-1}\frac{v''}{g(v')} \geq \frac{1}{n}v'$$

and integrating from any r_0 to r such that $0 < r_0 < r < R$

$$v(r) - v(r_0) \leq n(p-1) \int_{v'(r_0)}^{v'(r)} \frac{s^{p-1}}{g(s)} \, \mathrm{d}s < \infty$$

as $r \to R$. This means that $v(r)$ is bounded.

On the other hand, if we suppose that exist a finite constant $C > 0$ such that $v(r) \leq C$ when $r \to R$, then in $(0, R)$

$$((v')^{p-1})' \leq ((v')^{p-1})' + \frac{n-1}{r}(v')^{p-1} \leq f(C) + g(v').$$

Multiplying this inequality by v' and dividing by $g(v')$ we obtain, after integration from any r_0 to r such that $0 < r_0 < r < R$,

$$\int_{v'(r_0)}^{v'(r)} \frac{s^{p-1}}{g(s)} \, \mathrm{d}s \leq c(v(r) - v(r_0)).$$

Letting $r \to R$, since we know that $v'(r) \to \infty$ when $r \to R$, we obtain that the right-hand side of (3.25) is finite, as is desired. ■

3.4
Global existence and nonexistence of entire solutions

We prove Theorem 1 which ensures the existence or not of positive solutions of the equation

$$\Delta_p u = f(u) + g(|\nabla u|). \tag{3.29}$$

The first item is relative to the non existence of positive weak solutions. Comparing the solution of (3.11) with eventual solutions of our equation (3.29) will lead us to the nonexistence result.

Proof of Theorem 1 (i). Suppose by contradiction that the conclusion is false. That is, we assert that there exists $w \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $w \geq 0$, $w \neq 0$, satisfies

$$\Delta_p w \geq f(w) + g(|\nabla w|).$$

We can suppose, without loss of generality, that $w(0) > 0$. In fact, since $w \neq 0$, there exists a x_0 such that $w(x_0) > 0$. Then we can take $\widetilde{w}(x) = w(x + x_0)$ instead of w.

Let $v(r) = v(|x|) = u(x)$ be a radial solution of (P_+) defined in a maximal interval $(0, R)$ with $0 < R < +\infty$ which we obtained in Lemma 3.5 and Lemma 3.6, by setting $v_0 = \dfrac{w(0)}{2}$ in Lemma 3.6.

First we affirm that v is bounded when $r \to R$. Indeed, if $v(r) \to \infty$ as

$r \to R$ we can take $\epsilon > 0$ sufficiently small so that $u > w$ in $\partial B_{R-\epsilon}(0)$. Then, by using the comparison principle, we have that $u \geq w$ in $B_{R-\epsilon}(0)$ which contradicts $v_0 < w(0)$.

Now by applying the comparison principle once again in the whole ball $B_R(0)$ we obtain the existence of $\bar{x} \in \partial B_R(0)$ such that $w(\bar{x}) > u(\bar{x})$. Then we can take $a > 0$ such that the function $u_a = u + a$ satisfies $w(x) \leq u_a(x)$ for all $x \in \partial B_R(0)$ and $w(x_0) = u_a(x_0)$ for some $x_0 \in \partial B_R(0)$. Specifically, $a = \sup_{x \in \partial B_R(0)}(w(x) - u(x))$. We see that

$$\Delta_p u_a \leq f(u_a) + g(\nabla u_a) \quad \text{ in } B_R(0)$$

by the monotonicity of f. Then by the comparison principle we can infer

$$w(x) \leq u_a(x) \; \forall \; x \in B_R(0).$$

Since $w(x_0) = u_a(x_0)$, this obviously implies

$$\frac{u_a(x_0 + t\nu) - u_a(x_0)}{t} \geq \frac{w(x_0 + t\nu) - w(x_0)}{t},$$

for all small $t > 0$, where $\nu = -x_0/|x_0|$ is the interior normal to the boundary of the ball at x_0. When $t \to 0$, the left-hand side of this inequality tends to $-\infty$, by Lemma 3.6 (i). Hence, the right-hand side is unbounded, but this contradicts $w \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. Thus we can conclude that the only nonnegative sub-solution is the trivial one. ■

Remark 3.8 *We recall that we adopt the usual definition of solution for the p-Laplacian, in the weak-Sobolev sense. If we suppose we have a weak solution in the sense of viscosity, which is less common, the statement in the above proved theorem is also satisfied.*

Let us now give the detailed explanation of the statement given in Remark 3.1.

In the above proof of the Theorem 1 (i) we make fundamental use of the comparison principle. However, to our knowledge, for our type of equations with dependence on the gradient, the comparison principle is available only for functions in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. This forces us to make to assume our solutions are in this class. On the other hand, if a comparison principle in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ were available, then we can affirm the same non-existence result as above for sub-solutions in the Sobolev space $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, by imposing the additional integral condition

$$\int_1^\infty \frac{s^{2(p-1)}}{g(s)}\, \mathrm{d}s = \infty. \tag{3.30}$$

In fact, in this case we can repeat the previous proof until the comparison between $w(x)$ and $u_a(x)$ in $B_R(0)$.

We will now show that under (3.30) the radial function u in the previous proof is not in $W^{1,p}$ in a neighborhood of the boundary of the ball, so again by comparison, the same must be true for w, which is a contradiction.

Observe that if we multiply (P_+), i.e. the equation in (3.15), by $\dfrac{(v')^p}{g(v')}$ we have

$$\begin{aligned}
(v')^p &= \frac{\left((v')^{p-1}\right)'(v')^p}{g(v')} + \frac{n-1}{r}(v')^{p-1}\frac{(v')^p}{g(v')} - \frac{f(v)}{g(v')}(v')^p \\
&= (p-1)\frac{(v')^{2(p-1)}}{g(v')}v'' + \frac{n-1}{r}\frac{(v')^{2p-1}}{g(v')} - \frac{f(v)}{g(v')}(v')^p.
\end{aligned}$$

Now integrating from some $r_0 > 0$ to r, with $0 < r_0 < r < R$, it results

$$\begin{aligned}
\int_{r_0}^{r} (v')^p \, \mathrm{d}r = (p-1)\int_{r_0}^{r} &\frac{(v')^{2(p-1)}}{g(v')}v'' \, \mathrm{d}r + \\
&+ (n-1)\int_{r_0}^{r} \frac{(v')^{2p-1}}{rg(v')} \, \mathrm{d}r - \int_{r_0}^{r} \frac{f(v)}{g(v')}(v')^p \, \mathrm{d}r. \quad (3.31)
\end{aligned}$$

Since $v'(r) \to \infty$ as $r \to R$, taking $r \to R$ in (3.31) and substituting $s = v'$ in the first term of right-hand side gives

$$\begin{aligned}
\int_{r_0}^{R} (v')^p \, \mathrm{d}r = (p-1)\int_{v'(r_0)}^{\infty} &\frac{(s)^{2(p-1)}}{g(s)} \, \mathrm{d}s + \\
&+ (n-1)\int_{r_0}^{R} \frac{(v')^{2p-1}}{rg(v')} \, \mathrm{d}r - \int_{r_0}^{R} \frac{f(v)}{g(v')}(v')^p \, \mathrm{d}r. \quad (3.32)
\end{aligned}$$

Observe that by the monotonicity of f and g and since v is bounded when $r \to R$ in (r_0, R)

$$\begin{aligned}
f(v(r)) &\leq f(v(R)) \leq f(C) = C_0 \\
g(v'(r)) &\geq g(v'(r_0)) > 0,
\end{aligned}$$

then

$$\int_{r_0}^{R} \frac{f(v)}{g(v')}(v')^p \, \mathrm{d}r \leq \frac{C_0}{g(v'(r_0))}\int_{r_0}^{R} (v')^p \, \mathrm{d}r.$$

Substituting this in the equation (3.32) we obtain that

$$\int_{r_0}^{R} (v')^p \,\mathrm{d}r \geq (p-1)\int_{v'(r_0)}^{\infty} \frac{s^{2(p-1)}}{g(s)}\,\mathrm{d}s + (n-1)\int_{r_0}^{R} \frac{(v')^{2p-1}}{rg(v')}\,\mathrm{d}r - \frac{C_0}{g(v'(r_0))}\int_{r_0}^{R} (v')^p \,\mathrm{d}r. \quad (3.33)$$

Then,

$$\begin{aligned}\int_{r_0}^{R} (v')^p \,\mathrm{d}r &\geq \frac{1}{1+\frac{C_0}{g(v'(r_0))}}\left((p-1)\int_{v'(r_0)}^{\infty} \frac{s^{2(p-1)}}{g(s)}\,\mathrm{d}s + (n-1)\int_{r_0}^{R} \frac{(v')^{2p-1}}{rg(v')}\,\mathrm{d}r\right)\\ &\geq \frac{(p-1)}{1+\frac{C_0}{g(v'(r_0))}}\int_{v'(r_0)}^{\infty} \frac{s^{2(p-1)}}{g(s)}\,\mathrm{d}s.\end{aligned}$$

By the additional integral condition (3.30) we obtain that the integral on the right-hand side in the last inequality diverges, so

$$\int_{r_0}^{R} (v')^p \,\mathrm{d}r = \infty, \quad (3.34)$$

thus proving the statement in Remark 3.1.

Remark 3.9 *Observe that the statement* (3.25) *in Lemma 3.6 is not used in the above nonexistence proof. We included it in Lemma 3.6 in order to show that the radial solution does not always explode when r tends to R. This is because of the dependence on the right-hand side of the gradient. However, if g is such that $\int_1^{\infty} \frac{s^{p-1}}{g(s)}\,\mathrm{d}s = \infty$, so v actually explodes on the boundary of the ball, we obtain a much simpler proof of the nonexistence, since already the first comparison argument in the proof of Theorem 1 (i) gives a contradiction.*

We summarize the sufficient integral conditions that yield qualitative properties on v and v' for radial equations with dependence on the gradient:

Result in Text	*Integral Condition*	*Properties of v and v'*
Not (KO) (3.5)	$\int^{+\infty} \frac{s^{p-2}}{g(s)}\,\mathrm{d}s < +\infty$	$v'(r) \xrightarrow[r\to R]{} \infty$
Lemma 3.6(ii)	$\int^{+\infty} \frac{s^{p-1}}{g(s)}\,\mathrm{d}s < +\infty$	$v'(r) \xrightarrow[r\to R]{} \infty,$ *and* $v(r) \leq M$ *with* $0 < M < \infty$
Condition (3.30)	$\int^{+\infty} \frac{s^{2(p-1)}}{g(s)}\,\mathrm{d}s = +\infty$	$v'(r) \notin L^p(0, R)$

We now continue with the proof of the existence result in Theorem 1.

Proof of Theorem 1 (ii). We want to prove that we have a positive solution in the whole $\mathbb{R}^n$. For this purpose we are going to prove that v, the radial classical solution that we already proved to exist in a neighborhood of zero, is defined for all $r > 0$. Suppose by contradiction that the solution exists in a finite maximal interval $(0, R)$, where $R < \infty$ is finite.

First, we affirm that $v(r) \to \infty$ and $v'(r) \to \infty$ as $r \to R$.

In fact suppose that $v(r)$ is bounded as $r \to R$. By the continuity and monotonicity of v and f

$$((v')^{p-1})' \leq f(C) + g(v').$$

Dividing by $g(v')$ and integrating from r_0 to r with $0 < r_0 < r < R$ we obtain

$$(p-1)\int_{v'(r_0)}^{v'(r)} \frac{s^{p-2}}{g(s)}\,\mathrm{d}s \leq C(r - r_0)$$

and letting $r \to R$,

$$(p-1)\int_{v'(r_0)}^{\infty} \frac{s^{p-2}}{g(s)}\,\mathrm{d}s \leq C(R - r_0),$$

which is a contradiction with our condition (3.5). The conclusion about v' follows from Lemma 3.4 in the same way as in the proof of Lemma 3.6.

Now let us define

$$A(r) := \frac{r^{n-1}v'}{(F(v(r)))^{\frac{1}{p}}} \tag{3.35}$$

and separate the proof into several cases, according to the asymptotic behavior of $A(r)$ as $r \to R$.

Case I: Suppose that $A(r)$ is bounded when $r \to R$.

Then taking the integral between $R/2$ and any $r \in (R/2, R)$, we obtain

$$\left(\frac{R}{2}\right)^{n-1} \int_{R/2}^{r} \frac{v'}{F(v(s))^{\frac{1}{p}}} \,\mathrm{d}s \leq \int_{R/2}^{r} s^{n-1} \frac{v'}{F(v(s))^{\frac{1}{p}}} \,\mathrm{d}s = \int_{R/2}^{r} A(s) \,\mathrm{d}s \leq c(r - R/2).$$

Letting $r \to R$ we obtain that the term to the right is bounded. This is a contradiction with the hypothesis of the condition (3.4) since

$$\left(\frac{R}{2}\right)^{n-1} \int_{R/2}^{r} \frac{v'}{F(v(s))^{\frac{1}{p}}} \,\mathrm{d}s = \left(\frac{R}{2}\right)^{n-1} \int_{v(R/2)}^{v(r)} \frac{1}{F(s)^{\frac{1}{p}}} \,\mathrm{d}s.$$

.

Case II: Suppose that $A(r) \to \infty$ when $r \to R$.

Let $w = (v')^{p-1}$ and $H = F(v)^{\frac{1}{p}}$, thus the assumption of this case can be written as

$$\frac{H}{w^{\frac{1}{p-1}}} \xrightarrow[r \to R]{} 0 \tag{3.36}$$

and the problem with the specific equation as in (3.15) can be now rewritten as

$$\begin{cases} w' + \frac{n-1}{r} w & = p \frac{H^{p-1} H'}{w^{\frac{1}{p-1}}} + g\left(w^{\frac{1}{p-1}}\right) \\ \qquad w(0) & = 0. \end{cases} \tag{3.37}$$

By the properties shown in Lemma 3.4, we have that $w \geq 0$, which implies that

$$w' \leq p \frac{H^{p-1} H'}{w^{\frac{1}{p-1}}} + g\left(w^{\frac{1}{p-1}}\right). \tag{3.38}$$

On the other hand, we can fix $r_0 > 0$ such that $w \geq 1$ in (r_0, R), by the convergence to infinity of w when r tends to R. Let $S = S(t)$ be the solution of the initial value problem

$$\begin{cases} S'(t) & = (p-1) g\left(S^{\frac{1}{p-1}}(t)\right) \\ S(1) & = 1, \end{cases}$$

defined implicitly by

$$t = 1 + \int_1^{S(t)} \frac{\sigma^{p-2}}{g(\sigma)} \,\mathrm{d}\sigma, \qquad t \in [1, \infty).$$

From the hypothesis of g and the condition (3.5), we deduce that S is bijective from $[1, \infty)$ to $[1, \infty)$. Then, taking the inverse of this function we get

$$\left(S^{-1}(w)\right)' = \frac{w'}{S'\left(S^{-1}(w)\right)} = \frac{w'}{(p-1) g(w^{\frac{1}{p-1}})}.$$

Then, by (3.38) we have

$$\left(S^{-1}\left(w\right)\right)' \leq \frac{1}{p-1} + \frac{p}{p-1}\frac{H^{p-1}H'}{w^{\frac{1}{p-1}}g(w^{\frac{1}{p-1}})}. \tag{3.39}$$

On the other hand,

$$\left(S^{-1}\left(H^{p-1}\right)\right)' = \frac{H'H^{p-2}}{g\left(H\right)}.$$

By substituting this in (3.39), we get

$$\begin{aligned}\left(S^{-1}\left(w\right)\right)' &\leq \frac{1}{p-1} + \frac{p}{p-1}\frac{H^{p-1}}{w^{\frac{1}{p-1}}}\frac{g\left(H\right)}{g(w^{\frac{1}{p-1}})H^{p-2}}\left(S^{-1}\left(H\right)\right)' &(3.40)\\ &= \frac{1}{p-1} + \frac{p}{p-1}\frac{H}{w^{\frac{1}{p-1}}}\frac{g\left(H\right)}{g(w^{\frac{1}{p-1}})}\left(S^{-1}\left(H\right)\right)'. &(3.41)\end{aligned}$$

We also know by the monotonicity of f that H is increasing, hence we can choose r_1 such that $H \geq 1$ in (r_1, R).

