Pontifícia Universidade Católica
do Rio de Janeiro

Lucas Barbosa Gama

Quasi-periodicity and the positivity of Lyapunov exponents

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–Graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática.

Advisor: Prof. Silvius Klein

Rio de Janeiro
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Abstract

Barbosa Gama, Lucas; Klein, Silvius (Advisor). **Quasi-periodicity and the positivity of Lyapunov exponents**. Rio de Janeiro, 2018. 65p. Dissertação de Mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

The Benedicks and Carleson theorem states that for the quadratic family there exists a set of parameters, with positive measure, for which the Lyapunov exponent is positive at the critical point. In this dissertation we present a rigorous and detailed proof of this famous result. An important part of the proof is the study of the quasi periodic behavior of a set of orbits. In addition, a large deviation argument is used to show that parameters which do not satisfy the desired property form a small set. Such techniques have an intrinsic interest, as they have proven fruitful in the study of other problems in dynamical systems. Combining Benedicks-Carleson's theorem with Singer's theorem, we conclude that for a set of parameters with positive measure, the corresponding quadratic function does not admit periodic attractors, indicating its chaotic behavior. In this work we also study criteria for the positivity of the Lyapunov exponent of quasi-periodic Schrödinger cocycles, such as Herman's theorem. The study of the Schrödinger cocycles represents an important topic in mathematical physics. Moreover, some of the generalizations of such criteria use the techniques of Benedicks-Carleson.

Keywords

The quadratic family; Singer's theorem; Benedicks-Carleson's theorem; Lyapunov exponents; Large deviations; Quasi-periodic Schrödinger cocycles.

Resumo

Barbosa Gama, Lucas; Klein, Silvius. **Quase periodicidade e a positividade dos expoentes de Lyapunov**. Rio de Janeiro, 2018. 65p. Dissertação de Mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

O teorema de Benedicks e Carleson afirma que para a família quadrática existe um conjunto de parâmetros, com medida positiva, para os quais o expoente de Lyapunov é positivo no ponto crítico. Nesta dissertação apresentamos uma demonstração rigorosa e detalhada desse célebre resultado. Uma parte importante da demonstração é o estudo do comportamento quase periódico de um conjunto de órbitas. Além disso, um argumento de grandes desvios é utilizado para mostrar que os parâmetros que não satisfazem a propriedade desejada formam um conjunto pequeno. Tais técnicas apresentam um interesse intrínseco, já que têm se mostrado muito proveitosas para o estudo de outros problemas em sistemas dinâmicos. Combinando o teorema de Benedicks e Carleson ao teorema de Singer, conclui-se que para um conjunto de parâmetros com medida positiva, a função quadrática correspondente não admite atratores periódicos, indicando um comportamento caótico. Neste trabalho, também são estudados critérios para a positividade do expoente de Lyapunov de cociclos quase periódicos de Schrödinger, como o teorema de Herman. O estudo de cociclos de Schrödinger representa um importante tópico na área de física matemática. Mais ainda, algumas das generalizações de tais critérios utilizam as técnicas de Benedicks-Carleson.

Palavras-chave

A família quadrática ; O teorema de Singer; O teorema de Benedicks-Carleson; Expoentes de Lyapunov; Grandes desvios; Cociclos de Schrödinger quase periódicos.

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"In every thing give thanks: for this is the will of God in Christ Jesus concerning you."

The Holy Bible, 1 Thessalonians 5:18.

1
Introduction

The beginning of the Dynamical Systems theory can be attributed to Poincaré, who studied the three-body problem and had the idea of describing certain properties of the solutions of differential equations rather than explicitly finding them. It is natural, for instance, to try to identify recurrent trajectories, instability, sensitivity to initial conditions, etc. Concepts like these initiated the Chaos Theory.

In this thesis we study the Lyapunov exponent of certain types of dynamical systems. In the simplest situation, the Lyapunov exponent can be understood as follows.

Let $I \subset \mathbb{R}$ be an interval, let $f\colon I \to I$ be a differentiable function and let $x_0 \in I$ be a point. Consider the orbit of this point under the map f, that is, let $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\ldots$, $x_n = f(x_{n-1})\ldots$ Then we may take δx_0 to be a small deviation around x_0 and, since f is continuous, we could think of $\delta x_1 = \delta x_0 f'(x_0)$ as a small deviation around $x_1 = f(x_0)$. Inductively we get

$$\delta x_n = \delta x_0 \prod_{i=0}^{n} f'(x_i)\,.$$

Putting

$$L_n := \frac{1}{n} \log \left| \frac{\delta x_n}{\delta x_0} \right| = \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|\,,$$

the limit

$$L := \lim_{n\to\infty} L_n$$

is called the Lyapunov exponent of f at x_0.

Although this definition is not very general, it does provide some intuition on the nature of the concept of Lyapunov exponent. For example, if the Lyapunov exponent is positive, it means that δx_n is growing very quickly, that is, the initially small deviation δx_0 is growing rapidly as we iterate f, which essentially agrees with our intuition about chaos as sensitivity to initial conditions.

For most dynamical systems, certain types of behavior are difficult to describe even for simple sets of points in the phase space. A commonly used method is to study such properties in a probabilistic way.

In 1976, Hénon proposed a very simple two-dimensional system with interesting chaotic behaviour. This model was related to the quadratic map $f_a(x) = 1 - ax^2$ studied in this dissertation (but with an extra variable). It was also studied at the same time by Robert May, who suggested the quadratic family as a model for population dynamics.

Even though the maps of the quadratic family are very simple, establishing the positivity of the Lyapunov exponent at the critical point for a non-trivial set of parameters turns out to be an extremely technical problem. It was first solved by Benedicks and Carleson [1], [2] in 1985. We formulate their result below.

Theorem 1.0.1 *(Benedicks and Carleson) Let $f_a\colon [-1,1] \to [-1,1]$ be the quadratic family, i.e., $f_a(x) = 1 - ax^2$, where a is a parameter. There exists a positive measure set of parameters a contained in $(0,2]$, for which the Lyapunov exponent at the critical point $c = 0$ is positive.*

A detailed exposition of their proof, based in part on the approach in [3] is the first objective of this thesis. In the demonstration, we will evaluate the frequency of returns of the orbit to an interval close to zero. In addition, we will use a large deviations type argument to show that the set of points which do not satisfy the desired property has small Lebesgue measure. These kinds of arguments have proved fruitful elsewhere in dynamical systems.

Besides Benedicks-Carleson's theorem, other interesting results were obtained for the quadratic family. We mention the existence of absolutely continuous invariant measures with respect to the Lebesgue measure (Collet and Eckmann [4]), and the density of uniformly hyperbolic maps in a parameter interval (Lyubich [5]).

As we can see, despite its simplicity, this family gives us a very rich dynamics, which is already a good motivation to study it. Moreover, the quadratic family is a global transversal to the foliation space of non-regular smooth unimodal maps into topological classes ([5]). This fact has motivated the attempt to extend such results to more generic classes of functions.

In this thesis we also present Singer's theorem, which says that a certain type of attractor, the periodic sink of a function $f \in C^3([0,1])$ with negative Schwarzian derivative and finitely many critical points, must attract some critical point ([6]).

We recall that a periodic sink is a k-periodic point p such that the derivative of the k-th iterate of f at p has modulus less than 1.

Moreover, the Schwarzian derivative of a function f is defined as

$$\{f, x\} := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

We will see that if a point x is attracted by a periodic sink of a transformation f, then the Lyapunov exponent of f at x is less than or equal to zero. Thus for a positive measure set of parameters the quadratic family of transformations (which clearly satisfy the assumptions of Singer's theorem) does not have periodic sinks.

Theorem 1.0.2 *(Singer) Let f be a $C^3([0,1])$ function with finitely many critical points and negative Schwarzian derivative. If p is a periodic sink of f, then there exists a critical point c of f whose ω-limit is the orbit of p. In particular, f admits at most finitely many periodic sinks.*

We will also define the concept of Lyapunov exponent for linear cocycles (following [7]), which are skew-product maps acting on a vector bundle. An important example of a linear cocycle, on which we will focus in this thesis, is the quasi-periodic Schrödinger cocycle.

This type of cocycle plays an important role in mathematical physics as its iterates represent the transfer matrices associated to a discrete Schrödinger equation, while the corresponding Schrödinger operator is the discretized version of a quantum Hamiltonian.

More precisely, given a transformation T in X, a bounded observable $\phi\colon X \to \mathbb{R}$, an energy parameter $E \in \mathbb{R}$ and a coupling constant $\lambda > 0$, a Schrödinger cocycle is the pair $(T, A_{E,\lambda})$, where

$$A_{E,\lambda}(x) := \begin{bmatrix} \lambda(\phi(x) - E) & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.1}$$

When the phase space X is the torus $\mathbb{T}$ and the transformation T is an irrational translation, we call the corresponding linear cocycle quasi-periodic.

In this thesis, we present a proof of Herman's theorem ([8]), which gives a positive lower bound on the Lyapunov exponent of such an operator when the observable $\phi(x) = \cos(x)$. [1]

[1]We note that the Schrödinger operator given by this particular observable is called the almost Mathieu operator, whose study has been a very important and active research area.

Theorem 1.0.3 *(Herman) If $\phi(x) = \cos(x)$, then for all $E \in \mathbb{R}$ and $\lambda > 0$, the Lyapunov exponent $L(A_{E,\lambda})$ of the quasi-periodic Schrödinger cocycle* (1.1) *has the following lower bound:*

$$L(A_{E,\lambda}) \geq \log \frac{\lambda}{2}\,.$$

Therefore, a coupling constant greater than 2 *is sufficient to have the positivity of the Lyapunov exponent.*[2]

Herman's theorem was extended to more general observables ϕ (see [7] for a summary of related results). One such extension, due to Bjerklöv [10], uses an argument inspired by the proof of the Benedicks-Carleson theorem.

We believe that the methods presented in this dissertation may prove useful in the future, as we intend to study the quasi-periodic Schrödinger cocycle with different types of observables, to obtain other, more general versions of Herman's theorem.

The rest of this dissertation is organized as follows. In the first chapter we discuss the Benedicks and Carleson theorem, dividing the main technical results into the case of individual parameters and that of intervals of parameters. In Chapter 2 we present the proof of Singer's theorem and its application to the quadratic family. Finally, in the last chapter we present the proof of Herman's theorem.

[2]By work of Bourgain [9], this is also necessary.

2
The Benedicks-Carleson theorem

In this chapter, we formulate and present a proof of the Benedicks-Carleson theorem on the positivity of the Lyapunov exponent of the quadratic family, for a positive measure set of parameters.

The proof of this result first appeared in their papers [1] and [2]. Besides the original articles, we also followed the argument presented in [3] for a more general setting, namely that of Misiurewicz maps.

2.1
Definitions, statements and related results

We begin with the definition of the Lyapunov exponent of an interval map.

Definition 2.1.1 *The Lyapunov exponent of a map $f\colon [-1,1] \to [-1,1]$ at a point x is defined as*

$$L(f,x) = \liminf_{n\to\infty} \frac{1}{n} \log \left| (f^n)'(x) \right| .$$

By the chain rule we have that

$$\begin{aligned} L(f,x) &= \liminf_{n\to\infty} \frac{1}{n} \log \left| (f^n)'(x) \right| \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \left| f'(f^{n-1}x) \dots f'(f(x))\, f'(x) \right| && (2.1a) \\ &= \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left| f'(f^i x) \right| . && (2.1b) \end{aligned}$$

We will study the quadratic functions $f_a(x) = 1 - ax^2$ for parameters $a \in (0,2]$. Note that $c = 0$ is the only critical point of f_a, for each parameter a, and the corresponding critical value is $f_a(0) = 1$.

We are interested in the Lyapunov exponent of f_a at the critical value[1] $c_1 = 1$ and use the shorter notation $L(f_a)$ instead of $L(f_a, 1)$.

The Benedicks-Carleson theorem states the following.

[1]It would not be interesting to consider it at the critical point $c = 0$, since by the chain rule, $(f_a^n)'(0) = 0$.

Theorem 2.1.1 (Benedicks-Carleson) *Let $f_a \colon [-1,1] \to [-1,1]$ be the quadratic family, i.e., $f_a(x) = 1 - ax^2$, where a is a parameter. There exists a set $E \subset (0,2]$ of positive Lebesgue measure such that for every parameter $a \in E$, the Lyapunov exponent $L(f_a) > 0$.*

In fact we will prove a stronger statement. There are constants $\gamma > 0$ and $C > 0$ and there exists a set $E \subset (0,2]$ such that 2 is a Lebesgue density point of E and

$$|(f_a^n)'(1)| \geq C\, e^{\gamma\, n} \quad \text{for all } n \in \mathbb{N} \ \text{ and } \ a \in E\,. \tag{2.2}$$

The proof is divided into two main parts. In the first, we make certain assumptions on individual parameters that essentially guarantee the exponential growth in (2.2). In the second part, we "thicken" the "good" parameters previously defined, by considering small intervals of parameters with uniform behaviour. We then use a large deviations type argument to ensure the existence of a set of good parameters with a density point (thus of positive measure).

We note that an important related result was previously obtained by Jakobson in 1981. Jakobson's theorem motivated the work of Benedicks and Carleson. In order to formulate it, we need another definition.

Definition 2.1.2 *Let $I \subset \mathbb{R}$ an interval. A measure μ in I is absolutely continuous with respect to the Lebesgue measure λ and we write $\mu \ll \lambda$, if for every measurable set A,*

$$\lambda(A) = 0 \Rightarrow \mu(A) = 0.$$

Equivalently, $\mu \ll \lambda$ if there exists $\phi \in L^1(\mu)$ such that $d\mu = \phi d\lambda$.

Jakobson proved the existence of invariant absolutely continuous measures (with respect to Lebesgue) for the same family of functions.

Theorem 2.1.2 (Jakobson) *Let $f_a : [-1,1] \to [-1,1]$ be the quadratic family. There exists a set $E \subset (0,2]$ of positive Lebesgue measure such that for every parameter $a \in E$, there exists an invariant measure μ for f_a which is absolutely continuous with respect to the Lebesgue measure.*

It is well known (see [4]) that the positivity of the Lyapunov exponent of a map implies the existence of an absolutely continuous invariant measure. More recently, Ávila, de Melo and Lyubich (see [11]) proved that the reverse is also true.

All throughout the proof we will be using the following notation.

Definition 2.1.3 *The relation $a \approx b$ is defined by the following condition: there exist universal constants a_0, b_0 such that $a_0 b \leq a \leq b_0 b$.*

2.2 The intuitive idea of the proof

In the first part of the proof we will make some assumptions on individual parameters and prove that these assumptions are sufficient to ensure the exponential growth in (2.2).

Given a parameter a and an integer n, for brevity we denote

$$\xi_n(a) = f_a^n(0).$$

The goal is to introduce some natural conditions on a parameter a (chosen to be close enough to 2) that guarantee at time n the exponential growth

$$EX_n : |(f_a^j)'(1)| \geq C\, e^{\gamma j} \text{ for all } j \leq n.$$

Since $f_a'(x) = -2ax$, by the chain rule we have

$$|(f_a^j)'(1)| = |f_a'(f_a^{j-1}(1)) \dots f_a'(1)| = 2a|\xi_j(a)| \dots 2a|\xi_1(a)|\,.$$

It is then natural to examine the pattern of returns of the orbit $\{\xi_n(a)\}_{n\geq 1}$ to a small interval around 0.

Since the equation $\xi_n(a) = 0$ is polynomial in a, its solutions form a finite, hence negligible set. Therefore, after excluding a zero measure set of parameters a, we may assume that

$$\xi_n(a) \neq 0 \quad \text{for all } n \geq 1.$$

We impose a quantitative version of the above property. Namely, fixing some small constant $\alpha > 0$, for an integer n,

$$BA_n : |\xi_j(a)| > e^{-\alpha j} \text{ for every } j \leq n.$$

This is called the **Basic Assumption** on the parameter a.

We fix a large integer R and when $\xi_n(a) \in (-e^{-R}, e^{-R})$ we call the index n a **return**.

After the first return n_1, which we call **free**, there will be a period p_1, called **binding period**, such that the orbit will behave approximately as in the beginning. More than that, we will require some uniformity in the interval $(-e^{-r}, e^{-r})$, where r is the largest integer for which $\xi_{n_1}(a) \in (-e^{-r}, e^{-r})$.

The next return after this period is the second **free return**, n_2.

We repeat this process until the last free return before a fixed integer n.

Thus between any two free returns there is a binding period and we will quantify the time spent outside of it. As we will see, being outside the binding period is favourable to achieving an exponential growth of the derivative of the function. Therefore, the second assumption imposed on a will be a high frequency of being outside of a binding period. In other words, given a small constant $\delta > 0$, the **Frequency Assumption** refers to the following

$$FA_n : \frac{F_j(a)}{j} \geq 1 - \delta \text{ for every } j \leq n,$$

where $F_j(a)$ is the total amount of time spent by the orbit $\{\xi_n(a)\}_{n\geq 1}$ outside binding periods until the j-th iteration.

The figure below illustrates the terminology and type of behaviour just described. The orbit $\{\xi_n\}_{n\geq 1}$ was drawn as a continuous (instead of discrete) curve so that the "quasi-periodic" pattern of the binding periods is more visible.

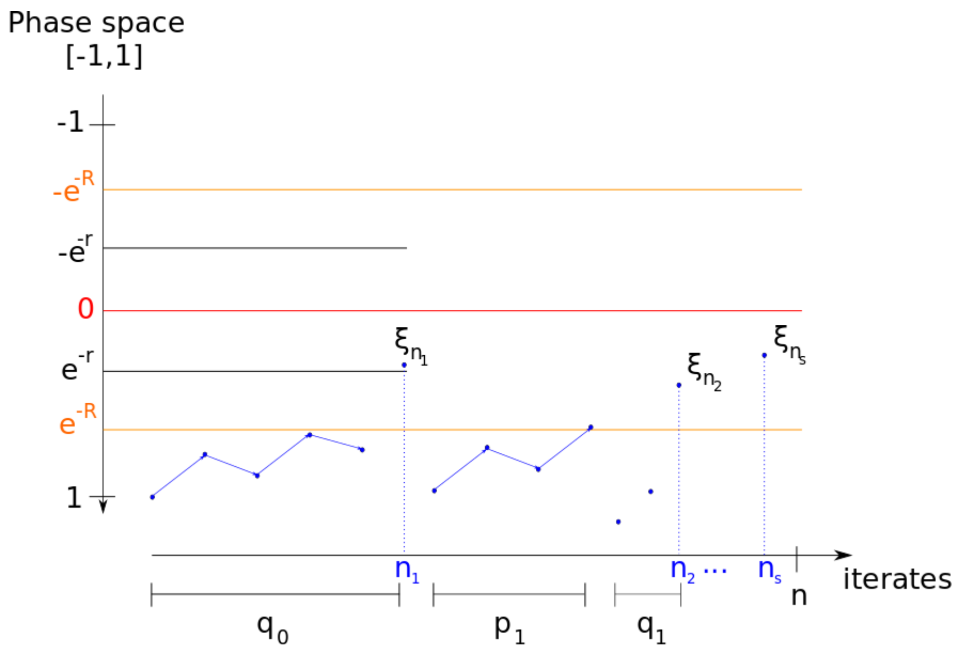

Figure 2.1: Return pattern of the orbit $\{\xi_n(a)\}$

It turns out that the basic and the frequency assumptions on a parameter a are sufficient to ensure the exponential growth EX_n.

In the second part of the argument we extend the above assumptions to small intervals of parameters, requiring a certain type of uniform bahaviour in each such interval. This is achieved inductively, by partitioning at each step n the intervals of the previous generation. The parameters in these intervals also satisfy the exponential growth of the derivative. In the end we use a large deviations type argument to show that the remaining points (after excluding parameters in each stage of the construction) form a positive measure set.

2.3
Assumptions on individual parameters

In this section we prove that under certain assumptions on parameters, the derivative of the iterates of the map f_a becomes exponentially large with the number of iterations. We begin with some definitions and notations.

For brevity, we denote

$$\xi_n(a) := f_a^n(0)$$
$$D_n(a) := (f_a^n)'(1)\,.$$

Since $f_a'(x) = -2ax$, by the chain rule, if we want the derivative to get large as we iterate the function, we have to study how the orbit of the point x approaches zero.

For some large $R \in \mathbb{N}$, define the set

$$U = (-e^{-R}, e^{-R}) = (-\delta, \delta)\,.$$

Moreover, we will also need to know more precisely how far or close to zero are the orbit points that return to U, and for this we define for all $r > R$,

$$U_r = (-e^{-r}, e^{-r}) \quad \text{and}$$
$$I_r = U_r \setminus U_{r+1}\,.$$

After returning to some interval U_r, the orbit of a point may behave for a while the same way as every other point in the interval U_r. We formalize this type of behaviour in the following definition.

Definition 2.3.1 *Given $n \geq 1$, if $\xi_n(a) \in U$ and $\xi_n(a) \neq 0$, consider the largest integer $r \geq R$ such that $\xi_n(a) \in U_r$. We define the binding period of the parameter a at n as*

$$p_n(a) = \max\{p\colon |f_a^j(x) - \xi_j(a)| \leq e^{-\beta j},\ \forall x \in U_r \text{ and } j \leq p\},$$

where $\beta > 0$ is a small constant to be specified later.

The points that enter U may go out after the binding period then enter again, and so on. As we will see, it will be helpful to know the frequency with which the orbit of a point enters U.

Definition 2.3.2 *Let $\xi_{n_1}(a) \in U$ be the first return of a to U, i.e.,*

$$n_1 = \min\{n > 0 \colon \xi_n(a) \in U\}\,.$$

We say that $\xi_{n_1}(a)$ is a free return. Inductively, define n_k to be the next return of a to U, after n_{k-1} and its binding period, i.e.

$$n_k := \min\{n > n_{k-1} + p_{n_{k-1}}(a) \colon \xi_n(a) \in U\}\,.$$

We also say that $\xi_{n_k}(a)$ is a free return.

Define $q_0(a) = n_1(a)$ to be the time spent until the first return, and inductively,

$$q_k(a) = n_{k+1} - (n_k + p_{n_k}(a))$$

the time spent out of binding periods between n_k and n_{k+1}.

The time spent out of binding periods until n will be

$$F_n(a) = q_0 + \ldots + q_s - (n_{s+1} - n)\,,$$

where n_s is the last return before n.

With these definitions, let us introduce the sets of parameters satisfying, respectively, the basic and the frequency assumptions, as well as the exponential growth of the derivative of the iterates.

Pick some constants $0 < \alpha < \beta$ and $\delta > 0$ close to 0.

Definition 2.3.3 *Define the following sets:*

$$\begin{aligned}
(BA)_n &:= \Big\{a \in [0,2] \colon |\xi_j(a)| \geq e^{-\alpha j} \text{ for all } j \in \{1, \ldots, n\}\Big\}\\
(FA)_n &:= \Big\{a \in [0,2] \colon F_j(a) \geq (1-\delta) j \text{ for all } j \in \{1, \ldots, n\}\Big\}\\
(EX)_n &:= \Big\{a \in [0,2] \colon |D_j(a)| \geq e^{cj} \text{ for all } j \in \{1, \ldots, n\}\Big\}
\end{aligned}$$

where c is sufficiently small. For example, $c < \frac{1}{40}$ will work. Moreover, we will take $\alpha \ll c$. When a parameter a is in some of these sets, for example BA_n, we also say that a satisfies the condition BA_n.

The following lemma will be used throughout the argument. It essentially says that for parameters a sufficiently close to 2, as long as the orbit points $\xi_1(a), \ldots, \xi_{k-1}(a)$ stay out of the critical interval U around 0, we have exponential growth of the derivative of the k-th iterate of f_a. The idea of the proof is to compare a *finite* piece of an orbit of the quadratic map f_a with the corresponding one for f_2. Since f_2 is conjugated to the doubling map, the exponential growth of the derivative of its iterates always holds.

Lemma 2.3.1 (Basic lemma) *Fix $c' < \log 2$ and $\delta > 0$. There are constants $C_0 > 0$ (universal) and $a_0(\delta) \in (0,2)$ such that for any $a \in [a_0, 2]$ and $x_0 \in [-1,1]$, if*

$$|f_a^j(x_0)| \geq \delta \quad \textit{for} \;\; j = 0, \ldots, k-1$$

while

$$|f_a^k(x_0)| \leq \delta\,,$$

then

$$|\partial_x f_a^k(x_0)| \geq C_0\, e^{c'k}\,.$$

Moreover, if

$$|f_a^j(x_0)| \geq \delta \quad \textit{for} \;\; j = 0, \ldots, k-1$$

(but we do not necessarily have that $|f_a^k(x_0)| \leq \delta$), then

$$|\partial_x f_a^k(x_0)| \geq C_0\, e^{c'k} \inf_{j=0,\ldots,k-1} |\partial_x f_a(f_a^j(x_0))|.$$

Additionally, given a neighbourhood $V \subset U$ there exists $K < \infty$ such that for each interval $[x,y]$ and integer n for which $f_2^n([x,y]) \subset V$, one has

$$\frac{|\partial_x f_2^n(x)|}{|\partial_x f_2^n(y)|} < K. \tag{2.3}$$

Proof. We perform the change of variables

$$x = \phi(\theta) = \sin\left(\frac{\pi}{2}\theta\right)$$

and define

$$\tilde{F}_a(\theta) := \phi^{-1} \circ f_a \circ \phi\,(\theta) = \frac{2}{\pi} \arcsin\left(1 - a\sin^2\left(\frac{\pi}{2}\theta\right)\right)\,.$$

The reason for considering this particular change of variables is that it conjugates f_2 with the doubling map (which clearly satisfies the exponential growth property we are seeking). Indeed,

$$\begin{aligned}
\tilde{F}_2(\theta) &= \frac{2}{\pi} \arcsin(1 - 2\sin^2(\frac{\pi}{2}\theta)) \\
&= \frac{2}{\pi} \arcsin(\cos(\pi\theta)) \\
&= \frac{2}{\pi} \arcsin(\sin(\pi\theta + \frac{\pi}{2}))
\end{aligned}$$

$$= 2\theta + 1.$$

(Note that (2.3) is immediate, since it trivially holds for the doubling map with $K = 1$, and hence it holds for f_2 via the conjugation, with a different upper bound K).