Since (3.36) is satisfied and g is monotone, we can say that exist $r_2 > r_1$ such that

$$\frac{p}{p-1}\frac{H}{w^{\frac{1}{p-1}}}\frac{g\left(H\right)}{g(w^{\frac{1}{p-1}})} \leq \frac{1}{p}$$

in (r_2, R). Then

$$(S^{-1}(w))' \leq 1 + \frac{1}{p}(S^{-1}(H))' \text{ in } (r_2, R).$$

Integrating from r_1 to $r \in (r_2, R)$

$$\begin{aligned}(S^{-1}(w))' &\leq S^{-1}(w(r_1)) + \frac{1}{p-1}(R-r_1) + \frac{1}{p}S^{-1}(H^{p-1}(r))\\ &- \frac{1}{p}S^{-1}(H^{p-1}(r_2)) + \int_{r_1}^{r_2}\frac{p}{p-1}\frac{H}{w^{\frac{1}{p-1}}}\frac{g(H)}{g(w^{\frac{1}{p-1}})}S^{-1}(H^{p-1})\\ &= C(r_1, r_2, p, R) + \frac{1}{p}S^{-1}(H^{p-1}(r))\end{aligned}$$

where $C(r_1, r_2, p, R)$ is a constant independent of r.

Since F is increasing we have that $H(r) \to \infty$ as $r \to R$. Hence, by the definition of S and the condition (3.5), $S^{-1}(H(r)) \to \infty$ as $r \to R$. Then we find an $r_3 \in (r_1, R)$ such that

$$S^{-1}(w) < S^{-1}(H^{p-1}) \text{ in } (r_3, R).$$

Since S^{-1} is increasing this means that $w < H$ in (r_3, R). But this is a contradiction with the fact of $\frac{H}{w^{\frac{1}{p-1}}} \to 0$ as $r \to R$.

Case III: Assume that we are in none of the above cases, i.e. $A(r)$, for

$r < R$, is neither bounded nor tends to infinity as $r \to R$.

This means that

$$\limsup_{r\to R} A(r) = +\infty \quad \text{and} \quad A_0 := \liminf_{r\to R} A(r) < +\infty.$$

Then, for each $\hat{A} > A_0$ we can find sequences $s_n, t_n \to R$ such that

$$A(s_n) = A(t_n) = \hat{A}, \quad A'(s_n) \geq 0, \quad A'(t_n) \leq 0.$$

The derivative of A is

$$\begin{aligned} A'(r) &= \frac{(r^{n-1}v')'F^{\frac{1}{p}}(v) - (r^{n-1}v')(F^{\frac{1}{p}}(v))'}{F^{\frac{2}{p}}(v)} \\ &= \frac{(r^{n-1}v')'}{F^{\frac{1}{p}}(v)} - \frac{r^{n-1}v'}{F^{\frac{2}{p}}(v)}\left(\frac{1}{p}F^{\frac{1}{p}-1}(v)f(v)v'\right) \\ &= \frac{(r^{n-1}v')'}{F^{\frac{1}{p}}(v)} - \frac{r^{n-1}(v')^2 f(v)}{pF^{\frac{1}{p}+1}(v)}. \end{aligned} \tag{3.42}$$

Since $v' \geq 0$, multiplying (3.42) by $(v')^{p-2}$

$$A'(r)(v')^{p-2} = \frac{(r^{n-1}v')'(v')^{p-2}}{F^{\frac{1}{p}}(v)} - \frac{r^{n-1}(v')^p f(v)}{pF^{\frac{1}{p}+1}(v)}.$$

Adding and subtracting the factor $(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}$ on the right-hand side of the previous equation gives

$$
\begin{aligned}
&A'(r)(v')^{p-2}\\
=&\frac{(r^{n-1}v')'(v')^{p-2}}{F^{\frac{1}{p}}(v)}+(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{(r^{n-1})'(v')^{p-1}+r^{n-1}(v')^{p-2}v''}{F^{\frac{1}{p}}(v)}+(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}\\
&\qquad\qquad-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{(r^{n-1})'(v')^{p-1}+(p-1)r^{n-1}(v')^{p-2}v''}{F^{\frac{1}{p}}(v)}-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{(r^{n-1})'(v')^{p-1}+r^{n-1}((v')^{p-1})'}{F^{\frac{1}{p}}(v)}-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{(r^{n-1}(v')^{p-1})'}{F^{\frac{1}{p}}(v)}-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{r^{n-1}(f(v)+g(v'))}{F^{\frac{1}{p}}(v)}-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}-\frac{r^{n-1}(v')^{p}f(v)}{pF^{\frac{1}{p}+1}(v)}\\
=&\frac{r^{n-1}f(v)}{F^{\frac{1}{p}}(v)}\left(1+\frac{g(v')}{f(v)}-\frac{1}{p}\frac{(v')^{p}}{F(v)}\right)-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}.
\end{aligned}
$$

Let $W(r):=1+\frac{g(v')}{f(v)}-\frac{1}{p}\frac{(v')^{p}}{F(v)}$, then,

$$A'(r)(v')^{p-2}=\frac{r^{n-1}f(v)}{F^{\frac{1}{p}}(v)}W(r)-(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}.$$

Now we deal with the cases $p\leq 2$ and $p\geq 2$ separately.

Case 1 $\quad p\leq 2$

When $p\leq 2$, it is clear that $(p-2)r^{n-1}\frac{(v')^{p-2}v''}{F^{\frac{1}{p}}}\leq 0$, thus

$$A'(r)(v')^{p-2}\geq\frac{r^{n-1}f(v)}{F^{\frac{1}{p}}(v)}W(r).$$

Setting $\bar{A}(r)=\frac{A(r)}{r^{n-1}}$, we have

$$
\begin{aligned}
W(r)&=1+\frac{g(\frac{A(r)}{r^{n-1}}F^{\frac{1}{p}})}{f(v)}-\frac{1}{p}\frac{A^{p}(r)}{r^{(n-1)p}}\\
&=1+\frac{g(\bar{A}(r)F^{\frac{1}{p}})}{f(v)}-\frac{1}{p}\bar{A}^{p}\\
&=1+\bar{A}^{p}(r)\left(\frac{1}{\bar{A}^{p}(r)}\frac{g(\bar{A}(r)F^{\frac{1}{p}})}{f(v)}-\frac{1}{p}\right).
\end{aligned}
$$

Note that $\bar{A}(t_n) \to \tilde{A} := \hat{A}R^{1-N}$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that $\bar{A}(t_n) \in (\tilde{A} - \epsilon, \tilde{A} + \epsilon)$. Then, by the monotonicity of g,

$$W(t_n) \geq 1 + \left(\bar{A}(t_n)\right)^p \left(\rho(\epsilon)^p \frac{g\left((\tilde{A}(r) - \epsilon)(F(v))^{\frac{1}{p}}\right)}{\left(\tilde{A} - \epsilon\right)^p f(v)} - \frac{1}{p}\right), \tag{3.43}$$

with $\rho(\epsilon) := \frac{\tilde{A}-\epsilon}{\tilde{A}+\epsilon} \to 1$ as $\epsilon \to 0$. Fix $\epsilon > 0$ sufficiently small and $\hat{A}$ large such that (3.6) be satisfied; then $W(t_n) > 0$ for n large. Consequently $A'(t_n) > 0$, a contradiction.

Case 2 $\quad p > 2$

If $p \geq 2$ then $(p-2)r^{n-1}\dfrac{(v')^{p-2}v''}{F^{\frac{1}{p}}} \geq 0$. Thus,

$$A'(r)(v')^{p-2} \leq \frac{r^{n-1}f(v)}{F^{\frac{1}{p}}(v)} W(r).$$

Then, analogously to the other case, since $\bar{A}(s_n) \to \tilde{A} := \hat{A}R^{1-N}$ for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\bar{A}(s_n) \in (\tilde{A} - \epsilon, \tilde{A} + \epsilon)$. Using the monotonicity of g

$$W(s_n) \leq 1 + \left(\bar{A}(s_n)\right)^p \left(\rho(\epsilon)^p \frac{g\left((\tilde{A}(r) + \epsilon)(F(v))^{\frac{1}{p}}\right)}{\left(\tilde{A} + \epsilon\right)^p f(v)} - \frac{1}{p}\right).$$

Fixing $\epsilon > 0$ sufficiently small and $\hat{A}$ large such that (3.7) be satisfied, then $W(s_n) < 0$. This is a contradiction with the established sign for $A'(s_n)$. This means that the third case for A is impossible too. Therefore, the radial classical solution is defined in whole $\mathbb{R}^n$. ∎

Remark 3.10 *Note that the limit conditions in* (3.6) *and* (3.7) *are different according to the value of p. Thus, for example, from this theorem we cannot conclude about the existence for the standard functions $f(t) = g(t) = t^q$ where $q \leq p - 1$ if $p < 2$, as we mentioned earlier.*

However, as we observed in Remark 3.2, conditions (3.6) *and* (3.7) *can be removed in the particular case when g has growth s^{p-1} at most, i.e*

$$\limsup_{s \to \infty} \frac{g(s)}{s^{p-1}} < +\infty. \tag{3.44}$$

In fact, by (3.37) *we have*

$$w' w^{\frac{1}{p-1}} - g(w^{\frac{1}{p-1}}) w^{\frac{1}{p-1}} \leq (H^p)'.$$

Now using the growth condition (3.44) *we obtain that*

$$w'w^{\frac{1}{p-1}} - cw^{\frac{p}{p-1}} \leq (H^p)'.$$

Letting $\widetilde{w} = w^{\frac{p}{p-1}}$, $\widetilde{H} = H^p$ *and* $c_1 = c\frac{p}{p-1}$ *the above inequality can be rewritten as*

$$\widetilde{w}' - cc_1\widetilde{w} \leq c_1\widetilde{H}'.$$

Consequently,

$$\begin{aligned} \widetilde{w} &\leq e^{cc_1}\int_{r_0}^{r} e^{-cc_1 s}c_1\widetilde{H}'(s)\,\mathrm{d}s \\ &\leq c_1Re^{cc_1}\widetilde{H}(s) \end{aligned}$$

for some $0 < r_0 < r$, *which implies that* $\dfrac{\widetilde{w}}{\widetilde{H}} \leq C(R)$, *that is* $\dfrac{w^{\frac{1}{p-1}}}{H} \leq C(p,R)$. *Thus we only have the first case in the above proof, in which no hypothesis other than the (KO) integral conditions is used.*

To focus on the important case of the power nonlinearity $g(t) = t^q$, $q > 0$, we have the following result.

Corollary 1 *Let* f *that satisfy* (3.1). *If* $q > p-1$, *the only non-negative weak solution in* $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ *of*

$$\Delta_p u = f(u) + |\nabla u|^q \quad \textit{in } \mathbb{R}^n \tag{3.45}$$

is the trivial one.

We study the solutions of

$$\Delta_p u = f(u) - g(|\nabla u|). \tag{P_-}$$

To prove the main result, Theorem 2 (i), first we prove the following lemma.

Lemma 3.11 *There exists an increasing smooth strict super-solution* $\bar{v}$ *of*

$$\begin{cases} ((v')^{p-1})' + \frac{n-1}{r}(v')^{p-1} & \leq f(v) - g(v') \\ \qquad\qquad\qquad\qquad v(0) & = \bar{v}_0 \\ \qquad\qquad\qquad\qquad v'(0) & = 0. \end{cases} \tag{3.46}$$

which ceases to exist at a finite R, *and satisfies* $\bar{v}(r) \to \infty$ *when* $r \to R$, *for* $\bar{v}_0$ *large enough.*

Proof. Suppose that $R \leq (\frac{p-1}{p})^{p-1}$, we search for a super-solution in the form

$$\bar{v}(r) = \phi(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}})$$

where ϕ is a decreasing function that is defined implicitly by

$$\int_{\phi(t)}^{\infty} \frac{\mathrm{d}s}{\Gamma^{-1}(F(s))} = t,$$

which is a super-solution of (3.46) if

$$\left(\frac{p}{p-1}\right)^p (p-1)r^{\frac{p}{p-1}} |\phi'|^{p-2} \phi'' + \left(\frac{p}{p-1}\right)^{p-1} n |\phi'|^{p-1} \leq f(\phi) - g\left(\frac{p}{p-1} r^{\frac{1}{p-1}} |\phi'|\right) \tag{3.47}$$

where the comma means the derivative with respect to $t = R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}$. Using the conditions (3.1) and by the hypothesis (3.8) ϕ is well defined and it can be verified that $\phi(t) \to \infty$ as $t \to 0$, $t > 0$; $\phi'(t) < 0$ and $\Gamma(|\phi'|) = F(\phi)$. Then, we can use the inequality

$$|\phi'|^{p-2} \phi'' + (\frac{p}{p-1})^{p-1} n |\phi'|^{p-1} + g(|\phi'|) \leq f(\phi) \tag{3.48}$$

instead of (3.47) to verify the character of super-solution of ϕ. Observe that

$$\frac{\Gamma^{-1}(F(s))}{s} \to \infty \quad \text{as} \quad s \to \infty, \tag{3.49}$$

hence

$$\frac{|\phi'(t)|}{\phi(t)} \to \infty \quad \text{as} \quad t \to 0.$$

Thus, we can affirm that exists $\epsilon > 0$ such that $\phi(t) \leq \frac{1}{2} |\phi'(t)|$ if $t \in (0, \epsilon)$. Now we take R such that $t \in (0, \epsilon)$ for $r \in (0, R)$.

Finally we check that ϕ satisfies (3.48), so is the desired function. By differentiating with respect to t in (3.49) we obtain

$$\phi'' = \frac{f(\phi) |\phi'|}{2g(2 |\phi'|) + (p-1)c(p, N) |\phi'|^{p-1}}. \tag{3.50}$$

We can see that $\phi'' > 0$ and

$$|\phi'|^{(p-2)} \phi'' \leq \frac{f(\phi)}{(p-1)c(p, N)}. \tag{3.51}$$

On the other hand, we have

$$F(t) = \int_0^t f(s) \, \mathrm{d}s \leq f(t)t$$

and

$$\Gamma(t) = \int_0^{2t} g(s) \, \mathrm{d}s + \frac{p-1}{p} c(p, N) t^p \geq t g(t) + \frac{p-1}{p} c(p, N) t^p$$

thus, by taking $c(p, N) = \left(\frac{p}{p-1}\right)^p n$, we obtain

$$g(|\phi'|) + \left(\frac{p}{p-1}\right)^{p-1} n\,|\phi'|^{p-1} \leq \frac{\Gamma(|\phi'|)}{|\phi'|} = \frac{F(\phi)}{|\phi'|} \leq f(\phi)\frac{\phi}{|\phi'|} \leq \frac{1}{2}f(\phi). \quad (3.52)$$

By adding (3.51) and (3.52), we get the expression (3.48) and conclude. ■

Using this result we can prove the Theorem 2.