It is reasonable to expect that the exponential growth property for the map f_a remains valid for *finite* time as long as the parameter a stays close enough to 2.

Note that

$$f_a = \phi \circ \tilde{F}_a \circ \phi^{-1},$$

so for all iterates i,

$$f_a^i = \phi \circ \tilde{F}_a^i \circ \phi^{-1}.$$

Let us compute explicitly the derivative of $\tilde{F}_a$.

$$\begin{aligned}
\partial_\theta \tilde{F}_a(\theta) &= \frac{2}{\pi} \frac{1}{\sqrt{1 - \left(1 - a\sin^2\left(\frac{\pi}{2}\theta\right)\right)^2}} \frac{-\pi}{2} 2a \sin\left(\frac{\pi}{2}\theta\right) \cos\left(\frac{\pi}{2}\theta\right). \\
&= \frac{-2a\sin\left(\frac{\pi}{2}\theta\right)\cos\left(\frac{\pi}{2}\theta\right)}{\sqrt{2a\sin^2\left(\frac{\pi}{2}\theta\right) - a^2\sin^4\left(\frac{\pi}{2}\theta\right)}} \\
&= \frac{-2a\sin\left(\frac{\pi}{2}\theta\right)\cos\left(\frac{\pi}{2}\theta\right)}{\sqrt{\left(a\sin^2\left(\frac{\pi}{2}\theta\right)\right)\left(2 - a\sin^2\left(\frac{\pi}{2}\theta\right)\right)}} \\
&= \frac{-2\sqrt{a}\,\mathrm{sgn}(\theta)\cos\left(\frac{\pi}{2}\theta\right)}{\sqrt{2 - a\sin^2\left(\frac{\pi}{2}\theta\right)}} \\
&= \frac{-\sqrt{2a}\,\mathrm{sgn}(\theta)\cos\left(\frac{\pi}{2}\theta\right)}{\sqrt{1 - \frac{a}{2}\sin^2\left(\frac{\pi}{2}\theta\right)}} \\
&= \frac{-\sqrt{2a}\,\mathrm{sgn}(\theta)\cos\left(\frac{\pi}{2}\theta\right)}{\sqrt{\cos^2\left(\frac{\pi}{2}\theta\right) + \sin^2\left(\frac{\pi}{2}\theta\right)\left(1 - \frac{a}{2}\right)}}.
\end{aligned}$$

Fix $x_0 \in [-1, 1]$ and $a \in (0, 2]$ and let $x_i = f_a^i(x_0)$ for all $i \geq 0$.

If $x_j \geq 1 - 2\delta^2$ holds for all indices $j = 0, \ldots, k-1$ (and thus x_j is close to 1) then we have nothing to prove. Indeed,

$$|\partial_x f_a^k(x_0)| = |(f_a)'(x_{k-1})| \ldots |(f_a)'(x_0)| = 2a\, x_{k-1} \ldots 2a\, x_0,$$

which grows exponentially.

Let us then assume that there is an index $j \in \{0, \dots, k-1\}$ such that

$$x_j < 1 - 2\delta^2 \quad \text{and} \quad x_i \geq 1 - 2\delta^2, \text{ if } i = 0, \dots, j-1.$$

Since for all indices $i \in \{0, \dots, k-1\}$ we have $|x_i| \geq \delta$, it follows that

$$x_{i+1} = 1 - a\, x_i^2 \leq 1 - a\, \delta^2.$$

In particular (and since a is close to 2, say $a \in (1, 2]$),

$$x_i \leq 1 - a\, \delta^2 \leq 1 - \delta^2 \quad \text{for} \quad i = j, \dots, k,$$

which shows that the piece of the orbit $\{x_i\}_{i \geq 0}$ from time $i = j$ to time $i = k$ stays away from 1.

This motivates splitting $f_a^k = f_a^{k-j} \circ f_a^j$, so

$$f_a^k = \phi \circ \tilde{F}_a^{k-j} \circ \phi^{-1} \circ f_a^j\,.$$

Then using the chain rule we have:

$$\partial_x f_a^k(x_0) = \phi'(\tilde{F}_a \circ \phi^{-1}(x_{k-1})) \prod_{i=j}^{k-1} \partial_\theta \tilde{F}_a(\phi^{-1}(x_i)) \, \frac{1}{\phi'(\phi^{-1}(x_j))} \prod_{i=0}^{j-1} (-2ax_i) \tag{2.4}$$

We will estimate one by one each of the four factors in the above product.

Since $\phi^{-1}(x) = \frac{2}{\pi} \arcsin(x)$, then

$$\begin{aligned} \phi'(\tilde{F}_a \circ \phi^{-1}(x_{k-1})) &= \phi'\left(\frac{2}{\pi} \arcsin(x_k)\right) \\ &= \frac{\pi}{2} \cos(\arcsin(x_k)) = \frac{\pi}{2} \sqrt{1 - x_k^2} \end{aligned}$$

In the first situation, when we know that $|x_k| = |f^k(x_0)| \leq \delta$, we can conclude from the above that

$$\phi'(\tilde{F}_a \circ \phi^{-1}(x_{k-1})) \geq \frac{\pi}{2} \sqrt{1 - \delta^2} > C_1,$$

for some universal constant $C_1 > 0$, as δ is close to zero.

The lower bound in the case when we do not necessarily have a return at time k, that is, when $|x_k|$ could still be greater than δ is a bit more tricky. Let us analyze the case when $x_k \geq 0$.

$$\sqrt{1 - x_k^2} = \sqrt{1 - x_k}\, \sqrt{1 + x_k} \geq \sqrt{1 - x_k}$$

$$= \sqrt{1-(1-ax_{k-1}^2)} = \sqrt{a}\,|x_{k-1}| = \frac{1}{2\sqrt{a}}\,|\partial_x f_a\,(x_{k-1})|$$
$$\geq \frac{1}{2\sqrt{2}} \inf_{j=0,\ldots,k-1} |\partial_x f_a(x_j))|. \tag{2.5}$$

The case when $x_k < 0$ can be treated similarly, except that depending on whether x_{k-1} is close to -1 or not, one might have to consider previous orbit points.

Let us now consider the second factor.

Note that uniformly in $x \in [0, 1-\delta^2]$,

$$\left|\partial_\theta \tilde{F}_a(\phi^{-1}(x))\right| = \frac{\sqrt{2a}\,|\cos(\phi^{-1}(x))|}{\sqrt{\cos^2(\phi^{-1}(x)) + \sin^2(\phi^{-1}(x))\,(1-\frac{a}{2})}} \to 2 \quad \text{as} \quad a \to 2.$$

Since $|x_i| \leq 1-\delta^2$ for $i = j, \ldots, k-1$, we may then conclude that for all these indices, $\left|\partial_\theta \tilde{F}_a(\phi^{-1}(x_i))\right|$ is close to 2, provided that a is close enough to 2. That is, there exists $a_0 = a_0(\delta)$ such that if $a \in (a_0, 2)$, then

$$|\partial_\theta \tilde{F}_a(\phi^{-1}(x_i))| \geq e^{c'} \qquad \text{for } i = j, \ldots, k-1.$$

Moving on to the next factor in the product (2.4), as before, since $|x_j| \geq \delta$, we obtain

$$\frac{1}{\phi'(\phi^{-1}(x_j))} \geq \frac{2}{\pi}\frac{1}{\sqrt{1-\delta^2}} > C_2,$$

for some universal constant $C_2 > 0$.

Finally, consider the fourth factor in (2.4). By our splitting of the iterates f_a^k, we ensured that

$$|x_i| \geq 1-2\delta^2 \qquad \text{for} \quad i = 0, \ldots, j-1.$$

It then follows that

$$\left|\prod_{i=0}^{j-1}(-2ax_i)\right| \geq e^{cj}.$$

Putting all the estimates together, we may now conclude that in the case $|x_k| \leq \delta$,

$$|\partial_\theta f_a^k(x_0)| \geq C_1\, e^{c'(k-j)}\, C_2\, e^{c'j} = C_1\, C_2\, e^{c'k}\,.$$

The final estimate in the case $|x_k| \geq \delta$ is the same, except that one uses the lower bound provided by (2.5) on the first factor of the product (2.4).

This completes the proof of the basic lemma. ■

The next lemma is concerned with the study of the binding periods. It relates the binding period p with the associated return level r, that is, we show that $p \approx r$. Moreover, we show that the piece of the orbit corresponding to a binding period still leads to some exponential growth, although a slower one, as long as we start with a point at the edge of the critical interval of level r.

Lemma 2.3.2 *There exists $C_0 > 0$ such that for R sufficiently large and a_0 close to 2, the following statement holds. Let $\xi_m(a)$ be a return, r the largest integer number s.t. $\xi_m(a) \in U_r$ and $n \geq m$. If $a \in (BA)_n \cap (EX)_{n-1} \cap [a_0, 2]$, then*

1. $p \approx r : c_0 r \leq p_m(a) \leq \frac{3r}{c} \leq \frac{3\alpha}{c} m < \frac{n}{100}$
2. $\frac{1}{C_0} \leq \frac{|Df_a^i(f_a(x))|}{|D_i(a)|} \leq C_0$ *for every* $x \in U_r$ *and* $i \leq p_m(a)$
3. $|\partial_x f_a^{p_m(a)+1}(x_0)| \geq e^{\frac{c}{4}p_m(a)}$ $\forall x_0 \in I_r$

Proof. Recall the notation 2.1.3, which will be used throughout this proof.

$$G \approx H \Leftrightarrow \exists G_0, H_0 > 0 \text{ such that } G_0 H \leq G \leq H_0 H,$$

or in other words, the quantity $\frac{G}{H}$ is bounded.

Let $m \leq n$ be a return time. Therefore, $|f_a^j(x) - f_a^j(y)| < 2e^{-\beta j}$ for every $x, y \in U_r$, $1 \leq j \leq p_m(a) = p$ and there exists $x \in U_r$ such that $|f_a^{p+1}(x) - f_a^{p+1}(0)| \geq e^{-\beta(p+1)}$. Hence, by definition of set $(BA)_n$, for every $x_0 \in U_r$, and $1 \leq j \leq p$, $f_a^j(x_0) \approx \xi_j(a)$.

Thus

$$|f_a(x_0). \dots .f_a^k(x_0)| \approx |\xi_1(a). \dots .\xi_k(a)|$$

for every $k \leq p$.

Therefore,

$$\begin{aligned}(2a)^k |f_a(x_0). \dots .f_a^k(x_0)| &= |f_a'(f_a^k(x_0)). \dots .f_a'(f_a(x_0)| = \\ &= |(f_a^k)'(f_a(x_0))| \approx (2a)^k |\xi_1(a). \dots .\xi_k(a)| \\ &= |D_k(a)|,\end{aligned} \tag{2.6}$$

for every $x_0 \in U_r$.

Besides that, since

$$2 \geq |f_a^{j+1}(x) - \xi_{j+1}(a)| = |f_a^j(f_a(x)) - f_a^j(\xi(a))|.$$

Then, if $x \in U_r$, by the mean value theorem, there exists $y \in f_a(U_r)$ such that

$$|f_a^j(f_a(x)) - f_a^j(\xi(a))| = |(f_a^j)'(y)(f_a(x) - \xi(a))| = |(f_a^j)'(y)ax^2|$$

and, by item 2.6 we have that:

$$|f_a^{j+1}(x) - \xi_{j+1}(a)| \approx |D_j(a)|\, ax^2\,. \tag{2.7}$$

Suppose that $x \in I_r$, i.e., $e^{-r-1} \le |x| \le e^{-r}$. Then, $|x| \approx e^{-r}$ and

$$D_j(a)ax^2 \approx D_j(a)e^{-2r}.$$

Now, as $2 \ge |f_a^{j+1}(x) - \xi_{j+1}(a)|$, then there exists c_o such that $2c_0 \ge D_j(a)e^{-2r}$ for every $j \le p$. Since $a \in (EX)_{n-1}$, we have that:

$$2c_0 \ge D_j(a)e^{-2r} \ge e^{cj-2r} \text{ for every } j \le \min\{n-1, p\} \tag{2.8}$$

Observe that if $\min\{n-1,p\} \ge \frac{3r}{c}$, then $2c_0 \ge e^r$ which gives us a contradiction, because we could take R sufficiently large. But since $\xi_m(a) \in U_r$ and $a \in (BA)_m$, we have that $e^{-\alpha m} \le |\xi_m(a)| \le e^{-r}$ and then

$$\alpha n \ge \alpha m \ge r.$$

Therefore, $\min\{n-1,p\}$ can not be $n-1$. Otherwise, we would obtain $r \le \frac{3\alpha r}{c} + \alpha$, which is absurd, because $\alpha \ll c$. Thus,

$$p \le \frac{3r}{c} \le \frac{3\alpha}{c} m \le \frac{n}{100}.$$

For the other inequality, take $x_0 \in U_r$ such that

$$e^{-\beta(p+1)} \le |f_a^{p+1}(x_0) - f_a^{p+1}(0)|.$$

As we did before, by the mean value theorem, for some $y \in f_a(U_r)$, we have that

$$e^{-\beta(p+1)} \le |(f_a^p)'(y)\, ax_0^2|.$$

Therefore,

$$e^{-\beta(p+1)} \le (2a)^p \prod_{i=0}^{p-1} |f^i(y)| ax_0^2 \le 4^p ax_0^2 \le 4^p 2e^{-2r}.$$

Hence, since r is large enough,

$$p \ge \frac{2r - \log 2 - \beta}{\beta + \log 4} = \frac{2r + \log 2}{\beta + \log 4} - 1 \ge \frac{2r}{\log 8} - \frac{2}{3} \ge \frac{r}{\log 8}\,. \tag{2.9}$$

Therefore, $p \sim r$.