Proof of Theorem 2 (i). Suppose by contradiction that we have a non trivial solution w of (P_-), defined in the whole $\mathbb{R}^n$. Let $\tilde{v}$ be a solution of (3.46) with $\tilde{v}_0 = \frac{w(0)}{2}$ – we know such a solution exists by Lemma 3.5.

Now we show that $\tilde{v}$ is defined in the whole $\mathbb{R}^n$. By Lemma 3.4 we know that $\tilde{v}' > 0$ and $((\tilde{v}')^{p-1})' \geq 0$. Then

$$\tilde{v}' \leq \frac{r}{n-1} f(\tilde{v}).$$

Thus if $\tilde{v}$ exists in some maximal interval $(0, \bar{R})$, with $\bar{R} < \infty$, and becomes infinite at $\bar{R}$, we have

$$w(x) < \tilde{v}(|x|) \quad \text{in} \quad \partial B_{r_0}(0),$$

for $0 < r_0 < \bar{R}$ sufficiently near to $\bar{R}$. Using the comparison principle in $B_{r_0}(0)$, we obtain that

$$w(x) < \tilde{v}(|x|) \quad \text{in} \quad B_{r_0}(0).$$

This is a contradiction with the initial hypothesis that $\tilde{v}(0) = \tilde{v}_0 < w(0)$.

Therefore $R = \infty$ and $\tilde{v}(r) \to \infty$ as $r \to \infty$, since $\tilde{v}$ is increasing and $((\tilde{v'})^{p-1})' \geq 0$. By using again the comparison principle we have

$$w(x) \not\leq \tilde{v}(|x|) \quad \text{in} \quad \partial B_r(0),$$

for all $r \in (0, \infty)$. This implies that there exists a sequence $x_n \in \mathbb{R}^n$ with $|x_n| \to \infty$ such that $w(x_n) \to \infty$. Fixing n_0 large enough we obtain $w(x_{n_0}) > \bar{v}_0$, where $\bar{v}_0$ is the number obtained in Lemma 3.11.

Repeating the above argument for $\bar{v}(|x - x_{n_0}|)$ instead of $\tilde{v}(|x|)$, with $\bar{v}$ the function of Lemma 3.11, we obtain a contradiction. ■

Remark 3.12 *The integral condition* (3.8) *is sharp – we can compute that*

$$\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) \geq (p-1)\,|u|^{p-2}\,u - |\nabla u|^p$$

has the non-negative nontrivial solution $u = \exp x_1$.

Proof of Theorem 2 (ii). Let v be a solution of

$$
\begin{cases}
((v')^{p-1})' + \frac{n-1}{r}(v')^{p-1} & = f(v) - g(v') \\
v(0) & = v_0 > 0 \\
v'(0) & = 0
\end{cases}
\tag{3.53}
$$

in a maximal interval $(0, R)$, $\quad 0 < R < \infty$. By the results in Lemma 3.4 we have that $f(v) - g(v') \geq 0$, hence

$$
g^{-1}(f(v)) \geq v'. \tag{3.54}
$$

Also by the properties of v we know that if v is defined in a maximal interval $(0, R)$ with $R < \infty$, $v(r) \to \infty$ as $r \to R$ and since v satisfy (3.53)

$$
((v')^{p-1})' \leq f(v)
$$

from which we get

$$
v' \leq (\frac{p}{p-1})^{\frac{1}{p}}(F)^{\frac{1}{p}}. \tag{3.55}
$$

By integrating (3.54) and (3.55) we obtain that

$$
\max\left\{\int_{v_0}^{v(r)} \frac{\mathrm{d}s}{(F(s))^{\frac{1}{p}}}, \int_{v_0}^{v(r)} \frac{\mathrm{d}s}{g^{-1}(f(s))}\right\} \leq (\frac{p}{p-1})^{\frac{1}{p}} r, \quad r \in (0, R).
$$

Letting $r \to R$ we get a contradiction with the assumption of the theorem. Then, a solution of the problem is $v(x) = u(|x|)$ for all $x \in \mathbb{R}^n$. ■

Remark 3.13 *To obtain* (3.54) *we only use that the radial expression of the operator is nonnegative. Thus if we know that* $f(v) - g(v') \geq 0$, *regardless of the operator, we obtain the integral condition* $\displaystyle\int_1^\infty \frac{1}{g^{-1}(f(s))}\,\mathrm{d}s = \infty$ *as a sufficient condition for the existence of solution.*

Analyzing the particular case of the nonlinearity $g(t) = t^q$, $q > 0$ we have $g^{-1}(t) = t^{\frac{1}{q}}$ and $\Gamma(s) = \frac{(2s)^{q+1}}{q+1} + (\frac{p}{p-1})^{p-1} n s^p$. Then

- If $0 < q \leq p - 1 \quad \Gamma(s) \sim$ cte s^p when $s \to \infty$. Thus $\Gamma^{-1}(s) \sim$ cte $s^{\frac{1}{p}}$ when $s \to \infty$.
- If $q > p - 1 \quad \Gamma(s) \sim$ cte s^{q+1} when $s \to \infty$. Thus $\Gamma^{-1}(s) \sim$ cte $s^{\frac{1}{q+1}}$ when $s \to \infty$.

This implies the following corollary.

Corollary 2 *Let f be a continuous function satisfying* (3.1). *Then*

(i) If $0 < q \leq p - 1$, the equation

$$
\Delta_p u = f(u) - |\nabla u|^q \quad \textit{in } \mathbb{R}^n \tag{3.56}
$$

admits a positive solution if and only if the condition (3.4) *is satisfied.*

(ii) If $q > p - 1$, the equation (3.56) *has at least one positive solution if*

$$\int_1^\infty \frac{1}{f(s)^{\frac{1}{q}}} = \infty$$

and any solution vanishes identically if

$$\int_1^\infty \frac{1}{F(s)^{\frac{1}{q+1}}} < \infty.$$

By taking $f(t) = t^m,\ m > 0$ in (3.56), the first affirmation is satisfied if and only if $m < q$ with $q > p - 1$. On the other hand, if $q \leq p - 1$, there is a positive solution if and only if $m > q$.

4
A priori bounds for quasilinear equations

This chapter deals with the second problem presented in Chapter 1. Recall that we are interested in obtaining a priori L^∞–estimates for quasilinear elliptic problems like

$$-Qu = H(x,u) \tag{4.1}$$

where Q is a quasilinear operator whose principal part contains the p-Laplacian operator. Specifically, we are going to deal with

$$-Q(u) := \begin{cases} i) - \Delta_p u \;\text{ and }\; u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty(\bar{\Omega}) \\ ii) - \Delta_p u + b(x)\,|Du|^{p-1} \;\text{ and } u \in W^{1,\infty}_{\text{loc}}(\Omega) \cap L^\infty(\bar{\Omega}), 1 < p \le 2 \end{cases} \tag{4.2}$$

with $b(x)$ a continuous and bounded function.

A priori estimates are important and interesting per se, and are useful in establishing existence results via degree theory. The latter is particularly relevant for equation which do not have variational structure. We note that the problem (4.1) is not variational, in particular because of the gradient term in the left-hand side. Even more important is that our subsequent results apply to elliptic systems (and possibly even systems of inequalities) which do not have variational nature. When H is superlinear at infinity, as we are going to choose in the case of super-solutions, a priori bounds become even more relevant.

In addition to the general notations in the first chapter here we will also use the following.

c, c_0, c_i, $i = 1..5$: Denote positive constants which depend on appropriate quantities, as well as in $\|b\|_{L^\infty}$, each time when we refer to an equation with the operator Q as in (4.2) $ii)$

$\mathcal{Q}(r) := \mathcal{Q}_{x_0}(r)$: The cube of $\mathbb{R}^n$ with center x_0 and ratio r

$B_R^+ = \{x \in \mathbb{R}^n : |x| < R, x_n > 0\}$: The half ball in $\mathbb{R}^n$

$B_R^0 = \partial B_R^+ \cap \{x_n = 0\}$: The portion of the boundary included in $\{x_n = 0\}$

$e \in \mathbb{R}^n$ stands for a vector in $\mathbb{R}^n$ which is a multiple of $(0,\dots,0,1)$

L_d^p: The weighted Lebesgue spaces $L_d^p(\Omega) := L^p(\Omega; d\,\mathrm{d}x)$ where d is the distance to the boundary

Also, when we write inf or sup we always mean the essential infimum or supremum of the function.

4.1 Introduction

Most a priori estimates for elliptic partial equations of second order to be found in the PDE literature were obtained from two classical techniques. The first one, introduced by Brezis and Turner, uses and requires a variational characterization of the problem. To apply this technique, it is necessary that the equation be defined in a bounded domain with prescribed boundary conditions on the whole boundary of the domain. Also a superlinearity condition on the nonlinearity is imposed which in case of x-dependence of the right-hand side needs to be uniform in x in the whole domain. This is due to the fact that, in this type of approach, the first step always uses the first eigenfunction of the operator as a test function, in order to obtain L^1_d–estimates for the nonlinearity $f(x,u)$. Among the huge number of papers devoted to a priori bounds for variational problems, in what follows we quote some of the most influential works which develop variants of this technique.

For the semilinear case $p = 2$

In [17] Brezis and Turner proposed the method and considered in addition a growth condition on the nonlinearity. They combined the L^1_d–estimates with Hardy-Sobolev type inequalities to obtain H^1–estimates, leading to uniform estimates via bootstrap arguments.

In [18] de Figuereido, Lions and Nussbaum implemented one different method which also needs some additional hypothesis on the growth of the locally Lipschitz continuous nonlinearity. They joined the L^1_d–estimates with Pohozaev inequalities to obtain H^1–estimates leading to uniform estimates. The hypotheses on the growth of the nonlinearity in this case were less restrictive and included a greater number of functions; however, to get this generalization they also needed to assume the convexity of the domain or more global hypotheses on the nonlinearity and its primitive.

More recently in [19], Polacik, Quittner and Souplet found another way of using the weighted spaces L^1_d as well an iterative bootstrap argument which leads to important generalizations of the above quoted results, for systems of equations.

For the quasilinear case where p is not necessarily 2.

To our knowledge, the most general results to date were obtained very recently by Damascelli and Pardo [20]. Using the method in [18] they gave almost necessary and sufficient conditions for a priori estimates when the equation is defined in a strictly convex smooth domain with a Dirichlet condition on the whole boundary. They work with a nonnegative nonlinearity $f(u)$ (strictly positive when $p>2$) with subcritical growth at infinity but including more functions than those with subcritical power. Additional conditions on the nonlinearity, different in each of the cases $1 < p < N$, $p = N$ and $p > N$ give the same prove in the three cases. These conditions are sufficiently weak to include more nonlinearities than those allowed in the papers quoted above.

The second technique is based on an argument of scaling (or “blow up”) by which the problem of obtaining a priori estimates reduces to showing non existence results of Liouville type in unbounded domains. To apply this method it is necessary that the nonlinearity possesses a precise power growth at infinity ($f(u) \sim u^q$ as $u \to \infty$). In the following we summarize of the most relevant works which use this method.

<u>For the semilinear case $p = 2$</u>

In [21, 22] Gigas and Spruck introduced this second technique for the first time in $\mathbb{R}^n$, $n \geq 2$, and in Riemannian manifolds in general. Their method consists in blowing up a sequence of positive solutions around points where they assume their global maximums. Then, the a priori bounds can be deduced if the limiting equation does not admit any positive solutions on both the entire space and the half space.

In [23] the authors proposed a different method in which the rescaling was done around points obtained from the “doubling lemma” (Lemma 5.1 in that reference) which says that the function cannot double its relative size, and leads to uniform estimates for the solutions of equations or systems of equations in terms of a power of the distance to the boundary. In this method, in particular the use of the “doubling lemma” and other topological results, dispenses the need of a boundary condition.

<u>For the quasilinear case where p is not necessarily 2</u>

For equations with the p-Laplacian, until the work of Zou (see [24]), there were not in general Liouville theorems on the half space. Thus the Gidas-Spruck method could not be applied without some additional hypotheses in the problem.

In particular, in [25] (and [26] for systems) the problem of Liouville on the half-space was avoided for the singular case of the operator, $1 < p \leq 2$, by imposing the Lipschitz continuity of the nonlinearity, which depends only on u, and the equation is defined in a strictly convex domain. The latter hypothesis allows the moving plane method to prove that maxima are away from the boundary. Thus the estimate follows from the results of Mitidieri and Pohozaev in [27] and [28].

In [29] the same problem was avoided, for $1 < p < N$, by using the blow up method around a fixed point of the domain (instead of the points of maxima) and by using Harnack-type inequalities to compare the values of the functions at different points in the domain.

Zou's work ([24]), mentioned above, together with the results in [30], allowed the authors to use the blow up method without changes in case $1 < p < n$ for nonlinearities $f = f(x, u, Du)$ which growth with respect to u as a subcritical power at infinity and satisfy other additional hypotheses.

Very recently a new method for obtaining a priori estimates, qualitatively different from the above, was developed. It was first sketched in [31] and then developed in [32], for viscosity solutions of uniformly elliptic fully nonlinear equations defined in a smooth domain. This method combines boundary estimates, more precisely, the boundary weak Harnack inequality (BWHI), the boundary local maximum principle (BLMP) and the boundary quantitative strong maximum principle (BQSMP). Here, we first give a partial extension of these boundary estimates to the singular/degenerate case of $p \neq 2$, and then use the method from [31] and [32] to obtain new results on a priori bounds for positive solutions of singular/degenerate equations. The novelty goes in the following two directions.

First, unlike what is required for variational methods, we do not need to consider boundary conditions on the whole boundary of the domain, and moreover, the method allows additional terms depending on the gradient. Among other things, we also allow indefinite nonlinearities, that is, right-hand sides which can vanish at some points in the domain.

Second, in contrast to the scaling methods, we do not rescale nor use Liouville type theorems, and consequently it is not necessary to impose a condition on the growth of nonlinearity in u, to behave like a precise power as $u \to \infty$. Moreover, the domain being convex is irrelevant. We remark that the scaling method cannot be applied to indefinite nonlinearities without additional assumptions on the subsets of the domain where the nonlinearity vanishes or is positive. Thus, we obtain a priori estimates for weak solutions in the Sobolev sense in general regular bounded domains and including more types of nonlinearities than only those of power growth.

On the other hand, when applied to nonlinearities which allow the use of the scaling method, our results are in general weaker than those given by that method. Indeed, for instance for the equation $-\Delta u = u^q$ in Ω, $u = 0$ on $\partial\Omega$, Gidas and Spruck obtained their results for $q < \dfrac{n+2}{n-2}$, $n > 2$ while our results applied to this equation give a priori bounds only for $q < c + 1$ where c is an small constant as we are going to specify later.

We also observe that our proofs of the BWHI, the BQSMP and the BLMP, that are not previously found in the literature for degenerate operators, need some modifications of the boundary estimates for viscosity solutions obtained in [33], because of the non additivity of the p-Laplacian operator.

Next, we state in detail our main results on uniform a priori estimates. We assume we are given a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with $\partial\Omega \in C^{1,1}$ and denote $d := d(x) = \operatorname{dist}(x, \partial\Omega)$. The additional assumptions on the nonlinearities will be introduced separately, because our results can treat independently sub and super-solutions, and each of these require different hypotheses.