To prove the second item, take $i \leq p_m(a)$ and note that

$$\frac{|Df_a^i(f_a(x))|}{|D_i(a)|} = \frac{|(2a)^i \prod_{j=1}^i f_a^j(x)|}{|(2a)^i \prod_{j=1}^i \xi_j(a)|} = \prod_{j=1}^i \frac{|f_a^j(x)|}{|\xi_j(a)|}.$$

By the definition of $p_m(a)$,

$$|f_a^j(x) - \xi_j(a)| < e^{-\beta j}$$

and then,

$$\left| \frac{|f_a^j(x)|}{|\xi_j(a)|} - 1 \right| \leq \frac{|f_a^j(x) - \xi_j(a)|}{|\xi_j(a)|} \leq e^{(\alpha-\beta)j}.$$

The last inequality holds because $j \leq i \leq p_m(a)$. By previous item, $p_m(a) < n$, and then we may use the fact that $a \in (BA)_j \supset (BA)_n$.

Therefore, there exists K such that

$$\sum_{j=1}^i \left| \frac{|f_a^j(x)|}{|\xi_j(a)|} - 1 \right| \leq K. \tag{2.10}$$

And this is sufficient, because $x < e^{x-1}$ for every x. Thus, for every j,

$$\frac{|f_a^j(x)|}{|\xi_j(a)|} < e^{\frac{|f_a^j(x)|}{|\xi_j(a)|}-1}$$

and then,

$$\prod_{j=1}^i \frac{|f_a^j(x)|}{|\xi_j(a)|} \leq e^{\sum_{j=1}^i \frac{|f_a^j(x)|}{|\xi_j(a)|}-1} \leq e^K = K'.$$

The other inequality holds, because

$$\sum_{j=1}^i \left| \frac{|\xi_j(a)|}{|f_a^j(x)|} - 1 \right| = \sum_{j=1}^i \left| \left(\frac{|\xi_j(a)|}{|f_a^j(x)|} \right) \left(\frac{|f_a^j(x)|}{|\xi_j(a)|} - 1 \right) \right|$$

and by 2.10, $\frac{|f_a^j(x)|}{|\xi_j(a)|}$ converges to 1, therefore, $\sum_{j=1}^i \left| \frac{|\xi_j(a)|}{|f_a^j(x)|} - 1 \right|$ is also bounded, and we may apply the same argument.

For the third item, note that

$$|\partial_x f_a^{p_m(a)+1}(x_0)| = |\partial_x f_a^{p_m(a)}(f_a(x_0))2ax_0|$$

By the previous item,

$$|\partial_x f_a^{p_m(a)+1}(x_0)| \leq C_0 |D_p(a)| 2ae^{-r}$$

and by definition of $p_m(a)$, if $x \in I_r$, then $|f_a^p(x) - \xi_p(a)| \leq e^{-\beta p}$ and $|f_a^{p+1}(x) - \xi_{p+1}(a)| \geq e^{-\beta(p+1)}$.

By mean value theorem,

$$4|f_a^p(x) - \xi_p(a)| > |2af_a^p(x')||f_a^p(x) - \xi_p(a)| = |f_a^{p+1}(x) - \xi_{p+1}(a)| \geq e^{-\beta(p+1)}.$$

Using the previous item and mean value theorem, we also conclude that

$$|f_a^{p+1}(x) - \xi_{p+1}(a)| \approx |Df_a^p(f_a(x'))||f_a(x') - \xi_1(a)| \approx |D_p(a)|e^{-2r}.$$

Therefore, $e^{-\beta p} \approx |D_p(a)|e^{-2r}$. We can write it also as $|D_p(a)|^{\frac{1}{2}} \approx e^{-\beta p/2+r}$. Moreover, by chain rule,

$$|Df_a^{p+1}(x)| = |Df_a(x)||Df_a^p(x)(f_a(x))| \approx e^{-r}|Df_a^p(x)(f_a(x))|$$

and by the first item of this lemma,

$$|Df_a^{p+1}(x)| \approx e^{-r}|D_p(a)|.$$

Finally, we conclude that

$$\begin{aligned} |Df_a^{p+1}(x)| &\geq Ce^{-r}|D_p(a)|^{\frac{1}{2}}|D_p(a)|^{\frac{1}{2}} \geq Ce^{-r}|D_p(a)|^{\frac{1}{2}}e^{-\beta p/2+r} \\ &\geq Ce^{-\beta p/2+cp/2} \geq e^{cp/4}. \end{aligned}$$

■

We want to show that a point in $(BA)_n$ and $(FA)_n$ is in $(EX)_n$, but firstly we also assume that it satisfies the $(EX)_{n-1}$ condition.

Lemma 2.3.3 $(BA)_n \cap (FA)_n \cap (EX)_{n-1} \subset (EX)_n$

Proof. If $a \in (BA)_n \cap (FA)_n \cap (EX)_{n-1}$ and $n_1, \ldots, n_s$ are the returns of a before n, by the chain rule,

$$|D_n(a)| = |Df_a^{n_1-1}(1)|(\Pi_{i=1}^{s-1}|Df_a^{n_{i+1}-n_i}(\xi_{n_i}(a))|)(|Df_a^{n+1-n_s}(\xi_{n_s}(a))|)$$

For each $i \in \{1, \ldots, s-1\}$, we may write,

$$Df_a^{n_{i+1}-n_i}(\xi_{n_i}(a)) = Df_a^{q_i}(\xi_{n_i+p_i+1}(a))Df_a^{p_i+1}(\xi_{n_i}(a))$$

And by 2.3.1 and 2.3.2,

$$|Df_a^{n_{i+1}-n_i}(\xi_{n_i}(a))| \geq Ce^{c'q_i+\frac{c}{4}p_i}.$$

Therefore,

$$|D_n(a)| \geq C^s e^{c'(n_1-1)} e^{c' F_n(a) + c\frac{n-F_n(a)}{4}} |Df_a^{n+1-n_s}(\xi_{n_s}(a))|$$

For $n+1-n_s$ we must consider the two possibilities: it is in binding period or not. Firstly suppose that $n > n_s + p_s$, i.e., it is out of binding period.

Then, we use the following result:

$$|Df_a^{n+1-n_s}(\xi_{n_s}(a))| \geq C e^{q_s c'} \inf_{n_s+p_s \leq j \leq n} |\xi_j(a)|.$$

With the BA property we conclude,

$$|D_n(a)| \geq C^{s+1} e^{c'(n_1-1)} e^{c' F_n(a) + c\frac{n-F_n(a)}{4}} e^{-\alpha n}.$$

On the other hand, if $n \leq n_s + p_s$, i.e., it is in binding period, we write the derivative as follows:

$$|Df_a^{n+1-n_s}(\xi_{n_s}(a))| = |Df_a(\xi_{n_s}(a))||Df_a^{n-n_s}(\xi_{n_s+1}(a))|.$$

Again, using the BA condition,

$$|Df_a^{n+1-n_s}(\xi_{\nu_s}(a))| \geq e^{-\alpha n_s} |Df_a^{n-\nu_s}(\xi_{\nu_s+1}(a))|.$$

By the previous lemma,

$$\begin{aligned} |Df_a^{n+1-n_s}(\xi_{\nu_s}(a))| &\geq C e^{-\alpha n_s} |D_{n-n_s}(a)| \\ &\geq C^2 e^{-\alpha n} e^{c'(n-n_s)} \\ &\geq C^2 e^{-\alpha n} \end{aligned}$$

and then, if we take the n-th derivative,

$$|D_n(a)| \geq C^{s+2} e^{c'(n_1-1)} e^{c' F_n(a) + c\frac{n-F_n(a)}{4}} e^{-\alpha n}.$$

Since for each return i, by the previous lemma, $\sum p_i \geq sCR$, then $sCR \leq n - F_n$ and therefore, $\frac{s}{n-F_n}$ goes to zero as $R \to \infty$. Then, as $C < 1$ and $n - F_n$ goes to ∞ faster than s, we can assume that $C^{s+2} e^{c'(n_1-1) + c\frac{n-F_n(a)}{4}} \geq 1$.

This is because $e^{c\frac{n-F_n(a)}{4}}$ goes to infinity faster than $C^{s+2} = e^{-(s+2)|\log C|}$ goes to zero.

Therefore,

$$|D_n(a)| \geq e^{c'(F_n(a))} e^{-\alpha n} \geq e^{c'(1-\delta)n - \alpha n} \geq e^{cn}$$

if we take $\alpha < c'(1-\delta) - c$. ■

The following proposition shows that a parameter that satisfies conditions $(BA)_n$ and $(FA)_n$, also satisfy $(EX)_n$. In other words, these assumptions are sufficient to the desired exponential growth of the derivative.

Proposition 2.3.4 $(BA)_n \cap (FA)_n \cap (a_0, 2] \subset (EX)_n$

Proof. For $n = 1$, it follows from 2.3.3. Suppose

$$(BA)_k \cap (FA)_k \cap (a_0, 2] \subset (EX)_k$$

Then,

$$(BA)_{k+1} \cap (FA)_{k+1} \cap (a_0, 2] \subset (BA)_k \cap (FA)_k \cap (a_0, 2] \subset (EX)_k$$

Therefore, by the previous lemma,

$$\begin{aligned}(BA)_{k+1} \cap (FA)_{k+1} \cap (a_0, 2] &= (BA)_{k+1} \cap (FA)_{k+1} \cap (a_0, 2] \cap (EX)_k \\ &\subset (BA)_{k+1} \cap (FA)_{k+1} \cap (EX)_k \subset (EX)_{k+1}\end{aligned}$$

■

Since some results are easier for the derivative with respect to a, we will use the following lemma to compare the derivatives.

Note that $f_a^k(x)$ is an algebraic expression that depends on a and x. By $\partial_a f_a^k(x)$ we mean the derivative of this expression with respect to a at x.

Lemma 2.3.5 *Fix $c < \log 2$. There exist $a_0, x_0, N(a_0), M = M(c, a_0, x_0)$ such that, if $a \in [a_0, 2]$, $x_1 \in [x_0, 1]$ and $|\partial_x f_a^i(x_1)| \geq e^{ci}$ for $i = N, \ldots, k-1$, then*

$$\frac{1}{M} \leq \left| \frac{\partial_a f_a^k(x_1)}{\partial_x f_a^k(x_1)} \right| \leq M.$$

Proof. For the first inequality, note that using the chain rule

$$\partial_x f_a^k(x) = -2a f_a^{k-1}(x) \partial_x f_a^{k-1}(x)$$

and

$$\partial_a f_a^k(x) = -(f_a^{k-1}(x))^2 - 2a f_a^{k-1}(x) \partial_a f_a^{k-1}(x).$$

Therefore,

$$\frac{\partial_a f_a^k(x_1)}{\partial_x f_a^k(x_1)} = \frac{\partial_a f_a^{k-1}(x_1)}{\partial_x f_a^{k-1}(x_1)} + \frac{f_a^{k-1}(x_1)}{2a \partial_x f_a^{k-1}(x_1)}$$

,

and then, if we take $x_1 = 1$,

$$1 > \left|\frac{\partial_a f_a^j(1)}{\partial_x f_a^j(1)}\right| = \left|\frac{-1}{2a} + \ldots + \frac{-1}{(2a)^{j-1}}\right| \geq \frac{1}{4}. \tag{2.11}$$

and, since

$$\left|\frac{\partial_a f_a^k(x_1)}{\partial_x f_a^k(x_1)} - \frac{\partial_a f_a^N(x_1)}{\partial_x f_a^N(x_1)}\right| = \left|\sum_{i=N}^{k-1} \frac{f_a^i(x_1)}{2a\partial_x f_a^i(x_1)}\right|$$

if we take N sufficiently large and x_0 close to 1,

$$\left|\frac{\partial_a f_a^k(x_1)}{\partial_x f_a^k(x_1)} - \frac{\partial_a f_a^N(x_1)}{\partial_x f_a^N(x_1)}\right| \leq \frac{1}{16}, \tag{2.12}$$

and

$$\left|\frac{\partial_a f_a^N(x_1)}{\partial_x f_a^N(x_1)} - \frac{\partial_a f_a^N(1)}{\partial_x f_a^N(1)}\right| \leq \frac{1}{16}. \tag{2.13}$$

From 2.11, 2.12 and 2.13, follows that:

$$M = 8 > \left|\frac{\partial_a f_a^k(x_1)}{\partial_x f_a^k(x_1)}\right| \geq \frac{1}{M}. \tag{2.14}$$

■

2.4 Assumptions on sets

Instead of only analyzing the behaviour of individual parameters, as we want to end up with a positive measure set of parameters, we will have to extend this approach and study certain small intervals of parameters.