The first result is about Lebesgue estimates for positive super-solutions.

Let $f : \Omega \times [0,\infty) \to [0,\infty)$ to be a continuous and bounded function in Ω, that satisfies the following superlinearity condition

$$\lim_{s\to\infty} \frac{f(x,s)}{s^{p-1}} = \infty \tag{f}$$

uniformly in $x \in K$, for some $K \subset \Omega$ with positive measure.

Theorem 4 (L^{ε_0}–estimate) *Assume u is a nonnegative weak super-solution of*

$$-Qu \geq f(x,u) \quad \text{in } \Omega.$$

Then,

$$\int_\Omega \left(\frac{u}{d}\right)^{\varepsilon_0} \leq c, \tag{4.3}$$

where $\varepsilon_0 = \varepsilon_0(n,p)$ and $c := c(\varepsilon_0, n, p, |K|)$ are positive constants.

We want to highlight that this theorem is valid for super-solutions, without a requirement for u to be a sub-solution or purely a solution. The only hypothesis that is needed is the superlinearity condition (f) which does not restrict the function to have any precise power growth behavior as $u \to \infty$. Moreover, it includes nonlinearities which are "indefinite" in x, i.e, f may vanish on a nontrivial subset of the domain.

The second theorem combines the above result with an estimate for sub-solutions, in order to obtain a uniform a priori bound of u together with a gradient bound on the boundary of the domain. A distinctive feature is that u does not need to be a sub-solution and a super-solution of the same equation. We can even allow that a function of u rather than u be a sub-solution of some equation.

Let $\xi : [0, \infty) \to [0, \infty)$ be a continuous and increasing bijection such that

$$\limsup_{s\to\infty} \frac{\xi(s)}{s^\alpha} < \infty \tag{4.4}$$

for some $\alpha > 0$ and let $g : \Omega \times [0, \infty) \to [0, \infty)$ be a continuous function such that for some $b_0 > 0$

$$g(x, s) \leq \frac{b_0}{d^\gamma(x)} (1 + s^m) \tag{g}$$

with $d := d(x) = \operatorname{dist}(x, \partial\Omega)$, γ and m positive constants that satisfy

$$\gamma \leq \begin{cases} \varepsilon_0 \frac{p(1-\varepsilon)}{n} & 1 < p \leq n \\ \varepsilon_0, & p > n \end{cases} \qquad \text{and} \qquad m \leq \begin{cases} \varepsilon_0 \frac{p(1-\varepsilon)}{\alpha n} + (p-1) & 1 < p \leq n \\ \frac{\varepsilon_0}{\alpha} + (p-1), & p > n \end{cases}$$

for some ε_0, $\varepsilon \in (0, 1]$ and $\alpha > 0$.

Remark 4.1 *The value of ε_0 in these assumptions corresponds to the exponent of the Lebesgue estimate in Theorem 4, while ε comes from the condition for $c(x)$ in Theorem 8 below.*

Theorem 5 (A priori interior uniform bounds) *Assume that u is a weak solution in Ω of*

$$-Qu \geq f(x, u) \tag{4.5}$$

$$-Q(\xi(u)) \leq g(x, \xi(u)) \tag{4.6}$$

where f and g satisfy the structural conditions (f) and (g) respectively. Then,

$$\|u\|_{L^\infty(\Omega)} \leq c \quad \text{and} \quad \left\| \frac{\xi(u)}{d} \right\|_{L^\infty(\Omega)} \leq c.$$

Standard examples of nonlinearities in which these results are valid are power functions like

$$f(u) = u^q \quad \text{and} \quad g(u) = u^r$$

with $q > p - 1$ and $r \leq c(\varepsilon, \varepsilon_0, p, n\alpha) + (p - 1)$, where c is as in the expression of m in the assumptions of (g).

Observe that it is unavoidable to make some restriction on the growth at infinity of the nonlinearity g. Indeed, as is known from the example constructed by Souplet in [34] there cannot be an a priori bound in the uniform norm for positive solutions of the Dirichlet problem

$$-\Delta u = a(x)u^r,$$

in a smooth bounded domain when $r > \frac{n+1}{n-1}$.

Also, the above theorems are satisfied in the case of nonlinearities

$$f(u) = u^q(\ln(1+u))^k, \qquad g(u) = u^r(\ln(1+u))^k, \quad \forall\ k > 0 \tag{4.7}$$

with $q \geq 1$ and $r < \begin{cases} \varepsilon_0 \frac{p(1-\varepsilon)}{\alpha n} + (p-1), & 1 < p \leq n \\ \frac{\varepsilon_0}{\alpha} + (p-1), & p > n. \end{cases}$

Remark 4.2 *It is important to notice that such types of nonlinearities arise naturally when we deal with equations of the form*

$$-\Delta_p u \pm |\nabla u|^p = h(x) \quad \textit{with } 1 < p < n$$

after a Kazdan-Kramer change of variables[1]*. The same change of the variables permits us to consider more general equations*

$$-Qu = H(x, u, \nabla u)$$

with Q defined as in (4.2) *and H such that $H(x,u,\nabla u) \leq h(x,u) \pm |\nabla u|^p$ where h corresponds to the function f, in the case of super-solutions (like in* (4.5)*), or g in the case of sub-solutions (like in* (4.6)*).*

In the following we state several boundary estimates, which will play an important role in the proofs of the above a priori estimates. These boundary estimates are global extensions of the classical interior weak Harnack inequality (WHI) and local maximum principle (LMP) which can be found in [36, Theorem 1.2], [2, Chapter 7], and [16, Chapter 8]. Such extensions for uniformly elliptic operators were recently obtained by Sirakov in [33]. For more general operators, the only result we are aware of is the very recent work [37], in which the authors obtained the BWHI for a particular equation involving the p-Laplacian operator, on a flat domain.

[1]The Kazdan-Kramer change of variables used in [35] allows us to pass from the quasilinear Dirichlet problem

$$-\Delta_p u = \beta(u)\,|\nabla u|^p + f(x) \quad \text{in } \Omega$$

to

$$-\Delta_p v = f(x)(1+g(v))^{p-1} \quad \text{in } \Omega$$

after the change of variable $v(x) = \psi(u(x)) = \displaystyle\int_0^{u(x)} \exp\frac{\gamma(\theta)}{p-1}\,\mathrm{d}\theta$, $\gamma(t) = \int_0^t \beta(\theta)\,\mathrm{d}\theta$. The new function g then behaves at infinity like in (4.7).

Here we prove variants of the results in [33] for homogeneous equations with a p-Laplacian type operator, which are both useful in themselves and help us to prove our theorems on a priori bounds.

Next we summarize these boundary extensions.

Theorem 6 (Boundary weak Harnack inequality, BWHI) *Assume that u is a nonnegative weak super-solution of*

$$-Qu \geq 0 \quad in\ B_2^+.$$

Then, there exist constants $\varepsilon := \varepsilon(n,p) > 0$ and $c := c(n,p,\varepsilon) > 0$, such that,

$$\inf_{B_1^+} \frac{u}{x_n} \geq c \left(\int_{B_{3/2}^+} \left(\frac{u}{x_n} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}. \tag{4.8}$$

Theorem 7 (Boundary quantitative strong maximum principle, BQSMP) *Assume that u is a nonnegative weak super-solution of*

$$-Qu \geq C\chi_E > 0 \quad in\ B_2^+,$$

where χ_E is the characteristic function in a ball E in B_2^+. Then, there exists a constant $c := c(n,p) > 0$, such that,

$$\inf_{B_1^+} \frac{u}{x_n} \geq c\,(C)^{\frac{1}{p-1}}. \tag{4.9}$$

Theorem 8 (Boundary local maximum principle, BLMP) *Assume that u is a weak sub-solution of*

$$-Qu - c(x)\,|u|^{p-2}\,u \leq 0 \quad in\ B_2^+, \tag{4.10}$$

with $u \leq 0$ in B_2^0 and $c(x)$ a function in the Lebesgue space

$$c(x) \in L^{\beta}, \quad \beta = \begin{cases} \frac{n}{p(1-\varepsilon)}, & 1 < p \leq n \\ 1+\varepsilon, & p > n \end{cases} \qquad for\ some\ \ \varepsilon \in (0,1),$$

for all $x \in B_2^+$. Then, there exists a constant $c := c(n,p,r,\varepsilon) > 0$, such that,

$$\sup_{B_1^+} \frac{u^+}{x_n} \leq c \left(\int_{B_{3/2}^+} \left(u^+\right)^r \right)^{\frac{1}{r}} \tag{4.11}$$

for any $r > 0$.

The proofs of these results are a combination of their interior versions and some different tools for boundary estimates which will be well explained in the next section.

4.2
Principal tools to prove a priori bounds

First we state two versions of the comparison principle which are adapted to our study of a priori bounds.

4.2.1
Comparison theorems

Recall that the operator Q in (4.1) has two possible forms according to the chosen regularity of u and the value of p. We distinguish these two type of problems in order to be able to apply some comparison principles. In the first case, corresponding to the operator Q in (4.2) (i) we use [2, Corollary 3.4.2], which we recall next.

Theorem 9 (Comparison principle of weak solutions of class $W^{1,p}_{\text{loc}}(\Omega)$) *Assume that $B = B(x, z)$ is non-increasing in the variable z. Let u and v be solutions of class $W^{1,p}_{\text{loc}}(\Omega)$ in Ω of*

$$\Delta_p u + B(x, u) \geq 0$$
$$\Delta_p v + B(x, v) \leq 0,$$

If $u \leq v$ in $\partial\Omega$, then $u \leq v$ in Ω.

Also, in this case the strong maximum principle is satisfied , see [38].

On the other hand, if there is an additional term depending on the gradient we need to consider stronger regularity on u, as we assume in (4.2) (ii), in order to have the following result from [2, Corollary 3.5.2].

Theorem 10 (Comparison principle for singular elliptic inequalities) *Let u and v be solutions of class $W^{1,\infty}_{\text{loc}}(\Omega)$ in Ω of*

$$\Delta_p u + B(x, u, Du) \geq 0$$
$$\Delta_p v + B(x, v, Dv) \leq 0,$$

where $1 < p \leq 2$. Assume also that $B = B(x, z, \xi)$ is a locally Lipschitz with respect to ξ in $\Omega \times \mathbb{R}$ and non-increasing in the variable z. If $u \leq v + M$ in $\partial\Omega$, where M is constant, then $u \leq v + M$ in Ω.

In this case we can use the strong maximum principle, as stated in [39, Theorem 1′].

Observe the difference between Theorem 10 and Theorem 3. In the latter there is no restriction on p but we assume that at least one of the gradients of u or v does not vanish.

Next we introduce some important interior estimates for weak sub and super-solutions for which we obtain partial boundary extensions later. We also give some more estimates and existence results which are instrumental in the proofs of these boundary extensions.

4.2.2 Existence of solutions of some particular problems

Here we quote some Dirichlet problems for which a result of existence of weak solution, in the sense of the definition Definition 2.1, is available.

First, the existence of solution for the Dirichlet problem

$$\begin{cases} -\Delta_p u &= \chi_A \qquad \text{in } \Omega \\ \qquad u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

where A is a subset of Ω with positive Lebesgue measure was discussed and verified in [40].

If a dependence on the gradient is considered as in the following Dirichlet problem

$$\begin{cases} -\Delta_p u + b(x)\,|Du|^{p-1} &= \chi_A \qquad \text{in } \Omega \\ \qquad u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

where A is a subset of Ω with positive Lebesgue measure, more details on the existence of solution can be seen in [4].

We will need to evoke the existence of solution for the above problems in the proof of the quantitative maximum principle Theorem 13 below.

We will also need to refer to the existence of solution of the Dirichlet problem,

$$\begin{cases} -Qu - c(x)\,|u|^{p-2}\,u &= 0 \qquad \text{in } B_1^+ \\ \qquad u &= h \qquad \text{on } \partial B_1^+, \end{cases}$$

with $c(x)$ a Lebesgue function that satisfies

$$c(x) \in L^\beta, \quad \beta = \begin{cases} \frac{n}{p(1-\varepsilon)}, & 1 < p \leq n \\ 1+\varepsilon, & p > n \end{cases} \qquad \varepsilon \in (0,1],$$

and h a nonnegative smooth function, which was obtained in [41]. This result of existence will be used in the proof of the Lipschitz bound for the p-Laplacian below (Theorem 15).

4.2.3
Maximum of weak sub-solutions

We also recall the following result which concerns to the maximum of weak sub-solutions.

Lemma 4.3 (Maximum of weak sub-solutions) *Let u a weak sub-solu tion of* (4.1)*, then $v(x)$=$\max\{u, k\}$ where $k > 0$ is also a weak sub-solution.*

In [42, Theorem 1] the proof of this lemma is done for a maximum of two weak sub-solutions in a domain with smooth boundary. Since k is also a sub-solution of (4.1), it follows that it is sufficient to choose one solution as u and the other one as k.

4.2.4
Some important interior bounds

Now we present some important bounds which will be used throughout the text, the weak Harnack inequality, the local maximum principle and the quantitative strong maximum principle.

The weak Harnack inequality (WHI), in particular, is one of the fundamental estimates in the theory of elliptic PDE. It is considered as a quantitative version of the maximum principle and as an extension of the strong maximum principle. Its study goes back to the works [43] of Di Giorgi and [44] of Moser who established an iterative method to prove the inequality for uniformly elliptic differential equations of second order. The weak Harnack inequality was later extended by Serrin to general elliptic equations in divergence form with a nonlinearity depending also on the gradient, modelled by the p-Laplacian, see [45]. We also quote the classical paper of Trudinger [36] (in particular, theorem 1.2 in that paper).

The result is also proved in [2, Chapter 7] by Pucci and Serrin. We recall this estimate in the version presented in [2].

Theorem 11 (The local Weak Harnack inequality, WHI) *Let u be a nonnegative weak solution of*

$$-Qu \geq 0 \quad \textit{in } B_2.$$

Then there exists a constant $C := C(p, n)$ such that

$$\inf_{B_1} u \geq C r^{-\frac{n}{\gamma}} \|u\|_{L^\gamma(B_{3/2})}$$

for any $\gamma < \frac{n(p-1)}{n-p}$ if $p \leq n$ and $\gamma \leq \infty$ if $p > n$.

The mentioned WHI is obtained for nonnegative super-solutions and has local nature.

When we have, conversely, a weak nonnegative sub-solution an interior estimate for the maximum of the function is obtained, called the local maximum principle (LMP). The proof of this result is also based on the Moser technique and can be developed in parallel to the proof of the weak Harnack inequality.

The next theorem contains this estimate as it found in reference [2].

Theorem 12 (The local Maximum Principle, LMP) *Let u be a nonnegative weak solution of*

$$-Qu + c(x)\,|u|^{p-2}\,u \le 0 \quad in \quad B_2,$$

where $c(x)$ is a function in the Lebesgue space

$$c(x) \in L^{\beta}, \quad \beta = \begin{cases} \frac{n}{p(1-\varepsilon)}, & 1 < p \le n \;\; \varepsilon \in (0,1) \\ 1, & p > n. \end{cases}$$

Then there exists a constant $C := C(p,n)$ such that

$$\sup_{B_1} u \le Cr^{-\frac{n}{\gamma}}\,\|u\|_{L^{\gamma}(B_{3/2})}$$

for any $\gamma > 0$.