Let $\omega_0 = [a_0, 2)$. We will use the notation $|\cdot|$ to Lebesgue measure (for example $|\omega_0| = 2 - a_0$). As before, we define free returns and binding periods.

Definition 2.4.1 *Let $\omega \subset \omega_0$ be an interval and let $\nu_1(\omega)$ be the first return of some element of ω to U. In other words, $\nu_1(\omega)$ is the first integer such that $\xi_{\nu_1(\omega)}(\omega) \cap U \neq \emptyset$. We say it is a free return. Associated to a free return $\nu_i(\omega)$, there is a binding period defined as $p_{\nu_i(\omega)}(\omega) = \min_{a\in\omega} p_{\nu_i(\omega)}(a)$. The other free returns are defined as*

$$\nu_k(\omega) = \min\{\nu > \nu_{k-1}(\omega) + p_{\nu_{k-1}(\omega)}(\omega); \xi_\nu(\omega) \cap U \neq \emptyset\}$$

Remark 1 *If we take a singleton as ω, the above definition is equivalent to the previous cases (with individual parameters).*

Also, we will split each interval I_r into r^2 intervals of equal length denoted by $I_{r,1}, \ldots, I_{r,r^2}$. To have some control over the properties of an interval, we will break it into smaller ones as follows.

For each $n \geq N$, we will make a partition $\mathcal{E}_n$ of $\omega_0 \cap EX_{n-1}$. These partitions are constructed as follows: $\mathcal{E}_N = \{\omega_0 \cap EX_{N-1}\}$. For each $n > N$ and $\omega \in \mathcal{E}_{n-1}$, we will consider the following possibilities.

1. n is not a free return of ω. Since binding periods are small and being outside U is good, it will not be necessary to divide this interval and then we take $\omega \in \mathcal{E}_n$.

2. n is a free return of ω. In this case, we may have that $\xi_n(\omega)$ contains an interval $I_{r,r'}$, $(r \geq R-1)$ or just a subset.

   (a) If $\xi_n(\omega)$ does not contain an interval $I_{r,r'}$ $(r \geq R-1)$, then by continuity there are 2 intervals of the form $I_{r,r'}$ that contain $\xi_n(\omega)$. In this case we say that this return is *inessential* and we take $\omega \in \mathcal{E}_n$.

   (b) If $\xi_n(\omega)$ contains at least one interval $I_{r,r'}$, $(r \geq R)$, then, we say it is an *essential return* and subdivide it into sets ω' and $\omega_{r,r'}$ such that ω' is mapped outside U_r and $\omega_{r,r'}$ is mapped to $I_{r,r'}$ by ξ_n. Besides that, if one of the two intervals at the end of ω in this partition does not contains an entire interval $I_{r,r'}$, then we add it to its neighbour. Intervals $I_{r,r'}$ are called *host intervals* of $\omega_{r,r'}$.

   If an interval ω' obtained in this way is outside U_{R+1}, then we call it an *escape*. If besides this, ω' is such that $|\xi_n(\omega')| \geq \sqrt{|U|}$, then we say it is a *substantial escape* and we break it into subintervals: for each such subinterval ω'' we have $\frac{\sqrt{|U|}}{2} < |\xi_n(\omega'')| < \sqrt{|U|}$.

Since the set of points with dense orbit is dense and $\xi_n(2) = -1$ for every $n \geq 2$, we may take ω_0 in such a way that the first free return $\nu_1(\omega_0)$ satisfies the following property:

$$\xi_{\nu_1(\omega_0)}(\omega_0) \supset U.$$

In this case, we have an essential return. We will denote the essential returns as $\hat{\nu}$, instead of ν and we will use $\omega_{\hat{\nu}_i}(a)$ for the element in $\mathcal{E}_{\hat{\nu}_i}$ containing a. If $\hat{\nu}_i$ is an escape component, then it has no return to U_{R+1}.

In the following results we will show that the proposition 2.3.4 still holds if we use these intervals of the partition, but firstly we will define those sets in terms of these intervals.

Definition 2.4.2 *For each $a \in \omega_0 \cap (EX)_{n-1}$, let $\omega_n(a)$ be the set of $\mathcal{E}_n$ which contains a. The set $(BA)'_n$ will be the set of parameters a such that $\omega_n(a) \cap (BA)_n \neq \emptyset$.*

As we did in 2.4.1, we may also define $q_i(\omega)$ as the time spent outside binding period between two free returns and define F_n as before (sum of $q_i's$), but now for intervals.

Definition 2.4.3 *The set FA'_n will contain the intervals ω in $\mathcal{E}_n$ such that $F_n(\omega) \geq n(1-\delta)$.*

We want to prove a statement like 2.3.4 but for intervals. As in the previous section, we will show it by induction. More precisely, we will show that $(BA)'_{n-1} \cap (FA)'_{n-1} \cap (EX)_{n-2} \subset EX_{n-1}$. Firstly note that $(BA)_n$ and $(BA)'_n$ are essentially the same sets. Indeed, take $a \in (BA)_n$. By construction, if n is a free return, then $\omega_n(a)$ is an escape or there exists r such that, $e^{-r+1} \geq |\xi_n(b)| \geq e^{-r-1}$ for every $b \in \omega_n(a)$. In both cases we conclude that $|\xi_n(b)| \geq e^{\alpha n-2}$. Also, all parameters in an interval like $w_n(a)$ have the same return times before the $n-$th iterate. Then, to prove the statement it will be sufficient to compare the derivatives in an interval of the partition.

Proposition 2.4.1 *There exists $C > 0$ such that for R sufficiently large, there exists $\epsilon > 0$ with the following property: for each $n \geq N$ and $\omega \in \mathcal{E}_{n-1}$, which has a free return at n and $\omega \subset (BA)'_{n-1} \cap (EX)_{n-2}$, if $\xi_n(\omega) \subset U_{\frac{R}{2}}$, then, for every $a, b \in \omega$ and $k = 0, \ldots, n$,*

$$\frac{|\xi'_k(a)|}{|\xi'_k(b)|} \leq C.$$

Furthermore,

$$(BA)'_{n-1} \cap (FA)'_{n-1} \subset (EX)_{n-1}.$$

In order to prove this proposition, we will need some lemmas. The following lemma gives us a relation between the derivative with respect to a and to x.

Lemma 2.4.2 *The inequality $|\partial_a f_a^j(x)| \leq \sum_{i=0}^{j-1} |\partial_x f_a^{j-1-i}(f_a^i(x))|$ holds for every x.*

Proof. For $j = 1$ the inequality is trivial. Since,

$$\partial_a f_a^{j+1}(x) = \partial_a f_a(f_a^j(x)) + Df_a(f_a^j(x)).\partial_a f_a^j(x)$$

then, by induction,

$$|\partial_a f_a^{j+1}(x)| \leq 1 + |Df_a(f_a^j(x))| \sum_{i=0}^{j-1} |Df_a^{j-1-i}(f^i(x))|$$

$$= 1 + \sum_{i=0}^{j-1} |Df_a^{j-i}(f^i(x))|$$
$$= \sum_{i=0}^{j} |Df_a^{j-i}(f^i(x))|,$$

as we wanted to show. ■

Since we want to compare how the intervals are behaving to use mean value theorem later, it is useful to define the Hausdorff distance.

Definition 2.4.4 *The Hausdorff distance between two sets X and Y is defined as*

$$d_H(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d(x, y), \sup_{y\in Y} \inf_{x\in X} d(x, y)\}.$$

The following lemma gives us a bound on the variation of an interval I_r when we variate the parameters.

Lemma 2.4.3 *Let $\omega \in \mathcal{E}_{n-1}$ and assume that $\nu < n$ is a free return of ω. Then $\xi_\nu(\omega)$ is contained in $I_{r,r'}$, or the union of one interval like this with pieces of its neighbours.*

For each R, there exists $\epsilon > 0$ (the length of ω_0) such that if $\omega \subset (BA)'_{n-1} \cap (EX)_{n-2}$, then

$$d_H(f_a^j(I_r), f_b^j(I_r)) \leq \frac{1}{1000}|f_a^j(I_r)|,$$

$$d_H(f_a^j(U_r), f_b^j(U_r)) \leq \frac{1}{1000}|f_a^j(U_r)|, \text{ and}$$

$$d_H(\xi_{\nu+j}(\tilde{\omega}), f_a^j(\xi_\nu(\tilde{\omega})) \leq \frac{1}{1000}|f_a^j(\xi_\nu(\tilde{\omega})|, ,$$

for every $a, b \in \omega$, $\tilde{\omega} \subset \omega$ and $j \in \{1, \ldots, p(r, \omega) + 1\}$.

Proof. For the first inequality, take $x \in I_r$. By the Mean Value Theorem, there exists ã $\in [a, b]$ such that $|f_a^j(x) - f_b^j(x)| = |\partial_a f_{\tilde{a}}^j(x)||a - b|$. By the second item of 2.3.2, we may conclude that

$$\frac{1}{C_0} \leq \frac{|Df_{\tilde{a}}^i(f_{\tilde{a}}(x))|}{|D_i(\tilde{a})|} \leq C_0 \tag{2.15}$$

for every $i \leq p_\nu(\tilde{a})$. By lemma 2.4.2,

$$|\partial_a f_{\tilde{a}}^j(x)| \leq \sum_{i=0}^{j-1} |Df_{\tilde{a}}^{j-1-i}(f_{\tilde{a}}^i(x))|$$

$$= \left(\sum_{i=1}^{j-1} |Df_{\tilde{a}}^{j-1-i}(f_{\tilde{a}}^{i}(x))| + |D(f_{\tilde{a}}^{j-2} \circ f_{\tilde{a}}(x)| \right)$$
$$= \left(\sum_{i=1}^{j-1} |Df_{\tilde{a}}^{j-1-i}(f_{\tilde{a}}^{i}(x))| + |D(f_{\tilde{a}}^{j-2}(f_{\tilde{a}}(x))|Df_{\tilde{a}}(x)| \right)$$

and, since $Df_{\tilde{a}}^{j-1-i}(f_{\tilde{a}}^{i}(x)).Df_{\tilde{a}}^{i-1}(f_{\tilde{a}}(x)) = Df_{\tilde{a}}^{j-2}(f_{\tilde{a}}(x))$,

$$|\partial_a f_{\tilde{a}}^{j}(x)| \leq |Df_{\tilde{a}}^{j-2}(f_{\tilde{a}}(x))| \left(\sum_{i=0}^{j-2} \frac{1}{|Df_{\tilde{a}}^{i}(f_{\tilde{a}}(x))|} + 4 \right)$$

and hence, by 2.15,

$$|\partial_a f_{\tilde{a}}^{j}(x)| \leq C|D_{j-2}(\tilde{a})| = C2a|\xi_{j-2}(a)D_{j-1}(\tilde{a})| \leq C'|D_{j-1}(\tilde{a})|$$

Therefore,

$$|f_a^j(x) - f_b^j(x)| \leq C'|D_{j-1}(\tilde{a})||b-a|$$

and again, by 2.15 and second item of lemma 2.3.2, since $\tilde{a}$ is in EX_{n-1}, then for $j = 1, \ldots, p_\nu(a)+1$,

$$|f_a^j(x) - f_b^j(x)| \leq C' \frac{|f_{\tilde{a}}^j(I_r)|}{|f_{\tilde{a}}(I_r)|}|\omega| \leq C' \frac{|f_{\tilde{a}}^j(I_r)|}{|f_{\tilde{a}}(I_r)|}|\omega| \frac{e^{-r}/r^2}{|\xi_\nu(\omega)|}.$$

For the last inequality we used the fact that $\xi_\nu(\omega) \subset I_{r,r'}$. By lemma 2.3.5, the mean value theorem , and the fact that $|f_a(I_r)| \geq Ce^{-2r}$, there exists $a' \in \omega$ such that,

$$|f_a^j(x) - f_b^j(x)| \leq C'|f_{\tilde{a}}^j(I_r)| \frac{1}{D_{\nu-1}(a')} \frac{e^{-r}/r^2}{e^{-2r}} \leq C'|f_{\tilde{a}}^j(I_r)| \frac{e^{-c\nu+r}}{r^2}$$

Furthermore as it was observed, since $\omega \subset (BA)'_{n-1}$ then $|\xi_\nu(b')| \geq e^{-\alpha\nu-2}$ for every $b' \in \omega$, then $e^{-\alpha\nu-2} \leq e^{-r}$ and so

$$r \leq \alpha\nu + 2\,.$$

Then

$$e^{-c\nu+r} \leq e^{\nu(\alpha-c)+2}$$

which is very small, because $\alpha \ll c$.