Actually, the LMP was given in [2] under the hypothesis $\gamma > p-1$, but as is well known, the result can be extended to any $\gamma > 0$ by applying the same technique as in [46, chapter 4, pages 75-76].

Remark 4.4 *Observe that from the above two theorems, it is simple to verify the conclusions in balls of radios r, $3/2r$ and $2r$ centered in any point x_0. The full expression of the inequalities is obtained by the scaling $x \mapsto (x+x_0)r$ in the ball B_2. The same for any radii $r_1 < r_2 < r_3$ instead of r, $3/2r$ and $2r$. Also the conclusions hold if we have a domain Ω and $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, by a local covering argument.*

Another estimate that also quantifies the strong maximum principle is the known QSMP in which it is proved that the infimum of the function is strictly positive. This estimate in the case of the Laplacian appeared at first in the work of Brezis-Cabré [47]. In the case of the p-Laplacian operator it was obtained in [40], more precisely in the Theorem 3.3 of this reference. With the same proof as in [40], by using the appropriate comparison principles, strong maximum principles and existence results discussed in the previous sections, we obtain the following QSMP for the equation driven by the operator Q.

Theorem 13 (Interior quantitative maximum principle, QSMP) *Assume that K is a compact subset of a domain Ω, A is subset of Ω of positive measure. Let u be a nonnegative super-solution of*

$$-Qu \geq \chi_A \quad in \quad \Omega.$$

Then, there exists a constant $c_0 := c_0(p, n, K, \Omega) > 0$ such that

$$\inf_K u \geq c_0 > 0. \tag{4.14}$$

Remark 4.5 *Observe that the above theorem is also valid for weak super-solutions of the equation $-Qu \geq C\chi_A > 0$. In this case we obtain that there exists a constant $c_0 := c_0(p, n, K, \Omega) > 0$ such that*

$$\inf_K u \geq c_0 \, (C)^{\frac{1}{p-1}} .$$

4.2.5
Some boundary estimates

The next result corresponds to the boundary Harnack inequality for the p-Laplacian with lower order terms. Its proof was recently obtained in [42, Theorem 1] in a Reifenberg flat domain, of which smooth boundary is a particular case.

Theorem 14 (Boundary Harnack inequality) *Let u be a nonnegative weak solution of*

$$-Qu = 0 \quad in \quad B_2^+$$

with $u = 0$ on B_2^0. Then, there exists a constant $c := c(n, p)$ such that

$$\inf_{B_1^+} \frac{u}{x_n} \geq c \, u(0, \ldots, 0, 1/2). \tag{4.15}$$

Finally we are going to prove the following Lipschitz bound for our particular inequalities which is an easy consequence of the C^1–estimates for the p-Laplacian.

Theorem 15 (Lipschitz bound for the p-Laplacian) *Let u be a weak solution of*

$$-Qu - c(x) \, |u|^{p-2} \, u \leq 0 \quad in \;\; B_2^+,$$

with $u \leq 0$ *on* B_2^0 *and* $c(x)$ *a function in the Lebesgue space*

$$c(x) \in L^{\beta}, \quad \beta = \begin{cases} \frac{n}{p(1-\varepsilon)}, & 1 < p \leq n \\ 1+\varepsilon, & p > n \end{cases} \qquad \varepsilon \in (0,1).$$

Then, there exists a constant $C := C(n,p)$ *such that*

$$u(x) \leq C \left(\sup_{B^+_{\frac{3}{2}}} u^+ \right) x_n \quad in \;\; B_1^+. \tag{4.16}$$

Proof. By the maximum principle we know that $u \leq \sup_{\partial B_1^+} u^+$ in B_1^+. Let h a C^1–function in B_1^+ such that

$$h = \begin{cases} 0 \;\text{ on }\; B_1^0 \\ u \;\text{ on }\; \partial B_1^+ \setminus B_1^0 \end{cases}$$

with $\|h\|_{C^1(B_1^+)} \leq c \sup_{\partial B_1^+} u^+$. Let v the weak solution of

$$\begin{cases} -Qv - c(x)\,|v|^{p-2}\,v &= 0 \qquad \text{in } \; B_1^+ \\ \qquad\qquad\qquad\quad v &= h \qquad \text{on } \; \partial B_1^+ \end{cases}$$

given in the second section of this chapter with $c(x)$ the same Lebesgue function that in the original equation satisfied by u. Then, by applying the appropriate comparison principle in B_1^+ we have that $u \leq v$ in this set.

On the other hand by the C^1–estimates of the p-Laplacian for v and the assumptions on h we have

$$\|v\|_{C^1(B_1^+)} \leq c(\delta) \sup_{\partial B_1^+} u^+.$$

Hence,
$$\frac{u}{x_n} \leq \frac{v}{x_n} \leq c \sup_{\partial B_1^+} u^+ \leq c \sup_{B^+_{3/2}} u^+$$

as desired. ∎

4.3
Boundary weak Harnack estimates and quantitative strong maximum principles

In this section we are going to obtain the extension up to the boundary of the interior estimates presented in Subsection 4.2.4. To deal with it we proceed by locally straightening the boundary of the domain through a classical change of coordinates near a fixed boundary point.

4.3.1 Straightening the boundary of the domain

Let u a weak solution in the Sobolev sense of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x)\,|\nabla u|^{p-1} = f(x,u) \quad \text{in } \Omega. \tag{4.17}$$

That means that u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \Phi \,\mathrm{d}x + \int_\Omega b(x)\,|\nabla u|^{p-1}\,\Phi\,\mathrm{d}x = \int_\Omega f(x,u)\Phi\,\mathrm{d}x, \quad \forall \Phi \in C_c^\infty(\Omega) \tag{4.18}$$

as we stated earlier in Definition 3.2.

Let $\mathcal{Q}_r$ be the cube of center $y_0 := (y_0', y_{0,n})$ and side r. Define the vector of $\mathbb{R}^n$, $e := (0,0,\ldots,1/2)$. Consider the $C^{1,1}$-difeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, such that, for any point $x_0 \in \partial\Omega$ and some $r > 0$, $\varphi(B_r(x_0)\cap\Omega) \subset \mathcal{Q}_1$ and $\varphi(B_r(x_0)\cap\partial\Omega) \subset \{x \in \partial\mathcal{Q}_1 : x_n = 0\}$. We set $\psi = \varphi^{-1}$ and the change of variables (see Figure 4.1)

$$x = \psi(y), \quad y = \varphi(x), \tag{4.19}$$

so, $v(y) := u(\psi(y))$ in $\mathcal{Q}_1$ for $u(x) = v(\varphi(x))$ in $B_r(x_0)\cap\Omega$.

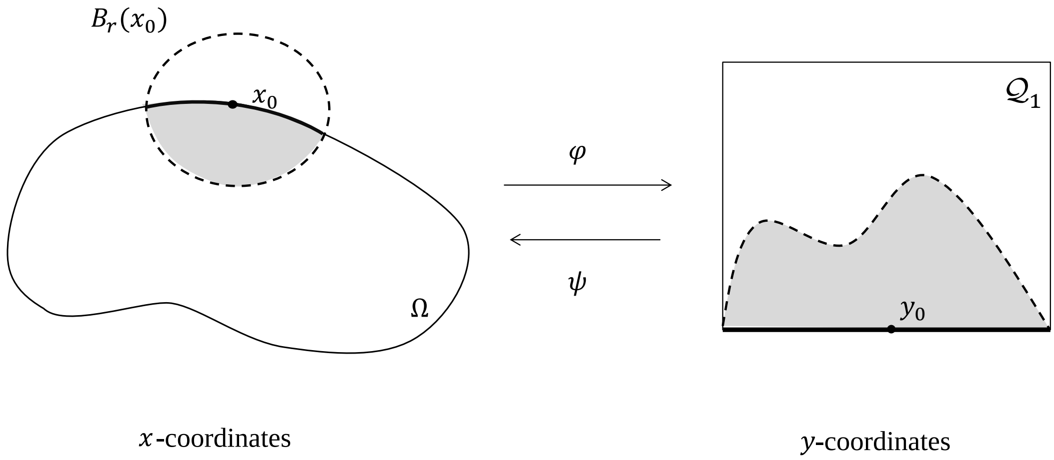

Figure 4.1: Change of variables by the diffeomorphism

Denote by $J\psi(y)$ the Jacobian matrix of ψ, thus equation (4.18) after the change of variables (4.19) becomes

$$
\begin{aligned}
\int_{\mathcal{Q}_1} &\left|\nabla_x u(\psi(y))\right|^{p-2} \nabla_x u(\psi(y)) \nabla_x \Phi(\psi(y)) \left|\det J\psi(y)\right| \mathrm{d}y \\
&+ \int_{\mathcal{Q}_1} \tilde{b}(y) \left|\nabla_x u(\psi(y))\right|^{p-1} \Psi(y) \left|\det J\psi(y)\right| \mathrm{d}y \\
&\qquad = \int_{\mathcal{Q}_1} \tilde{f}(y, v(y)) \Psi(y) \left|\det J\psi(y)\right| \mathrm{d}y, \quad (4.20)
\end{aligned}
$$

where,

$$
\begin{aligned}
\Psi(y) &:= \Phi(\psi(y)), \\
\tilde{f}(y, v(y)) &= f(\psi(y), u(\psi(y))), \\
\tilde{b}(y) &= b(\psi(y)).
\end{aligned}
$$

By simple computations we obtain that $u_{x_i} = \sum_{k=1}^{n} v_{y_k}(\varphi(x)) \varphi_{x_i}^k(x)$, thus,

$$
\begin{aligned}
\nabla_x u(\psi(y)) &= J^T \varphi(x) \nabla_y v(y) \\
&= \left(J^T \psi(y)\right)^{-1} \nabla_y v(y)
\end{aligned}
$$

and

$$
\nabla_x \Phi(\psi(y)) = \left(J^T \psi(y)\right)^{-1} \nabla_y \psi(y),
$$

where the symbol T denotes the transpose of the matrix. Hence, (4.20) can be rewritten only in the variable y as

$$
\begin{aligned}
\int_{\mathcal{Q}_1} &\left|\left(J^T \psi(y)\right)^{-1} \nabla v(y)\right|^{p-2} \left(\left(J^T \psi(y)\right)^{-1}\right)^T \cdot \\
&\cdot \nabla v(y) \left(J^T \psi(y)\right)^{-1} \nabla \psi(y) \left|\det J\psi(y)\right| \mathrm{d}y \\
&+ \int_{\mathcal{Q}_1} \tilde{b}(y) \left|\left(J^T \psi(y)\right)^{-1} \nabla v(y)\right|^{p-1} \Psi(y) \left|\det J\psi(y)\right| \mathrm{d}y \\
&\qquad = \int_{\mathcal{Q}_1} \tilde{f}(y, v(y)) \Psi(y) \left|\det J\psi(y)\right| \mathrm{d}y. \quad (4.21)
\end{aligned}
$$

Setting

$$
\begin{aligned}
A(y) &= \left(J^T \psi(y)\right)^{-1} \\
B(y) &= A^T(y) A(y) \\
C(y) &= \left|\det J\psi(y)\right|,
\end{aligned}
\quad (4.22)
$$

then, v is a weak solution in $\mathcal{Q}_1$ of

$$
-\operatorname{div}(C(y) \left|A(y) \nabla v\right|^{p-2} B(y) \nabla v) + \tilde{b}(y) C(y) \left|A(y) \nabla v\right|^{p-1} = C(y) \tilde{f}(y, v).
$$

Taking for simplicity $\det J\psi(y) = 1$ in the computations, this equation reduces to

$$-\operatorname{div}(|A(y)\nabla v|^{p-2} B(y)\nabla v) + \tilde{b}(y)\,|A(y)\nabla v|^{p-1} = \tilde{f}(y,v) \quad \text{in } \mathcal{Q}_1. \qquad (4.23)$$

Remark 4.6 *Observe that since φ and ψ are $C^{1,1}$ mappings, it is classical to check that we can find constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that*

$$\begin{aligned} c_1\,|\nabla v| \le |A(y)\nabla v| \le c_2\,|\nabla v|\,, \\ c_3\,|\nabla v| \le |B(y)\nabla v| \le c_4\,|\nabla v|\,, \\ C(y) \le c_5. \end{aligned} \qquad (4.24)$$

These inequalities imply that the structural conditions for the WHI (Theorem 11) remain true after transforming the above divergence equation with the help of the diffeomorphism.

Remark 4.7 *It is not difficult to verify that the function $\tilde{f}$ inherits the superlinearity condition of f in its second argument. Similarly, $\tilde{b}$ inherits the regularity and boundedness of b.*

Remark 4.8 *Also observe that the above computations remain true if u is a weak solution of the more general equation in divergence form*

$$-\operatorname{div} A(x,u,Du) + B(x,u,Du) = 0 \quad \textit{in } \Omega$$

which, after the same change of variables, transforms into

$$\begin{aligned} -\operatorname{div}\Big(C(y)\tilde{A}(\psi(y), v(y), \big(J^T\psi(y)\big)^{-1}\nabla v)\Big) \\ + C(y)\tilde{B}(\psi(y), v(y), \big(J^T\psi(y)\big)^{-1}\nabla v) = 0 \ \ \textit{in } \mathcal{Q}_1. \end{aligned}$$

4.3.2
Proof of the boundary weak Harnack inequality(BWHI)

In this section we prove the BWHI, Theorem 6. The proof follows the ideas in [33] and [37]. In the first of these two papers the estimate is obtained for viscosity solutions of general uniformly elliptic inequalities

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)\,|Du| \le f(x),$$

where b, f belong to suitable Lebesgue spaces and $\mathcal{M}^{-}_{\lambda,\Lambda}$ is the extremal Pucci operator

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M)(M) = \Lambda \sum_{\mu_i<0} \mu_i + \lambda \sum_{\mu_i>0} \mu_i = \inf_{\lambda I \leq A \leq \Lambda I} \operatorname{trace}(AM)$$

in a bounded smooth domain Ω, while in the second the BWHI is proved for weak solutions in the Sobolev sense of the particular inequality

$$-\Delta_p u + a(x) |u|^{p-2} u \geq 0,$$

where a is a nonnegative L^∞-function, in a domain with a flat boundary.

In what follows we give a complete proof of the estimate for our equation, in which the differences with respect to the above two works are shown for the sake of completeness.

First, we are going to prove the following version of the theorem, in which we assume that the super-solution is defined in the whole bounded domain Ω with $C^{1,1}$ boundary, as we stated at the beginning of the chapter. The result in B_2^+ follows by considering some Ω with smooth boundary such that $B^+_{3/2} \subset \Omega \subset B_2^+$.

Theorem 16 (BWHI in a domain with smooth boundary) *Assume that u is a nonnegative weak super-solution of*

$$-Qu \geq 0 \quad \textit{in} \;\; \Omega,$$

where Ω is a bounded $C^{1,1}-$domain. Then, there exist constants $\varepsilon > 0$ and $c := c(n, p, \gamma, \varepsilon, \Omega) > 0$, such that,

$$\inf_{\Omega} \frac{u}{d} \geq c \left(\int_{\Omega} \left(\frac{u}{d} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}. \tag{4.25}$$

This theorem is a consequence of the equivalent result in cubes, by locally straightening the boundary and covering it with balls in which such straightening is possible. It is important to recall that, because of the nonlinearity of the p-Laplacian, after the straightening procedure the operators in (4.2) change into new operators also in divergence form but different from the original ones.

Theorem 17 (BWHI in cubes) *Assume that v is a weak super-solution of*

$$-\widetilde{Q}v \geq 0 \quad \textit{in} \quad \mathcal{Q}_2 \tag{4.26}$$

with

$$
-\widetilde{Q}(v) := \begin{cases} (i) - \operatorname{div}(|A(y)\nabla v|^{p-2}\, B(y)\nabla v) & \text{if } v \in W^{1,p}_{\mathrm{loc}}(\mathcal{Q}_2) \cap L^\infty(\mathcal{Q}_2) \\ (ii) - \operatorname{div}(|A(y)\nabla v|^{p-2}\, B(y)\nabla v) + \tilde{b}(y)\,|A(y)\nabla v|^{p-1} & \\ \qquad\qquad\qquad \text{if } v \in W^{1,\infty}_{\mathrm{loc}}(\mathcal{Q}_2) \cap L^\infty(\mathcal{Q}_2),\ 1 < p \le 2 & \end{cases} \tag{4.27}
$$

where A and B are smooth $n \times n$ matrices satisfying (4.24) *and depending only on y. Then, there exist constants $\varepsilon = \varepsilon(n,p) > 0$ and $c = c(n,p,\varepsilon) > 0$, such that,*

$$
\inf_{\mathcal{Q}_1} \frac{v}{y_n} \ge c \left(\int_{\mathcal{Q}_{3/2}} \left(\frac{v}{y_n} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}.
$$

To prove this theorem we need some important lemmas: a growth lemma, Lemma 4.9, and the Lemma 4.11 below.

Lemma 4.9 (Growth Lemma) *Given $\nu > 0$, there exists $k = k(n,p,\nu) > 0$ such that, if u is a non-negative weak super-solution of* (4.26) *in $\mathcal{Q}_{3/2}$ and we have*

$$
|\{v(y) > y_n\} \cap \mathcal{Q}_1| \ge \nu, \tag{4.28}
$$

then $v(y) > k y_n$ in $\mathcal{Q}_1$.

Remark 4.10 *The constant k in the above lemma also depends on $\|b\|_{L^\infty}$ if we deal in* (4.26) *with Q as in* (4.2) *ii).*

The growth lemma also quantifies the maximum principle up to the boundary in the whole cube if we have (4.28), which says that the quantification is satisfied in a "representative" (i.e. with positive measure) part of the cube.

For the proof of the growth lemma we need to define some different cubes contained in a cube of side 2, and centered at points on the axis that corresponds to the n-th coordinate. These cubes are represented in the Figure 4.2 below for the particular case $n = 2$ as follows.

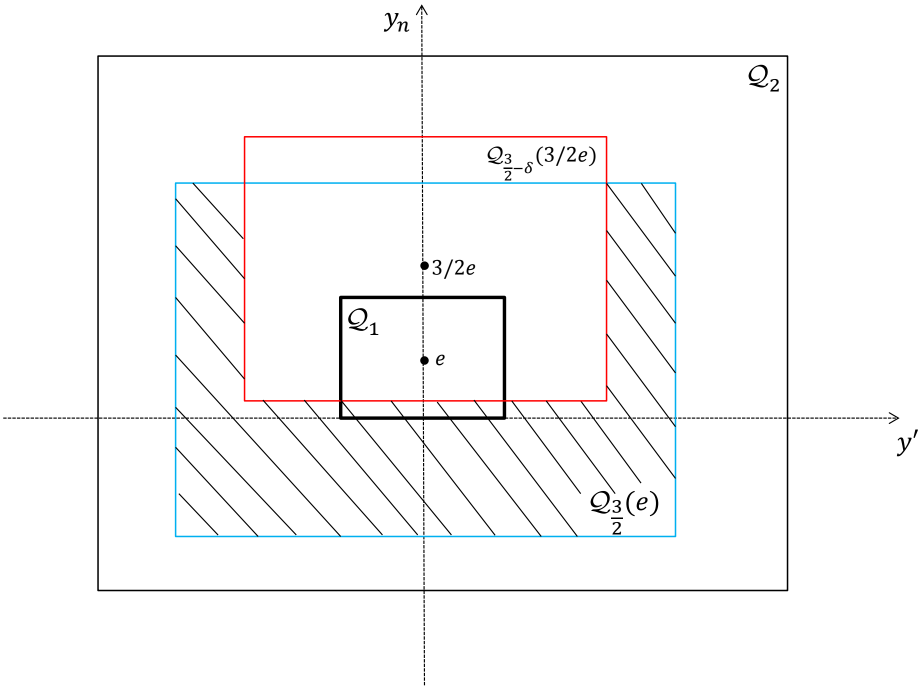

Figure 4.2: Auxiliary figure in the proof of Lemma 4.9 in case $n = 2$
The dashed region contains the flat portion of the boundary.
In light black $\mathcal{Q}_2(e)$ the cube of center $e = (0, 1/2)$ and side 2
In blue $\mathcal{Q}_{3/2}(e)$ the cube of center e and side $3/2$
In red $\mathcal{Q}_{3/2-\delta}(3/2e)$ the cube of center $3/2e$ and side $3/2 - \delta$
for some well chosen $\delta > 0$
In dark black $\mathcal{Q}_1(e)$ the cube of center e and side 1

For a better understanding we start by introducing the principal steps of the proof of this lemma. First, we prove the interior version of the lemma and subsequently the result on the flat portion of the boundary in which $\{y_n = 0\}$ (this part corresponds exactly to the boundary of the domain Ω after the process of straightening the boundary). We prove the interior result in $\mathcal{Q}_1(e)$ dealing with the intersection $\mathcal{Q}_{\frac{3}{2}-\delta}(e) \cap \mathcal{Q}_1(e)$. To focus on the flat portion of the boundary included in $\{y_n = 0\}$ we work in the region $\mathcal{Q}_{\frac{3}{2}}(e) \setminus \mathcal{Q}_{\frac{3}{2}-\delta}(e)$, the dashed region in the case of $n = 2$ in the above figure, to finally consider its intersection with $\mathcal{Q}_1$.

We continue with the details of the proof of the lemma.

Proof of Lemma 4.9. Take $c_1 = c_1(\nu, n) \in (0, 1/2)$ for which the set

$$S_\delta = \mathcal{Q}_{\frac{3}{2}}(e) \setminus \mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right)$$

has measure such that

$$|S_\delta| \leq \frac{\nu}{2} \quad \text{for any} \quad 0 < \delta \leq c_1.$$

Then, it is easy to verify that for every $\delta \in (0, c_1]$

$$\left|\{v(y) > y_n\} \cap \mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right)\right| \geq \frac{\nu}{2}.$$

In fact,

$$\{\{v(y) > y_n\} \cap \mathcal{Q}_1\} \subset \left\{\{v(y) > y_n\} \cap \mathcal{Q}_{\frac{3}{2}}(e)\right\} \subset \left\{\{v(y) > y_n\} \cap \mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right)\right\} \cup S_\delta,$$

thus, using the initial assumption (4.28), it holds

$$\begin{aligned}\left|\{v(y) > y_n\} \cap \mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right)\right| &\geq |\{v(y) > y_n\} \cap \mathcal{Q}_1| - |S_\delta| \\ &\geq \nu - \nu/2 = \frac{\nu}{2}\end{aligned}$$

as desired.

Also, by using the WHI (Theorem 11) we obtain that, for all $\delta \in (0, c_1]$ there exists a $k_\delta > 0$ such that

$$\frac{v}{y_n} \geq k_\delta \ \text{ in } \ \mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right). \tag{4.29}$$

Indeed, by the WHI on the cube $\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)$ we have

$$\inf_{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)} v(y) \geq C_\delta \left(\int_{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)} v^\gamma\right)^{\frac{1}{\gamma}}.$$

Thus, since $v \geq 0$ and by the previous results we have that

$$\begin{aligned}\left(\inf_{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)} v(y)\right)^\gamma &\geq C_\delta^\gamma \int_{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)} v^\gamma \\ &\geq C_\delta^\gamma \int_{\left\{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)\right\} \cap \{v \geq y_n\}} v^\gamma \\ &\geq C_\delta^\gamma \int_{\left\{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)\right\} \cap \{v \geq y_n\}} (y_n)^\gamma \\ &\geq C_\delta^\gamma \int_{\left\{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)\right\} \cap \{v \geq y_n\}} \left(\frac{\delta}{2}\right)^\gamma \\ &= C_\delta \left(\frac{\delta}{2}\right)^\gamma \left|\left\{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\tfrac{3}{2}(e)\right)\right\} \cap \{v \geq y_n\}\right| \\ &\geq C_\delta \left(\frac{\delta}{2}\right)^\gamma \frac{\nu}{2}.\end{aligned}$$

Observe that for any $y \in \mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)$ we have $\frac{\delta}{2} \leq y_n \leq \frac{3}{2}$. Using the superior bound of y_n and the above inequality we have

$$\inf_{\mathcal{Q}_{\frac{3}{2}-\delta}\left(\frac{3}{2}(e)\right)} \frac{v(y)}{y_n} \geq \frac{\delta}{3} C_\delta \left(\frac{\nu}{2}\right)^{\frac{1}{\gamma}}.$$

The statement (4.29) follows by taking $k_\delta = \frac{\delta}{3} C_\delta (\frac{\nu}{2})^{\frac{1}{\gamma}}$.Thus we have proven the growth lemma for the interior of $\mathcal{Q}_1$.

For the proof on the flat portion of the boundary which contains $\{y_n = 0\}$ we first observe that, since v solves (4.26), by the inequalities in (4.24) satisfied by A and B,

$$\begin{aligned} 0 &\leq \int_{\mathcal{Q}_2} |A(y)\nabla v|^{p-2} B(y)\nabla v \nabla \Psi + \tilde{b}(y) \, |A(y)\nabla v|^{p-1} \Psi \, \mathrm{d}y \\ &\leq c \left(\int_{\mathcal{Q}_2} |\nabla v|^{p-2} B(y)\nabla v \nabla \Psi + \tilde{b}(y) \, |\nabla v|^{p-1} \Psi \, \mathrm{d}y \right) \\ &\leq c \left(\int_{\mathcal{Q}_2} |\nabla v|^{p-2} \mathcal{V} \nabla \Psi + \tilde{b}(y) \, |\nabla v|^{p-1} \Psi \, \mathrm{d}y \right) \quad \forall \ \Psi \in C_c^\infty \end{aligned}$$

where $\mathcal{V} := \mathcal{V}(y) = \sum_{i=1}^{n} \frac{\partial v}{\partial y_i} (1, 1, \dots, 1)$ for $\widetilde{Q}$ as in (4.27) ii). If $\widetilde{Q}$ behaves as (4.27) i) we have a similar inequality without the term of the gradient and considering the respective regularity of v. That is, v is also a weak solution of

$$-\operatorname{div}(|\nabla v|^{p-2} \mathcal{V}) + c\tilde{b}(y) \, |\nabla v|^{p-1} \geq 0 \quad \text{in} \ \mathcal{Q}_2. \tag{4.30}$$

Now for a fixed $\gamma > 0$ we introduce the smooth function

$$\eta : \left[-\frac{3-2c_1}{4}, \frac{3-2c_1}{4} \right]^{n-1} \to \mathbb{R}$$

such that

$$\eta(y_1, \dots, y_{n-1}) := \begin{cases} 0 \ \text{ if } \ (y_1, \dots, y_{n-1}) \in \left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \\ \frac{c_1}{m} \ \text{ if } \ (y_1, \dots, y_{n-1}) \in \partial \left(\left[-\frac{3-2c_1}{4}, \frac{3-2c_1}{4} \right]^{n-1} \right) \end{cases},$$

$$0 \leq \eta(y_1, \dots, y_{n-1}) \leq \frac{c_1}{m} \ \text{for} \quad (y_1, \dots, y_{n-1}) \in \left[-\frac{3-2c_1}{4}, \frac{3-2c_1}{4} \right]^{n-1}$$

where $m > 0$ is a constant that will be chosen later, with c_1 as chosen in the beginning of the proof and sufficiently small so that $|\nabla \eta| < \frac{1}{2n}$ (this last consideration implies trivially that $\sum_{k=1}^{n-1} \frac{\partial \eta}{\partial y_k} \neq 1$ which will be necessary to prove that the next auxiliary function is a sub-solution).

Consider the auxiliary function

$$\omega_\delta(y) = \frac{1}{\delta}\left(y_n - \eta(y_1,\ldots,y_{n-1})\right)^2 + \left(y_n - \eta(y_1,\ldots,y_{n-1})\right)$$

defined in [2]

$$\Omega_\delta := \Big\{(y_1,\ldots,y_n) \in \Big[-\frac{3-2c_1}{4}, \frac{3-2c_1}{4}\Big]^{n-1} \times \Big[0, \frac{c_1}{2}\Big] : \ \eta(y_1,\ldots,y_{n-1}) \le y_n \le \frac{\delta}{2}\Big\}.$$

Note that $0 \le y_n - \eta(y_1,\ldots,y_{n-1}) \le \frac{\delta}{2}$ in Ω_δ, and

$$\begin{aligned}
\frac{\partial \omega_\delta}{\partial y_i} &= -(\frac{2}{\delta}(y_n - \eta(y_1,\ldots,y_{n-1})) + 1)\frac{\partial \eta}{\partial y_i}, \qquad \text{for } i = 1,2,\ldots,n-1\\
\frac{\partial \omega_\delta}{\partial y_n} &= \frac{2}{\delta}(y_n - \eta(y_1,\ldots,y_{n-1})) + 1.
\end{aligned}$$

thus,

$$\begin{aligned}
&-\operatorname{div}(|\nabla\omega_\delta|^{p-2}\mathcal{V}_\delta) + c\,\|b\|_{L^\infty(\Omega_\delta)}|D\omega_\delta|^{p-1}\\
&= -(p-1)\frac{2}{\delta}\left(\frac{2}{\delta}(y_n-\eta)+1\right)^{p-2}\left(\sum_{i=1}^{n-1}\left(\frac{\partial\eta}{\partial y_i}\right)^2+1\right)^{\frac{p-2}{2}}\left(\sum_{k=1}^{n-1}\frac{\partial\eta}{\partial y_i}-1\right)^2\\
&\quad+\left(\frac{2}{\delta}(y_n-\eta)+1\right)^{p-1}\sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\left(\left(\sum_{i=1}^{n-1}\left(\frac{\partial\eta}{\partial y_i}\right)^2+1\right)^{p-2}\left(\sum_{k=1}^{n-1}\frac{\partial\eta}{\partial y_i}-1\right)\right)\\
&\quad+c\,\|b\|_{L^\infty(\Omega_\delta)}|D\omega_\delta|^{p-1}\\
&\le -c(p-1)\frac{2}{\delta}\left(\sum_{k=1}^{n-1}\frac{\partial\eta}{\partial y_i}-1\right)^2\\
&\quad+2^{p-1}\left|\sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\left(\left(\sum_{i=1}^{n-1}\left(\frac{\partial\eta}{\partial y_i}\right)^2+1\right)^{p-2}\left(\sum_{k=1}^{n-1}\frac{\partial\eta}{\partial y_i}-1\right)\right)\right|\\
&\quad+c\,\|b\|_{L^\infty(\Omega_\delta)}2^{p-1}\left(\sum_{i=1}^{n-1}\left(\frac{\partial\eta}{\partial y_i}\right)^2+1\right)^{\frac{p-1}{2}}.
\end{aligned}$$

By the boundness of $\tilde{b}(y)$ and the properties of η, we can assure that there exists $c_2 = c_2(n,p) \in (0, c_1]$ such that, for all $0 < \delta \le c_2$,

$$-\operatorname{div}(|\nabla\omega_\delta|^{p-2}\mathcal{V}_\delta) + c\,\|b\|_{L^\infty(\Omega_\delta)}|D\omega_\delta|^{p-1} \le 0.$$

[2]Note that, on the part of the flat boundary of $\mathcal{Q}_1$ in which $(y_1,\ldots,y_{n-1}) \in [-\frac{1}{2},\frac{1}{2}]^{n-1}$ the values of ω_δ above coincide with the auxiliary function proposed in [40]. The small difference on the auxiliary function from [40] is due to the degenerate\singular character of the p-Laplacian operator.