If we take ϵ sufficiently small, and consequently ν large, and using the fact that a, b are arbitrary,

$$d_H(f_a^j(I_r), f_b^j(I_r)) \leq \frac{1}{2000}|f_{\tilde{a}}^j(I_r)| \leq \frac{1}{2000} \max_{a' \in \omega} |f_{a'}^j(I_r)|.$$

■

Corollary 2.4.1 *For every $a \in \omega$, the statement 2.3.2 holds if $p_m(a)$ is replaced by $p_r(\omega)$.*

Proof. The first and second items are trivial. For the third item, take $\tilde{a} \in \omega$ such that $p_m(\tilde{a}) = p_r(\omega)$. From the previous lemma, there exists K such that

$$\frac{|f_a^j(U_r)|}{|f_b^j(U_r)|} \leq \frac{1}{K}$$

for every $a, b \in \omega$ and $j = 1, \ldots, p_r(\omega) + 1$. Then,

$$|f_a^{p_r(\omega)+1}(U_r)| = |f_a^{p_m(\tilde{a})+1}(U_r)| \geq K|f_{\tilde{a}}^{p_m(\tilde{a})+1}(U_r)| \geq Ke^{-\beta p_m(\tilde{a})} = Ke^{-\beta p_r(\omega)}.$$

Hence, the proof of 2.3.2 can be done replacing $p_m(a)$ by $p_r(\omega)$.

■

By lemma 2.3.5 and mean value theorem, if ω is in $(BA)'_{n-1} \cap (EX)_{n-2}$, then $\frac{|\omega|}{|\xi_j(\omega)|}$ is exponentially small for $j \leq n-1$. In next lemma we will see that $\frac{|\xi_j(\omega)|}{|\xi_\nu(\omega)|}$ is also exponentially small, when $j < \nu \leq n-1$ and ν is a free return.

Lemma 2.4.4 *There exists a constant C such that for R sufficiently large, there exists ϵ such that if $|\omega_0| < \epsilon$, $\omega \in \mathcal{E}_{n-1}$ is in $(BA)'_{n-1} \cap (EX)_{n-2}$ and $\tilde{\omega} \subset \omega$, then for any consecutive free returns, $\nu < \nu' \leq n$ of ω, $j = 1, \ldots, p(r, \omega) + 1$,*

$$|\xi_{\nu+j}(\tilde{\omega})| \geq Ce^{cj}|\xi_{\nu+1}(\tilde{\omega})|$$

and $|\xi_{\nu+p(r,\omega)+1}(\tilde{\omega})| \geq e^{c\frac{p(r,\omega)}{4}}|\xi_\nu(\tilde{\omega})|$. Moreover,

$$|\xi_{\nu'}(\tilde{\omega})| \geq Ce^{(\nu'-j)c}|\xi_j(\tilde{\omega})|$$

for $j = \nu + p(r, \omega) + 1, \ldots, \nu'$ and

$$|\xi_{\nu'}(\tilde{\omega})| \geq 2|\xi_\nu(\tilde{\omega})|.$$

Proof. By the third inequality from lemma 2.4.3, $d_H(\xi_{\nu+j}(\tilde{\omega}), f_a^j(\xi_\nu(\tilde{\omega})) < \frac{1}{1000}|f_a^j(\xi_\nu(\tilde{\omega}))|$ for every $j \leq p(r, \omega) + 1$. Then, by mean value theorem and lemma 2.3.2, $|\xi_{\nu+j}(\tilde{\omega})| \geq C_0 e^{cj}|\xi_{\nu+1}(\tilde{\omega})|$ and $|\xi_{\nu+p(r,\omega)+1}(\tilde{\omega})| \geq e^{\frac{cp(r,\omega)}{4}}|\xi_\nu(\tilde{\omega})|$.

Now we will prove that

$$|\xi_{\nu'}(\tilde{\omega})| \geq Ce^{(\nu'-j)c}|\xi_j(\tilde{\omega})|$$

for $j = \nu + p(r, \omega) + 1, \ldots, \nu'$.

By lemma 2.4.2,
$$|\partial_a f_a^j(x)| \leq Ce^{4j}.$$

By lemma 2.3.5, we also have that

$$|a-b| \leq Ce^{-cj}|\xi_j(\tilde{\omega})|.$$

Therefore,

$$|f_a^{k-j}(\xi_j(b)) - f_b^{k-j}(\xi_j(b))| \leq Ce^{4(k-j)}e^{-cj}|\xi_j(\tilde{\omega})|.$$

Hence, if we take $a, b \in \tilde{\omega}$ as the end values of the interval, we get

$$\begin{aligned}
|\xi_k(\tilde{\omega})| &= |f_a^{k-j}(\xi_j(a)) - f_b^{k-j}(\xi_j(b))| \\
&\geq |f_a^{k-j}(\xi_j(a)) - f_a^{k-j}(\xi_j(b))| - |f_a^{k-j}(\xi_j(b)) - f_b^{k-j}(\xi_j(b))| \\
&\geq |\xi_j(\tilde{\omega})| \left(|Df_a^{k-j}(x_j) - Ce^{4(k-j)}e^{-cj}\right)
\end{aligned}$$

where $x_j \in \xi_j(\tilde{\omega})$.

Now suppose that $k - j \leq \frac{c}{c+4}j$. In this case, it is sufficient to take $\beta < c\left(1 - \frac{4}{c+4}\right)$. Indeed, in this case we would have that $e^{4(k-j)}e^{-cj} \leq e^{-\beta j}$, and then
$$d_H(\xi_k(\tilde{\omega}), f_a^{k-j}(\xi_j(\tilde{\omega})) \leq e^{-\beta j}|\xi_k(\tilde{\omega})|. \tag{2.16}$$

If $k - j > \frac{c}{c+4}j$, then we will split in sum of smaller terms. Take $W_1 \subset W_2 \subset W_3 \subset W_4$ small neighbourhoods of 0, of sizes $c', 2c', 3c'$ and $4c'$ for some small c', respectively.

Now choose $\nu + p + 1 = k_0 \leq k_1 \leq k_\mu = \nu'$ so that

$$k_{i+1} - k_i \leq \frac{c}{c+4}k_i.$$

It is possible because we can assume that the first return is sufficiently large. If we take c' sufficiently small, we may also assume that for every $m = k_i + 1, \ldots, k_{i+1} - 1$,
$$\xi_m(\tilde{\omega}) \cap W_4 = \emptyset$$

and $\xi_{k_{i+1}}(\tilde{\omega}) \cap W_4 = \emptyset \Rightarrow \frac{c}{2(c+4)}k_i \leq k_{i+1} - k_i$.

If $\xi_{k_{i+1}}(\tilde{\omega}) \cap W_2 \neq \emptyset$ and $\xi_{k_{i+1}}(\tilde{\omega})$ does not contain a component of $W_3 \backslash W_2$, then equation 2.16 implies that $f_a^{k_{i+1}-k_i}(\xi_{k_i}(\tilde{\omega})) \subset W_4$. Also, since $\xi_j(\tilde{\omega}) \cap U = \emptyset$ for $j = k_i, \ldots, k_{i+1} - 1$, then we have that $f_a^{j-k_i}(\xi_{k_i}(\tilde{\omega})) \cap U' = \emptyset$ for some U' smaller than U. Therefore, since it remains outside U' and enter in W_4 we may apply the basic lemma (2.3.1) to conclude that

$$|Df_a^{k_{i+1}-j}(\bar{x}_j)| \geq C_0 e^{c(k_{i+1}-j)},$$

for $j = k_i, \dots, k_{i+1}$.

Otherwise, if $\xi_{k_{i+1}}(\tilde{\omega}) \cap W_2 \neq \emptyset$ but $\xi_{k_{i+1}}(\tilde{\omega})$ contain a component of $W_3 \setminus W_2$, then $\nu' - k_{i+1}$ is bounded. Again, from the basic lemma follows that

$$|Df_a^{k_{i+1}-j}(\bar{x}_j)| \geq C_0 e^{c(\nu'-j)}$$

for $j = k_i, \dots, \nu'$.

Finally, from the equation 2.16, follows that if $\xi_{k_{i+1}} \cap W_2 = \emptyset$, then $f_a^m(\bar{x}_j) \in W_1$ for $m = 0, \dots, k_{i+1} - j$ and using again the basic lemma, we can conclude that

$$|Df_a^{k_{i+1}-j}(\bar{x}_j)| \geq C_0 e^{c(k_{i+1}-j)},$$

for $j = k_i, \dots, k_{i+1}$.

Using $(BA)'_{n-1}$, and the fact that α is small, this implies

$$|Df_a^{k_{i+1}-k_i}(\bar{x}_{k_i})| \geq C e^{c(k_{i+1}-k_i)}$$

this implies the second inequality and for the last one is a consequence of the fact that $\nu' - \nu \to \infty$ as $|\omega_0| \to 0$, and then we have proved the statements. ■

Finally, we will compare $D_k(a)$ through an interval of the partition, and with lemma 2.3.5, conclude the proposition.

Lemma 2.4.5 *There exists a constant C such that for R sufficiently large there exists $\epsilon > 0$ with the following property. If $\omega \in \mathcal{E}_{n-1}$ is such that $\omega \subset (BA)'_{n-1} \cap (EX)_{n-2}$ and n is a free return, then*

$$\frac{|D_k(a)|}{|D_k(b)|} < C$$

for all $k \in \{0, \dots, n-1\}$ and $a, b \in \omega$, when $\xi_n(\omega) \subset U_{\frac{R}{2}}$.

Proof. By construction of $\mathcal{E}_n$, every point in a set ω of this partition, have the same return times $\nu_1 < \nu_2 < \ldots \leq n$. Take s such that $\nu_s \leq k < \nu_{s+1}$. Then, $\nu_j(\omega)$ is a subset of I_{r_j, r'_j} and possibly two neighbours of the ends of intervals, for some $j = 1, \dots, s$ and $\nu_j \geq R(1 \leq \nu'_j \leq r_j^2)$.

From the previous lemma, and since $\nu_{j+1} - \nu_j$ is large when R is large, we can conclude that $|\xi_{\nu_j+1}(\omega)| \geq 2|\xi_{\nu_j(\omega)}|$. By lemma 2.3.5, $\frac{|\xi_n(\omega)|}{|\omega|} \leq Ce^{cn}$ and therefore, $|\omega| \leq Ce^{-cn}$.

As we want estimate

$$\frac{|D_k(a)|}{|D_k(b)|} = \left(\frac{a}{b}\right)^k \prod_{i=1}^{k} \frac{\xi_i(a)}{\xi_i(b)}$$

and $\frac{a}{b} = \frac{a-b}{b} + 1 \leq |\omega| + 1 \leq Ce^{-cn} + 1$, then $\left(\frac{a}{b}\right)^k$ is bounded and therefore, we have to prove that $\prod_{i=1}^k \frac{\xi_i(a)}{\xi_i(b)}$ is also bounded.

As we did in the proof of lemma 2.3.2, it is sufficient to show that

$$\sum_{i\geq 1}^{k} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b)|}$$

is bounded. In order to do this, we will consider firstly $k \leq k_0$, where $k_0 \leq n$ is the largest integer satisfying $|\xi_{k_0}(\omega)| \leq |U|$. Now we will split this sum up in the following parts, for $j = 1, \ldots, s$:

$$S_0'' = \sum_{i=1}^{\nu_0 - 1} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b)|},$$

$$S_j' = \sum_{i=\nu_j}^{\nu_j + p_j} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b)|}$$

and

$$S_j'' = \sum_{i=\nu_j + p_j + 1}^{\nu_{j+1} - 1} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b)|}$$

If $\nu_s + p_s \geq k$, we will take

$$S_s' = \sum_{i=\nu_s}^{k} \frac{|\xi_i(a) - \xi_i(b)|}{|\xi_i(b)|}$$

and

$$S_s'' = 0$$

instead.

By the previous lemma, if we take $i = \nu_j + p_j + 1, \ldots, \nu_{j+1}$, we can bound this terms as follows:

$$(\nu_{j+1} - p_j - \nu_j)|\xi_{j+1}(\omega)| \geq C \sum_{i=\nu_j + p_j + 1}^{\nu_{j+1}} Ce^{\gamma_0(\nu_{j+1} - i)} |\xi_i(a) - \xi_i(b)|.$$

Since this is the period which the iterates are outside of U, we also have that $|\xi_i(b)| \geq |U|$. Therefore,

$$S_j'' \leq \frac{C}{|U|} |\xi_{\nu_j + 1}(\omega)|.$$

From the previous lemma, it follows that $|\xi_k(\omega)| \geq 2|\xi_{\nu_j+1}(\omega)| \geq$

$4|\xi_{\nu_j}(\omega)|$. Hence,

$$\sum_{j=1}^{s} S_j'' \leq \sum_{j=1}^{s-1} \frac{C}{|U|}|\xi_{\nu_j+1}(\omega)| + \frac{C}{|U|}|\xi_k(\omega)| \leq \frac{C}{|U|}|\xi_{\nu_k}(\omega)|$$

and therefore the sum for the free orbit it bounded.

From lemmas 2.3.2 and 2.4.3, it follows that

$$\begin{aligned}|\xi_i(b) - \xi_i(a)| &\leq C|f_a^{i-\nu_j-1}(\xi_j(\omega))| \leq C|Df_a^{i-\nu_j-1}(x_{\nu_j+1})||\xi_{\nu_j+1}(\omega)| \\ &\leq C|D_{i-\nu_j-1}(a)||f_a(\omega_{\nu_j}(a))|\end{aligned}$$

for $\nu_j \leq i \leq \nu_j + p_j$.

By mean value theorem and the definition of p_j,

$$|D_{i-\nu_j-1}(a)||\xi_{\nu_j+1}(a)| \leq |f_a^{i-\nu_j}(\xi_{\nu_j}(a)) - \xi_{i-\nu_j}(a)| \leq e^{-\beta(i-\nu_j)}.$$

Therefore,

$$|\xi_i(b) - \xi_i(a)| \leq C\frac{|f_a(\omega_{\nu_j}(\omega))|e^{-\beta(i-\nu_j)}}{|\xi_{\nu_j+1}(a)|} \leq C\frac{|\omega_{\nu_j}(\omega)|e^{-\beta(i-\nu_j)}}{|U_{r_j}|}$$

where the last inequality follows from the definition of the function f_a.

Using $(BA)'_{n-1}$ and the definition of binding period, we conclude the following inequalities:

$$|\xi_i(b)| \geq |\xi_{i-\nu_j(b)}|/2 - |\xi_i(b) - \xi_{i-\nu_j}(b)| \geq e^{-\alpha(i-\nu_j)}/2 - e^{-\beta(i-\nu_j)} \geq \frac{e^{-\alpha(i-\nu_j)}}{4}$$

Therefore,

$$S_j' \leq \sum_{i=\nu_j}^{\nu_j+p_j} \frac{|\xi_i(b) - \xi_i(a)|}{|\xi_i(b)|} \leq C\sum_{i=\nu_j}^{\nu_j+p_j} \frac{|\omega_{\nu_j}(\omega)|e^{-\beta(i-\nu_j)}}{|U_{r_j}e^{-\alpha(i-\nu_j)}|} \leq C\frac{|\omega_{\nu_j}(\omega)|}{|U_{r_j}|}.$$

Let A_r be the set of indices $j < s$ such that $\omega_{\nu_j}(\omega) \cap I_r \neq \emptyset$. By the previous lemma, $|\omega_{\nu_{j+1}}(\omega)| \geq 2|\omega_{\nu_j}(\omega)|$, and then

$$\sum_{j\in A_r} \frac{|\omega_{\nu_j}(\omega)|}{|U_r|} \leq C\max_{j\in A_r} \frac{|\omega_{\nu_j}(\omega)|}{|U_r|}.$$

By definition of partition, each of $\omega_{\nu_j}(\omega)$ is contained in at most three of the r^2 intervals $I_{r,r'}$, and then

$$\frac{|\omega_{\nu_j}(\omega)|}{|U_r|} \leq \frac{3e^{-r}}{r^2e^{-r}} = \frac{3}{r^2}.$$

Therefore,

$$\sum_{j=1}^{s} S'_j \leq C \sum_{j=1}^{s} \frac{|\omega_{\nu_j}(\omega)|}{|U_r|} \leq C \sum_r \max_{j \in A_r} \frac{|\omega_{\nu_j}(\omega)|}{|U_r|} \leq C \sum_r \frac{3}{r^2} < 10C.$$

Since we supposed that $k \leq k_0$ we still have to prove the cases $k = k_0 + 1, \ldots n$.

Again, by previous lemma, $|\xi_i(\omega)|$ grows exponentially, and $|\xi_i(\omega)| \geq C_0|U|$ for $i = k_0, \ldots, n$. From this inequality and the way we divide the intervals in the partition, we conclude that $|\xi_i(\omega)|$ does not contain an interval I_r. Hence, $\xi_i(\omega) \cap U_{R+q} = \emptyset$ for some q sufficiently large and $i = k_0, \ldots, n-1$.

Since every interval eventually intersects the critical point 0, there exists $N_0(R)$ such that $0 \in f_2^j(\xi_{k_0}(\omega))$ for some $j \leq N_0(R)$ for each $\omega \subset \omega_0$. Note that we may take $N_0(R) < \infty$ because $|\xi_i(\omega)| \geq C_0|U|$. The same holds for $|\xi_i(\omega)|$, because we can take ω_0 sufficiently small. Therefore, $n - k_0 \leq N_0(R)$. Since we want to show that

$$\prod_{i=k_0}^{k} \frac{\xi_i(a)}{\xi_i(b)}$$

is bounded, now we only have to show that $\frac{\xi_i(a)}{\xi_i(b)}$ is bounded for $i = k_0, \ldots, k$. By the basic lemma (2.3.1), if $f_2^{k-k_0}(x, y)$ is a subset of a neighbourhood of 0, then for every $j = 0, \ldots, k - k_0$ we have the following:

$$\frac{|Df_2^j(x)|}{|Df_2^j(y)|} < K. \tag{2.17}$$

We can take ω_0 sufficiently small in order to obtain a small Hausdorff distance between $\xi_k(\omega)$ and $f_2^{k-k_0}(\xi_{k_0}(\omega))$, for $k = k_0, \ldots, n-1$. Since $\xi_n(\omega) \subset U_{R/2}$, we conclude that $f_2^{n-k_0}(\xi_{k_0}(\omega))$ is contained in a small neighbourhood of 0. Therefore the equation 2.17 holds and, using the fact that the Hausdorff distance between $\xi_k(\omega)$ and $f_2^{k-k_0}(\xi_{k_0}(\omega))$ is small, we conclude that

$$\frac{|Df_a^j(x)|}{|Df_b^j(y)|} < CK,$$

for every $a, b \in \omega$, $x, y \in \xi_{k_0}(\omega)$ and $j = 0, \ldots, n - k_0$. Hence, $\frac{\xi_i(a)}{\xi_i(b)}$ is also bounded as we wanted to show. ■

From lemma 2.3.5 and the lemmas we just proved, the proposition 2.4.1 follows.

Now we have to prove that $(BA)'_n$ and $(FA)'_n$ are satisfied by a large set of parameters. With the following result we will conclude that for each step n, the part of $(BA)'_{n-1}$ lost in $(BA)'_n$ is small.

Lemma 2.4.6 *For R sufficiently large, there exist ϵ, C_0 such that, for every*

$\omega \in \mathcal{E}_{n-1}$ contained in $(BA)'_{n-1} \cap (EX)_{n-2}$ and with a return at time n, if $|\omega_0| < \epsilon$, then

$$\frac{|\omega \setminus \bigcup_{r \geq \alpha n} \omega_{r,r'}|}{|\omega|} \geq 1 - e^{-\alpha n C_0}$$

for $n \geq N$.

Proof. Take $\nu < n$ a free return such that $|\xi_\nu(\omega)| \geq \frac{e^{-r}}{r^2}$, for $r \leq \alpha\nu < \alpha n$ and binding period p. By lemmas 2.4.4 and 2.3.2, we have the following:

$$\begin{aligned} |\xi_n(\omega)| &\geq e^{c\frac{p}{4}} |\xi_\nu(\omega)| \geq e^{c\frac{p}{4}} \frac{e^{-r}}{r^2} \\ &\geq \frac{e^{(-1+C_0)r}}{r^2} \geq e^{(-1+C_0/2)\alpha n}. \end{aligned}$$

By previous lemma, if we take $\tilde{\omega} \subset \omega$ as the largest interval such that $\xi_n(\tilde{\omega}) \subset U_{R/2}$, we also get the inequality for $|\xi_\nu(\tilde{\omega})|$. Using the mean value theorem, we may write $|\bigcup\limits_{r > \alpha n} \omega_{r,i}|$ and $|\tilde{\omega}|$ as

$$|\bigcup_{r > \alpha n} \omega_{r,i}| = \frac{|\xi_n(\bigcup_{r > \alpha n} \omega_{r,i})|}{|\xi_n'(\tilde{a})|} \leq \frac{e^{-\alpha n}}{|\xi_n'(\tilde{a})|}$$

and

$$|\tilde{\omega}| = \frac{|\xi_n(\tilde{\omega})|}{|\xi_n'(\tilde{b})|}.$$

By the proposition we have proved, we also have the following

$$\frac{|\bigcup_{r > \alpha n} \omega_{r,i}|}{|\omega|} \leq \frac{|\bigcup_{r > \alpha n} \omega_{r,i}|}{|\tilde{\omega}|} \leq C \frac{e^{-\alpha n}}{|\xi_n(\tilde{\omega})|} \leq C e^{-C_0 \alpha n/2} \leq e^{-C_0 \alpha n/4}.$$

Therefore,

$$\frac{|\omega \setminus \bigcup_{r \geq \alpha n} \omega_{r,r'}|}{|\omega|} \geq 1 - e^{-\alpha n C_0}.$$

■

Taking the summation of ω in both sides, we conclude that

$$\frac{|(BA)'_n|}{|(BA)'_n \cap (EX)_{n-1}|} \geq 1 - e^{-\alpha n C_0}.$$

Here we also used the fact that the part of ω in $(BA)'_{n-1} \cap (EX)_{n-1}$ which does not satisfy $(BA)'_n$ are the sets $\omega_{r,r'}$.

Now we will show that essential returns are frequent, which is not a good scenario per se, but the next lemma will help us to handle with.

Lemma 2.4.7 *For R sufficiently large, there exists $\epsilon > 0$ such that for any $\omega \in \mathcal{E}_{\hat{\nu}}$ with an essential return and host interval $I_{r,r'}$, $r \geq R - 1$ at time $\hat{\nu}$, if*

$\hat{\nu}'$ is the next return of ω and $\omega \subset (EX)_{\hat{\nu}'-1} \cap (BA)'_{\hat{\nu}'-1}$, then

$$\hat{\nu}' - \hat{\nu} \leq \frac{4r}{c}$$

and if the equality holds, then ω has a substantial escape at time $\hat{\nu}'$. Moreover, if $\omega' \subset \omega$ is such that $\xi_{\hat{\nu}'}(\omega') \subset I_{\hat{r},\hat{r}'}$, then

$$|\omega'| \leq Ce^{6\frac{\beta r}{c}} e^{-\hat{r}} |\omega|.$$

Proof. If ω has an escape at time $\hat{\nu}$, then replace it by the subset $\tilde{\omega}$ for which $\xi_{\hat{\nu}}(\tilde{\omega}) = I_{R,R^2}$. Let $\hat{\nu} = \nu_0 < \ldots < \nu_k = \hat{\nu}'$ be the returns between $\hat{\nu}$ and $\hat{\nu}'$. Since $\hat{\nu}$ is the next essential return, the other returns are inessential.

For each ν_j define the respectively binding period p_j and q_j. By lemma 2.4.3, the intervals $\xi_{\nu_j+p_j}(\omega)$ and $f_a^{p_j}(\xi_{\nu_j}(\omega))$ are very close.

Besides that, $|f_a^{p_j}(I_{r_j})| > Ce^{-\beta p_j}$ and therefore,

$$\frac{|\xi_{j+p_j}(\omega)|}{|\xi_j(\omega)|} \geq \frac{Ce^{-\beta p_j}}{|I_{r_j,r_j'}|} \geq Ce^{-\beta p_j} r_j^2 e^{r_j}$$

and, since $p_j \leq 3\frac{r_j}{c}$, then

$$\frac{|\xi_{j+p_j}(\omega)|}{|\xi_j(\omega)|} \geq e^{(1-4\beta/c)r_j} e^{\beta p_j/2}.$$

By lemma 2.4.4,

$$\begin{aligned} |\xi_{\nu_j+p_j}(\omega)| &\geq Ce^{cq_j} |\xi_{\nu_j+p_j}(\omega)| \\ &\geq Ce^{cq_j} e^{(1-4\beta/c)r_j} e^{\beta p_j/2} |\xi_j(\omega)| \\ &\geq e^{cq_j} e^{(1-5\beta/c)r_j} |\xi_j(\omega)| \end{aligned}$$

because $q_j + p_j \to \infty$ as $\epsilon \to 0$.