On the other hand, defining $v_\delta(y) = \frac{2v(y)}{k_\delta}$, since $\frac{2}{k_\delta} > 1$ we can compute that

$$-\operatorname{div}(|\nabla v_\delta|^{p-2}\mathcal{V}_\delta) + c\,\|b\|_{L^\infty(\Omega_\delta)}\,|Dv_\delta|^{p-1} \geq 0 \quad \text{in} \;\; \Omega_\delta.$$

Besides, by the above computations we observe that

- for $y_n = \frac{\delta}{2}$,
$$v_\delta(y) = \frac{2v(y)}{k_\delta} \geq \frac{2}{k_\delta}k_\delta y_n = \delta \geq \omega_\delta,$$
- for $y_n = 0$ and $(y_1, \ldots, y_{n-1}) \in [-\frac{1}{2}, \frac{1}{2}]^{n-1}$ we have that $\omega_\delta = 0$ and
$$v_\delta(y) \geq 0 = \omega_\delta,$$
- and for $y_n = 0$ and $(y_1, \ldots, y_{n-1}) \in \left([-\frac{1}{2}, \frac{1}{2}]^{n-1}\right)^c$, by taking $m \geq 2$ such that $\frac{c_1}{m} \leq \delta$ we have
$$v_\delta(y) \geq 0 \geq \omega_\delta.$$

It follows that $v_\delta(y) \geq \omega_\delta$ on $\partial\Omega_\delta$. Then, applying the appropriate comparison principle, it follows that for all $\delta \in (0, c_2]$, $v_\delta(y) \geq \omega_\delta$ in Ω_δ. In particular for $\delta = \frac{c_2}{2}$, in $\Omega_{\frac{c_2}{2}} \cap \mathcal{Q}_1$ (where $\eta = 0$) it holds

$$v(y) \geq \frac{1}{2}k_{\frac{c_2}{2}}\omega_{\frac{c_2}{2}}(y) = \frac{1}{2}k_{\frac{c_2}{2}}\left(\frac{2}{c_2}y_n^2 + y_n\right) \geq \frac{1}{2}k_{\frac{c_2}{2}}y_n$$

as we wanted to show. ∎

Lemma 4.11 *Assume the same hypotheses of the growth lemma and also that*

$$\inf_{\mathcal{Q}_1} \frac{v(y)}{y_n} \leq 1.$$

Then,
$$\left|\{v/y_n > M^j\} \cap \mathcal{Q}_1\right| \leq (1-\mu)^j,$$
for all $j \in \mathbb{N}$, for some $M > 1$ and some $\mu \in (0,1)$ which depend only on n.

To prove this lemma we need to recall the propagating Ink Spot Lemma ([48, Lemma 6]).

Lemma 4.12 *Let $A \subset B \subset \mathcal{Q}_1$ be two open sets. Assume there exists $\alpha \in (0,1)$ such that*
$$|A| \leq (1-\alpha)\,|\mathcal{Q}_1| \tag{4.31}$$
and for any $x_0 \in \mathcal{Q}_1$ such that $\mathcal{Q} = \mathcal{Q}_\rho(x_0) \subset \mathcal{Q}_1$, $\rho > 0$, we have

$$\textit{if } |\mathcal{Q} \cap A| \geq (1-\alpha)\,|\mathcal{Q}|, \quad \textit{then} \quad \mathcal{Q} \subset B. \tag{4.32}$$

Then,

$$|A| \le (1 - c_0\alpha)\,|B|\,,$$

for some constant $c_0 = c_0(n) \in (0,1)$.

Proof of Lemma 4.11. Let $0 < k < 1$ be the constant from Lemma 4.9 where $\alpha \in (0,1)$ will be chosen later. Fix $\gamma > 0$, c_1 and c_2 and set

$$M \ge \max\left\{\frac{1}{k}, \frac{4}{c_1}(1-\alpha)^{-\frac{1}{\gamma}}\right\}.$$

First, observe that replacing $v(y)$ by $kv(y)$ we have $\inf_{\mathcal{Q}_1} \dfrac{kv(y)}{y_n} \le k$. Then by Lemma 4.9

$$|\{v(y)/y_n > M\} \cap \mathcal{Q}_1| \le |\{kv(y) > y_n\} \cap \mathcal{Q}_1| \le \nu < 1 - \alpha.$$

Thus, the result is valid for $j = 1$ and $\mu \le \alpha$.

We fix $\mu = c_0\alpha$ where $c_0 < 1$ is the constant for Lemma 4.12. We also define the sets

$$\begin{aligned} A &= \{v(y)/y_n > M^j\} \cap \mathcal{Q}_1 \\ B &= \{v(y)/y_n > M^{j-1}\} \cap \mathcal{Q}_1. \end{aligned}$$

Since $M > 1$, it is obvious that $|B| \le |A|$ and

$$|A| = \left|\{v(y)/y_n > M^j\} \cap \mathcal{Q}_1\right| \le |\{v(y)/y_n > M\} \cap \mathcal{Q}_1| \le 1 - \alpha.$$

This bound for the measure of the set A is exactly the condition (4.31). In order to apply Lemma 4.12 let us fix a cube $\mathcal{Q} = \mathcal{Q}_\rho(x_0) \subset \mathcal{Q}_1$, such that

$$|\mathcal{Q} \cap A| \ge (1 - \alpha)\,|\mathcal{Q}| \tag{4.33}$$

and prove that $\mathcal{Q} \subset B$, that is, $u/x_n > M^{j-1}$.

Let us denote $y_0 = (y_0', y_{0,n})$ with $y_0' = (y_{0,1}, \ldots, y_{0,n-1}) \in \mathbb{R}^{n-1}$. We rescale the variables by setting

$$z = (z', z_n) = \frac{(y' - y_0', y_n)}{\rho'} \quad \text{where } \rho' := 2y_{0,n},$$

and

$$w(z) = \frac{v(y)}{\rho'} = \frac{1}{\rho'} v(y_0' + \rho' z', \rho' z_n), \;\; \tilde{\tilde{b}}(z) = \tilde{b}(y), \;\; \tilde{A}(z) = A(y), \;\; \tilde{B}(z) = B(y).$$

Then, w is a nonnegative solution of

$$-\operatorname{div}(\left|\widetilde{A}(z)\nabla w\right|^{p-2}\widetilde{B}(z)\nabla w)+\widetilde{\widetilde{b}}(z)\left|\widetilde{A}(z)\nabla w\right|^{p-1}\geq 0 \quad \text{in} \quad \mathcal{Q}_1.$$

in the cube $\mathcal{Q}_{\frac{2}{\rho'}}\left(\frac{-y_0'}{\rho'},\frac{1}{2\rho'}\right)\supset\mathcal{Q}_2$ for Q as in (4.2) $ii)$ (for $i)$ we can take $b\equiv 0$).

Moreover the condition (4.33) is equivalent to

$$\left|\{w(z)/z_n>M^j\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|\geq(1-\alpha)\left|\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|=(1-\alpha)\left(\frac{\rho}{\rho'}\right)^n \tag{4.34}$$

and we need to show that $w/z_n>M^{j-1}$ in $\mathcal{Q}_{\frac{\rho}{\rho'}}(e)$.

Now we separate the proof in two cases.

First case: Suppose $\rho\geq\frac{\rho'}{4}$.

Observe that,

$$\left|\{w(z)/M^j>z_n\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|\geq\left|\{w(z)/M^j>z_n\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|\geq(1-\alpha)\left(\frac{1}{4}\right)^n\geq\nu.$$

Thus Lemma 4.9 implies that $w(z)/M^j>kz_n$ in $\mathcal{Q}_1$. Since $\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\subset\mathcal{Q}_1$ by the definition of M we obtain $v(y)/y_n>M^{j-1}$ in $\mathcal{Q}_{\frac{\rho}{\rho'}}(e)$ as we wanted.

Second case: if $\rho<\frac{\rho'}{4}$

Since $M>0$, we have that v/M^j is also a super-solution for our problem. Then applying the WHI (Theorem 11) we have

$$\inf_{\mathcal{Q}_{\frac{\rho}{\rho'}}(e)}\frac{w(z)}{M^j}\geq C_1\left(\left(\frac{\rho}{\rho'}\right)^{-n}\int_{\mathcal{Q}_{\frac{\rho}{\rho'}}(e)}\left(\frac{w}{M^j}\right)^{\gamma}\mathrm{d}z\right)^{\frac{1}{\gamma}},$$

for γ as in that theorem (Theorem 11).

Note that in $\mathcal{Q}_{\frac{\rho}{\rho'}}(e)$, $\frac{1}{4}<z_n<1$. Using also (4.34) we deduce that

$$\left|\left\{w(z)/M^j>\frac{1}{4}\right\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|\geq(1-\alpha)\left(\frac{\rho}{\rho'}\right)^n.$$

Then,

$$\begin{aligned}
\inf_{\mathcal{Q}_{\frac{\rho}{\rho'}}(e)}\frac{w(z)}{M^j} &\geq C_1\left(\left(\frac{\rho}{\rho'}\right)^{-n}\int_{\left\{w(z)/M^j>\frac{1}{4}\right\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)}\left(\frac{w}{M^j}\right)^{\gamma}\mathrm{d}z\right)^{\frac{1}{\gamma}}\\
&\geq C_1\left(\left(\frac{\rho}{\rho'}\right)^{-n}\left(\frac{1}{4}\right)^{\gamma}\left|\left\{w(z)/M^j>\frac{1}{4}\right\}\cap\mathcal{Q}_{\frac{\rho}{\rho'}}(e)\right|\right)^{\frac{1}{\gamma}}\\
&\geq \frac{C_1}{4}(1-\alpha)^{\frac{1}{\gamma}}.
\end{aligned}$$

By the above result, the definition of M and since $z_n \leq 1$ in $\mathcal{Q}_{\frac{\rho}{\rho'}}(e)$

$$\begin{aligned} \frac{w(z)}{M^{j-1}} &\geq M\frac{C_1}{4}(1-\alpha)^{\frac{1}{\gamma}} \\ &\geq 1 \geq z_n. \end{aligned}$$

■

Proof of Theorem 17.
It is standard to deduce from Lemma 4.11 that there exist constants $\varepsilon = \varepsilon(n,p) > 0$ and $c = (n,p,\varepsilon)$, such that for all $t > 0$,

$$|\{v/y_n > t\} \cap \mathcal{Q}_1| \leq C \min\{1, t^{-2\varepsilon}\} \quad \text{for } t > 0.$$

Indeed (we give the proof for the reader's convenience), define the real non increasing function

$$f(t) = |\{v/y_n > t\} \cap \mathcal{Q}_1|.$$

Let M and μ be the constants obtained in Lemma 4.11 and set

$$C = \max\{(1-\mu)^{-1}, M^{2\varepsilon}\} > 1 \quad \text{and} \quad \varepsilon = -\frac{1}{2}\frac{\ln 1-\mu}{\ln M} > 0.$$

Then, if $t \in [0, M]$,

$$\begin{aligned} f(t) \leq 1 &\leq \max\{(1-\mu)^{-1}, M^{2\varepsilon}\}M^{-2\varepsilon} \\ &\leq C\min\{1, t^{-2\varepsilon}\}. \end{aligned}$$

Now, if $t > M > 1$, without lost of generality we can take $t \in [M^j, M^{j+1}]$, for some $j \in \mathbb{N}$, thus

$$\frac{\ln t}{\ln M} - 1 \leq j \leq \frac{\ln t}{\ln M}.$$

By the monotonicity of f and Lemma 4.11 we have

$$f(t) \leq f(M^j) \leq (1-\mu)^j,$$

also, since $1-\mu < 1$,

$$f(t) \leq (1-\mu)^{\ln t/\ln M - 1}.$$

Observe that,

$$\begin{aligned}\ln(1-\mu)^{\ln t/\ln M-1} &= \left(\frac{\ln t}{\ln M}-1\right)\ln(1-\mu)\\ &= \ln t\frac{\ln(1-\mu)}{\ln M}-\ln(1-\mu)\\ &\leq -2\varepsilon\ln t+\ln C\\ &= \ln\left(Ct^{-2\varepsilon}\right),\end{aligned}$$

thus,

$$\begin{aligned}(1-\mu)^{\ln t/\ln M-1} &\leq Ct^{-2\varepsilon}\\ &= C\min\left\{1,t^{-2\varepsilon}\right\} \quad \text{if } t\geq 1.\end{aligned}$$

So the claim is proved.

Hence, using [16, Lemma 9.7 i],

$$\begin{aligned}\int_{\mathcal{Q}_1}\left(\frac{v(y)}{y_n}\right)^{\varepsilon}\mathrm{d}y &= \varepsilon\int_0^{\infty}t^{\varepsilon-1}\left|\left\{\frac{v(y)}{y_n}>t\right\}\cap\mathcal{Q}_1\right|\mathrm{d}t\\ &\leq C\varepsilon\int_0^{\infty}t^{\varepsilon-1}\min\{1,t^{-2\varepsilon}\}\,\mathrm{d}t\\ &\leq C.\end{aligned}$$

For each $\beta>0$ we introduce the function

$$\widetilde{v}=\frac{v}{\inf_{\mathcal{Q}_1}\frac{v(y)}{y_n}+\beta}$$

and conclude by applying the above inequality to $\widetilde{v}$ and by letting $\beta\to 0$. ■

Remark 4.13 *As we observed previously, the straightening of the boundary remains valid for a more general equation as long as the boundary of the domain is smooth. The only change needed in the extension up to the boundary of the WHI appears in the proof of the fact that the function is a weak super-solution of the equation* (4.30)*. Thus, we can obtain the estimate up to the boundary for a super-solution of*

$$-\operatorname{div}A(x,u,Du)+B(x,u,Du)=0 \quad \textit{in } \Omega$$

with $\partial\Omega\in C^1$ if we impose in addition to the structural conditions, which are required for the interior estimates, the following condition

$$\widetilde{A}(y,v,Dv)\leq c\,|\nabla v|^{p-2}\,\mathcal{V}(y)$$

with $\mathcal{V}$ a vector only depending on y, after the change of variables resulting from the straightening.

4.3.3 Proof of the boundary quantitative strong maximum principle(BQSMP)

The proof follows directly from the following lemma.

Lemma 4.14 *The weak solution of*

$$\begin{cases} -Q\omega & = \chi_{B_{1/2}} \quad \text{in } B_1 \\ \quad \omega & = 0 \qquad \text{on } \partial B_1. \end{cases} \tag{4.35}$$

with Q as in (4.2) is such that

$$\omega \geq c_0 d$$

where $\chi_{B_{1/2}} = \begin{cases} 1 & \text{in } B_{1/2} \\ 0 & \text{on } B_1 \setminus B_{1/2}, \end{cases}$ *represent the characteristic function of the subset $B_{1/2}$, $d = \mathrm{dist}(x, \partial B_1)$ and $c_0 := c_0(n,p) > 0$.*

Proof of Lemma 4.14. Suppose that the conclusion fails, this is, there exist a sequence of functions ω_k and a sequence of points $y_k \in B_1$ with $\dfrac{\omega_k(y_k)}{d(y_k)} \to 0$ as $k \to \infty$,where ω_k is a weak solution of

$$\begin{cases} -Q\omega_k & = \chi_{B_{1/2}} \quad \text{in } B_1 \\ \quad \omega_k & = 0 \qquad \text{on } \partial B_1. \end{cases} \tag{4.36}$$

A sub-sequence of y_k converges, to a point which may be in the interior or on the boundary of B_1.

<u>Case I</u>: y_k converges to an interior point y_0 in B_1.

Observe that by the $C^{1,\alpha}$–estimates of the p-Laplacian ([49, 50, 51, 52]) we know that ω_k is bounded in $C^{1,\alpha}(B_1)$ for some $0 < \alpha < 1$. Therefore we can extract a sub-sequence, still denoted by ω_k, which converges to a function ω in $C^{1,\alpha}(B_1)$, with $0 < \alpha < 1$. Furthermore by the interior WHI in B_2 we have that there exists a constant $c > 0$ that for any $\gamma > \frac{n(p-1)}{n-p}$ if $p \leq n$ and $\gamma \leq \infty$ if $p > n$,

$$\int_{B_1} \omega_k \leq c \left(\inf_{B_1} \omega_k \right)^{\gamma} \leq c \left(\omega_k(y_k) \right)^{\gamma}.$$

Passing to the limit when $k \to \infty$, we obtain that $\omega \equiv 0$ in B_1 since $\omega_k \geq 0$ in B_2, but this contradicts that ω satisfies (4.35).

<u>Case II</u>: y_k converges to a point y_0 in ∂B_1.

Without lost of generality take $y_0 = e := (0, \ldots, 0, 1)$, that is (according to (4.35)),

$$\frac{\partial \omega_k}{\partial e}(e) \to 0 \ \text{ as } \ k \to \infty. \tag{4.37}$$

Take ϕ_k a sequence of weak solutions of the problem

$$\begin{cases} -Q\phi_k & = 0 \quad \text{in } \ B_{1/2}(e/2) \\ \quad \phi_k & = \psi_k \ \ \text{on } \ \partial B_{1/2}(e/2), \end{cases}$$

where ψ_k is a sequence of functions with the same regularity of ω_k in $B_{1/2}(e/2)$ such that $0 \le \psi_k \le \omega_k$ on $\partial B_{1/2}(e/2)$, in particular

$$\begin{aligned} \psi_k &= 0 \quad \text{on } \ \partial B_{1/2}(e/2) \cap \{x : \ x_n > 3/4\} \\ \psi_k &= \omega_k \quad \text{on } \ \partial B_{1/2}(e/2) \cap \{x : \ x_n < 1/2\}, \end{aligned}$$

and $\|\psi_k\|_{C^{1,\alpha}(B_{1/2}(e/2))} \le c \, \|\omega_k\|_{C^{1,\alpha}(B_{1/2}(e/2))}$. Thus,

$$\begin{aligned} -Q\phi_k &= 0 \le -Q\omega_k \qquad \text{in } \ B_{1/2}(e/2) \\ \phi_k &\le \omega_k \qquad\qquad\quad \text{on } \ \partial B_{1/2}(e/2). \end{aligned}$$

Applying the appropriate comparison principle according to the expression of Q we obtain that $\phi_k \le \omega_k$ in $B_{1/2}(e/2)$.

Also, by the interior QSMP, and the boundary condition on ϕ_k

$$\phi_k = \omega_k \ge c_0 \ \text{ on } \ \partial B_{1/2}(e/2) \cap \{x : \ x < 1/2\}.$$

Now using the $C^{1,\alpha}$–estimates of the p-Laplacian ([49, 50, 51, 52]) on ϕ_k we have that there exists a constant $c > 0$ such that $\|\phi_k\|_{C^1(B_{1/2}(e/2))} \le c$, thus there exists $d_0 > 0$ such that

$$\phi_k \ge c_0/2 \ \text{ in } \ B_{1/2}(e/2) \cap \{x : \ x_n < 1/4\} \cap \Big\{x : \ \operatorname{dist}(x, B_{1/2}(e/2)) < d_0\Big\}.$$

Hence, by applying the interior WHI in $B_{\frac{1-d_0}{2}}(e/2)$ we have that

$$\inf_{B_{\frac{1-d_0}{2}}(e/2)} \phi_k \ge c_1 > 0,$$

and in particular, $\phi_k(e/2) \ge c_1 > 0$.

On the other hand, since $\phi_k(e) = 0 \le \omega_k(e)$, it is obvious that

$$-\frac{\omega_k(e + te) - \omega_k(e)}{t} \ge -\frac{\phi_k(e + te) - \phi_k(e)}{t}$$

for all small $t < 0$.

Using the boundary Harnack inequality (4.15) and the fact that $\phi_k(e)$=0, we have

$$-\frac{\omega_k(e+te)-\omega_k(e)}{t} \geq c_2\phi_k(e/2) \geq c_2c_1 > 0 \quad \text{for all } t < 0,$$

and this means that,

$$\frac{\omega_k(e+te)-\omega_k(e)}{t} < 0,$$

which directly contradicts (4.37). Therefore $\omega \geq c_0 d$ and the lemma is proved. ∎

Now we are able to obtain the QSMP up to the boundary.

Proof of the BQSMP, Theorem 7. First, consider that u is a nonnegative weak super-solution of

$$-Qu \geq \chi_A \quad \text{in } \Omega,$$

with Ω an smooth domain and A a subset of Ω with positive measure. Then we are going to prove that there exists a constant $c := c(n,p) > 0$, such that, $\inf\limits_{\Omega} \dfrac{u}{d} \geq c$.

By simplicity we choose $\Omega = B_1$ and $A = B_{1/2}$. Now let ω the weak solution of the Dirichlet problem (4.35). Hence we have

$$\begin{cases} -Qu & \geq \chi_{B_{1/2}} = -Q\omega & \text{in } B_1 \\ \quad u & \geq \omega & \text{on } \partial B_1. \end{cases}$$

Then, using the appropriate comparison principle in B_1 we obtain that $u \geq \omega$. Then the conclusion in B_1 follows by Lemma 4.14.

To conclude in B_2^+ it is sufficient to consider an smooth domain Ω such that $\Omega \subset B_2^+$ and $A \subset B_1^+$. ∎

4.3.4 Proof of the boundary local maximum principle (BLMP)

Before giving the proof, we want to observe that the constant ε in the expression of β in the case of $p > n$ in the theorem is only needed in order to guarantee the hypothesis in [41] for the existence of the solution of (4.10). The interior version of the theorem is satisfied imposing a weak Lebesgue regularity on $c(x)$ as we enunciated in Theorem 12.

Proof of the BLMP, Theorem 8. Since $u^+ := \max\{u, 0\}$ by Lemma 4.3 u^+ also satisfies (4.10). Then extending u^+ as zero in $B_2 \setminus B_2^+$ we get a sub-solution for the problem in B_2.

Applying first the LMP (Theorem 12) and then the Lipschitz bound for the p-Laplacian (Theorem 15) with an appropriate rescaling of the radii as we describe in the Remark 4.4, we obtain the desired result. ■

4.4 The uniform estimates

In this last section we are going to obtain the main a priori estimates for non coercive equations. Although all the results have been stated by assuming the smoothness of the whole boundary we want to highlight that if we assume a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with only a portion of the boundary being $C^{1,1}$, each one of the results is satisfied in a bounded domain Ω' such that $\overline{\Omega'} \subset \Omega \cup T$, where T denotes the $C^{1,1}$–smooth relative open portion of the boundary of Ω.

4.4.1 Proof of the Lebesgue estimate, Theorem 4

By locally straightening the boundary we are going to prove the L^{ε_0}–estimate for a nonnegative weak solution v of

$$\widetilde{Q}v \geq \widetilde{f}(y,v) \quad \text{in } \mathcal{Q}_2, \tag{4.38}$$

which implies directly the desired estimate (Theorem 4), by straightening the boundary of Ω. We denote $\widehat{K}$ the image of K by the diffeomorphism.

Proof of the L^{ε_0}–estimate in cubes. In the following we write $y{=}(y',y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote the cube $\mathcal{Q}_R = \mathcal{Q}'_R \times (0,R)$ where $\mathcal{Q}'_R = \{-R/2 < y' < R/2\} \subset \mathbb{R}^{n-1}$.

Denote $I_v = \inf_{\mathcal{Q}_1} \frac{v(y',y_n)}{y_n}$, thus we have $v \geq I_v y_n$ for all $y \in \mathcal{Q}_1$. First observe that by the non negativity of v, I_v is well defined.

Take $\widehat{\mathcal{Q}}$ a cube in $\widehat{K} \subset \mathcal{Q}_1$ such that the side of $\widehat{\mathcal{Q}}$ is larger than $\varepsilon_1 > 0$ and $\operatorname{dist}(\partial\widehat{\mathcal{Q}}, \partial\mathcal{Q}_1) > \varepsilon_1$, where ε_1 is a positive constant which depends only on the measure of K.

By the superlinearity condition on $\tilde{f}$, then $\inf_{s\in[A,\infty)} \tilde{f}(y,s) > 0$ for all $y \in \widehat{\mathcal{Q}}$, for some $A \in \mathbb{R}$, $A \geq 1$. If $I_v \leq A/\varepsilon_1$ we can conclude after applying the boundary weak Harnack inequality (Theorem 6). Then let us assume in the following that $I_v \geq A/\varepsilon_1$. Since $I_v > 0$ for all $y \in \widehat{\mathcal{Q}}$ we have that $A \leq \varepsilon_1 I_v < I_v y_n < I_v$. Thus we can observe that v also satisfies

$$\widetilde{Q}v \geq \inf_{s\in[y_n I_v,\infty)} \tilde{f}(y,s)\chi_{\widehat{\mathcal{Q}}} \geq \begin{cases} 0, \text{ in } \mathcal{Q}_1 \setminus \widehat{\mathcal{Q}} \\ \inf_{y\in\widehat{\mathcal{Q}}, s\in[y_n I_v,\infty)} \tilde{f}(y,s) > 0. \end{cases}$$

Now, by dividing the last inequality by $\inf_{y\in\widehat{\mathcal{Q}},s\in[y_nI_v,\infty)}\tilde{f}(y,s)$, we obtain that

$$\widetilde{Q}\left(\frac{v}{\left(\inf_{y\in\widehat{\mathcal{Q}},\,s\in[y_nI_v,\infty)}\tilde{f}(y,s)\right)^{\frac{1}{p-1}}}\right)\geq\chi_{\widehat{\mathcal{Q}}}\quad\text{in}\quad\mathcal{Q}_1.$$

Then, applying the BQSMP (Theorem 7) to this inequality, we obtain that there exists a constant $c:=c(n,p)$ such that

$$I_v\geq c\left(\inf_{y\in\widehat{\mathcal{Q}},\,s\in[y_nI_v,\infty)}\tilde{f}(y,s)\right)^{\frac{1}{p-1}},\tag{4.39}$$

and so, since for all $y\in\widehat{\mathcal{Q}},\ \varepsilon_1<y_n<1$,

$$\inf_{y\in\widehat{\mathcal{Q}}\ s\in[\varepsilon_1I_v,\infty)}\tilde{f}(y,s)\leq c(I_v)^{p-1}=c\varepsilon_1^{1-p}(\varepsilon_1I_v)^{p-1}.$$

Then by the superlinearity condition on $\tilde{f}$ we have that I_v is bounded by above. On the other hand, by the BWHI (Theorem 6) we have that v is also bounded from below by an integral which depends on v, specifically

$$\left(\int_{\mathcal{Q}^\star}\left(\frac{v}{y_n}\right)^{\varepsilon_0}\right)^{\frac{1}{\varepsilon_0}}\leq cI_v\leq c$$

where $\mathcal{Q}^\star$ is such that $\mathcal{Q}_1\subset\mathcal{Q}^\star\subset\mathcal{Q}_2$. ■

4.4.2
Proof of the L^∞–estimate, Theorem 5

similar to the previous subsection we are going to prove the L^∞–estimate in cubes, with the appropriate change in the inequalities of the Theorem 5.
Proof. Let $C,\ s_1\geq s_0$ be such that

$$s_0\leq\xi(s)\leq Cs^\alpha,\quad\text{if } s\geq s_1.$$

Since $s_1\geq s_0$ is a fixed number, we have that $\xi(s_1)$ satisfies the inequality (4.6). Then, by Lemma 4.3 the function $\tilde{v}=\max\{v,s_1\}$ is also a weak sub-solution of this inequality. Set

$$c(x)=\frac{\tilde{g}^+(x,\xi(\tilde{v}))}{\xi^{p-1}(\tilde{v})},\tag{4.40}$$

then, by the assumptions on (g) and the condition (4.4) on ξ we have

$$
\begin{aligned}
c(x) \leq \frac{b_0}{d^\gamma}\frac{(1+\xi^m(\tilde v))}{\xi^{p-1}(\tilde v)} \leq \frac{b_0}{d^\gamma}\left(\left(\frac{1}{s_0}\right)^{p-1} + \xi^{m-(p-1)}(\tilde v)\right) \\
\leq \frac{b_0}{d^\gamma}\left(\left(\frac{1}{s_0}\right)^{p-1} + \tilde v^{\alpha(m-(p-1))}\right),
\end{aligned}
$$

so, by the values of γ and m in (g), $c(x) \in L^\beta$ with $\beta = \begin{cases} \frac{n}{p(1-\varepsilon)}, & 1 < p \leq n \\ 1, & p > n \end{cases}$,

$\varepsilon \in (0,1)$.

Thus, $\xi(\tilde v)$ is a weak sub-solution of

$$
-\widetilde{Q}(\xi(\tilde v)) - c(x)\,|\xi(\tilde v)|^{p-2}\,\xi(\tilde v) \leq 0 \quad \text{in} \quad \mathcal{Q}_2,
$$

so we can apply the BLMP, Theorem 8. Also, by (4.4) and the previous L^{ε_0}–estimate we can deduce

$$
\begin{aligned}
\sup_{\mathcal{Q}_1} \frac{\xi^+(\tilde v)}{d} &\leq c\left(\int_{\mathcal{Q}^\star} \xi^+(\tilde v)^r\right)^{\frac{1}{r}} \\
&\leq c\left(\int_{\mathcal{Q}^\star} (\tilde v^+)^{\varepsilon_0}\right)^{\frac{1}{\varepsilon_0}} \leq c,
\end{aligned}
$$

where $\mathcal{Q}^\star$ is such that $\mathcal{Q}_1 \subset \mathcal{Q}^\star \subset \mathcal{Q}_2$.

We conclude that $\frac{\xi(\tilde v)}{d}$ is bounded in $\mathcal{Q}_1$ which implies that $\left\|\frac{\xi(v)}{d}\right\|_{L^\infty(\mathcal{Q}_1)} \leq c$ and $\|v\|_{L^\infty(\mathcal{Q}_1)} \leq c$ by the monotonicity of ξ and the definition of $\tilde v$. The theorem is proved. ■

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