Since the first return is essential, $|\xi_{\nu_j}(\omega)| \approx \frac{e^{-r}}{r^2}$, and then

$$|\xi_{\nu_1}(\omega)| \geq e^{cq_0} e^{(1-5\beta/c)r} \frac{e^{-r}}{r^2} \geq e^{cq_0 - \frac{5\beta}{c}}.$$

Therefore, $q_0 \leq \frac{5\beta r}{c}$. Moreover, for $2 \leq j \leq k$,

$$|\xi_{\nu_j}(\omega)| \geq |\xi_{\nu_1}(\omega)| \prod_{m=1}^{j-1} e^{cq_m} e^{\left(1-\frac{5\beta}{c}\right)r_m} \geq e^{cq_0 - \frac{5\beta}{c}} \prod_{m=1}^{j-1} e^{cq_m} e^{\left(1-\frac{5\beta}{c}\right)r_m}.$$

Then, for $j = 1, \ldots, k$

$$|\xi_{\nu_j}(\omega)| \geq e^{-5\beta r/c} \tag{2.18}$$

and

$$\sum_{m=1}^{k}(cq_m + (1-5\beta/c)r_m) \geq 5\beta r/c \Rightarrow |\xi_{\nu_k}(\omega)| \geq 1.$$

Therefore, since $p_j \leq \frac{3r_j}{c}$, either $|\xi_{\nu_k}(\omega)| \geq 1$, or

$$\begin{aligned}
\nu_k - \nu_0 = \sum_{m=0}^{k-1}(p_m+q_m) &\leq \frac{3r}{c} + 5\frac{\beta r}{c^2} + \sum_{m=1}^{k-1}(\frac{3r_m}{c}+q_m) \\
&\leq \frac{1}{c}\left(3r + \frac{5\beta r}{c} + \sum_{m=1}^{k-1}(3r_m + cq_m)\right) \\
&\leq \frac{1}{c}\left(3r + \frac{5\beta r}{c} + 16\frac{\beta r}{c}\right) \leq \frac{4r}{c}.
\end{aligned}$$

Now suppose that $\omega' \subset \omega$ is so that $\xi_{\hat{\nu}'}(\omega') \subset I_{\hat{r},\hat{r}'}$. Then, if $\xi_{\hat{\nu}'}(\omega) \subset U_{R/2}$ (if it is false, just shrink the interval ω), then we may use proposition 2.4.1 and get

$$\frac{|\omega'|}{|\omega|} \leq C\frac{|\xi_{\hat{\nu}'}(\omega')|}{|\xi_{\hat{\nu}'}(\omega')|}$$

and by equation 2.18, we get the result:

$$\frac{|\omega'|}{|\omega|} \leq Ce^{5\beta r/c}e^{\hat{r}}$$

as we wanted to show. ■

From this lemma, follows that, if $\hat{\nu} = \hat{\nu}_0 < \ldots < \hat{\nu}_s \leq n$ are subsequent essential return of a, then:

$$\frac{|\omega_{\hat{\nu}_s}(a)|}{|\omega_{\hat{\nu}}(a)|} \leq C^s e^{\sum_{i=1}^{s}(6\beta r_{i-1}/c - r_i)} \leq C^s e^{-7/8\sum_{i=1}^{s} r_i + 6\beta r/c}.$$

With the following function, we will estimate the F_n. Let $\hat{\nu}$ be an essential return of a.

Then we define the function E as,

$$E(a;\hat{\nu}) = \inf\{k > 0; \omega_{\hat{\nu}+k}(a) \textit{ has a substantial escape at time } \hat{\nu}+k\}$$

Since we want to prove that the derivatives are large, it would be useful to verify that the function defined above is small, i.e., a substantial escapes happens shortly after an essential return, as we will see on the following theorem.

Lemma 2.4.8 *For R sufficiently large, there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ and every $\omega \in \mathcal{E}_{\hat{\nu}}$ which has an essential return at time $\hat{\nu} < n$ and is contained in $(BA)'_{n-1} \cap (EX)_{n-1}$ and host interval $I_{r,r'}$ one has*

$$\int_{\{a\in\omega;\frac{6r}{c}\leq E(a;\hat{\nu})\leq n-\hat{\nu}\}} e^{cE(a;\hat{\nu})}da \leq e^{-\frac{r}{8}}|\omega|$$

Proof. To prove this inequality, we will consider the level sets of the function E, that is, $\{a \in \omega; E(a;\hat{\nu}) = t\}$, but it will also be relevant to know how many essential returns occurred before the substantial escape and how close to the origin these returns got. For this, we will define the set

$$\begin{aligned} A_{s,B} = &\{a \in \omega; a \text{ has the first substantial escape at the } (s+1)\text{-th} \\ &\text{essential return following } \hat{\nu} \text{ and for which } \sum_{i=1}^{s} r_i = B\}. \end{aligned}$$

Here, r_j are the integers for which I_{r_j,r'_j} are the host intervals for the s returns.

Now, we will estimate the number of components of $\mathcal{E}_n$ in $A_{s,B}$. Since $r_i \geq R$, then $B \geq sR$ and therefore $A_{s,B} = \emptyset$ for $\frac{s}{B} > \frac{1}{R}$. Suppose that $\frac{s}{B} \leq \frac{1}{R}$. There are $\binom{B+s-1}{s-1}$ different ways to sum s non-negative integers and result in B. This is easily seen if we take $B + s - 1$ spaces and choose $s - 1$ spaces to take out. Then we may count the left spaces between empty spaces and sum these numbers.

Therefore, the number of components in $A_{s,B}$ is

$$2^s\binom{B+s-1}{s-1}\Pi_{i=1}^{s} r_i^2$$

where 2^s and $\Pi_{i=1}^{s} r_i^2$ were taken considering the possibilities of the side of intervals I_{r_i} and components I_{r_i,r'_i} respectively. Using Stirling's Formula (see [12]), $n! \approx (\frac{n}{e})^n\sqrt{2\pi n}$, and the fact that $e^{\frac{t}{16}} \geq t^2$ for t large, there are at most

$$C.2^s\frac{(s+B-1)^{s+B-1}}{B^B(s-1)^{s-1}}\sqrt{\frac{s+B-1}{(s-1)B}}e^{\left(\frac{1}{16}\right)\sum_{i=1}^{s} r_i}$$

components. Rearranging the terms, we get

$$C2^s\left(\frac{s+B-1}{B}\right)^{B+\frac{1}{2}}\left(\frac{s+B-1}{s-1}\right)^{s-1}\frac{1}{\sqrt{s-1}}e^{\frac{B}{16}}$$

and since $\frac{S}{B} \leq \frac{1}{R}$,

$$\left(\frac{s+B-1}{B}\right)^{B+\frac{1}{2}} = \left[\left(1+\frac{s-1}{B}\right)^{\frac{B}{s-1}}\right]^{\frac{(s-1)(B+\frac{1}{2})}{B}} \leq e^s$$

for R sufficiently large. Moreover, since $\frac{B}{s} \geq R$,

$$\begin{aligned}
&2eC\left(\frac{s+B-1}{s-1}2e\right)^{s-1} = 2eC\left(2e+\frac{2eB}{s-1}\right)^{s-1} \\
&\leq 2eC\left(1+\frac{10B}{s-1}\right)^{s-1} = \left\{(2eC)^{\frac{1}{B}}\left[\left(1+\frac{10B}{s-1}\right)^{\frac{s-1}{10B}}\right]^{10}\right\}^B \\
&= (1+o(R))^B.
\end{aligned}$$

And then there are at most

$$e^{\frac{B}{16}}.(1+o(R))^B$$

components of $A_{s,B}$. Let $\hat{\omega_s}$ be the largest one. So we get,

$$|A_{s,B}| \leq e^{\frac{B}{16}}(1+o(R))^B|\hat{\omega_s}|.$$

By the previous lemma,

$$E(a;\hat{\nu}) \leq \hat{\nu}_{s+1} - \hat{\nu} = (\hat{\nu}_1 - \hat{\nu}) + \ldots + (\hat{\nu}_{s+1} - \hat{\nu}_s) \leq \frac{(4B+4r_0)}{c}.$$

Now define the sets $L_{s,B,t} = \{a \in \omega; E(a;\hat{\nu}) = t\} \cap A_{s,B}$. They are empty if $s > \frac{B}{R}$ or $n - \hat{\nu} \geq t \geq \frac{4R+4r_0}{c}$.

Therefore,

$$\begin{aligned}
|\{a \in \omega; E(a;\hat{\nu}) = t \leq n-\hat{\nu}\}| &= \sum_{s,B}|L_{s,B,t}| \leq \sum_{s\leq\frac{B}{R},B\geq\frac{ct}{4}-r_0}|A_{s,B}| \\
&\leq \sum_{B\geq\frac{ct}{4}-r_0}\sum_{s=1}^{\frac{B}{R}} e^{\frac{B}{16}}(1+o(R))^B|\hat{\omega_s}|
\end{aligned}$$

and using the previous lemma equation, we get

$$\begin{aligned}
|\{a \in \omega; E(a;\hat{\nu}) = t \leq n-\hat{\nu}\}| &\leq \sum_{B\geq\frac{ct}{4}-r_0}\sum_{s=1}^{\frac{B}{R}} C^s e^{\frac{B}{16}}(1+o(R))^B e^{-\frac{7}{8}B+6\beta\frac{r_0}{c}}|\omega| \\
&\leq \sum_{B\geq\frac{ct}{4}-r_0}(1+o(R))^B e^{-\frac{3}{4}B+6\beta\frac{r_0}{c}}|\omega|
\end{aligned}$$

$$\leq \sum_{B \geq \frac{ct}{4} - r_0} e^{(o(R) - \frac{3}{4})B} e^{6\beta \frac{r_0}{c}} |\omega|$$
$$\leq C e^{(o(R) - \frac{3}{4})(\frac{ct}{4} - r_0)} e^{6\beta \frac{r_0}{c}} |\omega|$$

If $n - \hat{\nu} \geq t \geq \frac{6r_0}{c}$, then

$$|\{a \in \omega; E(a;\hat{\nu}) = t \leq n - \hat{\nu}\}| \leq e^{-\frac{t}{20}} |\omega| \leq C e^{(o(R) - \frac{3}{4}) \frac{t}{12}} e^{6\frac{\beta r_0}{c}} |\omega|$$
$$\leq e^{-\frac{t}{20}} |\omega|$$

and since we took $c < \frac{1}{40}$, it follows that

$$\int_{\{a \in \omega; \frac{6r_0}{c} \leq E(a;\hat{\nu}) \leq n - \hat{\nu}\}} e^{cE(a;\hat{\nu})} da \leq \sum_{t \geq \frac{6r_0}{c}} e^{ct} |\{a \in \omega; E(a;\hat{\nu}) = t\}|$$
$$\leq \sum_{t \geq \frac{6r_0}{c}} e^{ct} e^{-\frac{t}{20}} |\omega| \leq |\omega| \sum_{t \geq \frac{6r_0}{c}} e^{-\frac{t}{20}}$$
$$\leq |\omega| e^{-\frac{3r_0}{10c}} \frac{1}{1 - e^{-\frac{1}{20}}} \leq e^{-\frac{r_0}{8}} |\omega|.$$

■

For $E(a;\hat{\nu}) \leq \frac{6r_0}{c}$ we will use only the trivial inequality,

$$\int_{\{a \in \omega; E(a;\hat{\nu}) \leq \frac{6r_0}{c}\}} e^{cE(a;\hat{\nu})} da \leq e^{\frac{r_0}{6}} |\omega|. \tag{2.19}$$

Let $e_{i+1}(a)$ be the smallest integer such that $\omega_{\bar{\nu}_i}$ has a substantial return at time $\bar{\nu}_i + e_{i+1}(a)$. Then, for $i \geq 0$, let

$$E_i(a) = \begin{cases} 0, \text{if } a \text{ escapes at time } \bar{\nu}_i(a) \\ e_{i+1}(a), \text{otherwise.} \end{cases}$$

Also, define

$$T_n(a) = \sum_{i=0}^{s-1} E_i(a),$$

where s is the maximal integer such that $e_1(a) + \ldots + e_s(a) \leq n$. Then $T_n(a)$ is constant on $\omega \in \mathcal{E}$. Now we will estimate F_n in terms of T_n.

Lemma 2.4.9 *For R sufficiently large, if $a \in \omega \in (EX)_{n-1} \cap (BA')_{n-1}$ has a substantial escape at time $\bar{\nu}_i(a)$ and if the next return occurs at time $\bar{\nu}_{i+1}(a) \leq n$, then it also has a substantial escape. In particular, $F_n(a) \geq n - T_n(a)$.*

Proof. Indeed, by definition $n - T_n(a)$ is the sum of E_i for the indices that we have a substantial escape at time $\bar{\nu}_a(a)$. Then, $\xi_{\bar{\nu}_a(a)}(\omega_{\bar{\nu}_i(a)})$ is outside U and contains an interval of length $\geq \sqrt{|U|}$. As we showed before, if $\nu_i'(a)$ is the next return, then

$$|\xi_{\nu_i'(a)}(\omega_{\bar{\nu}_i(a)})| \geq \sqrt{|U|}$$

That is, this return has a substantial escape. Therefore, we can write $\nu_i'(a)$ as $\nu_i'(a) = \bar{\nu}_i(a) + E_i(a) = \bar{\nu}_{i+1}(a)$. Furthermore, $\xi_k(\omega_{\bar{\nu}_i})$ stays outside U for $\bar{\nu}_i \leq k \leq \bar{\nu}_i + E_i(a)$. Since this part of the orbit is not in binding period, we got a part of a free orbit of length $E_i(a)$. Hence, $F_n(a) \geq n - T_n(a)$. ■

The following lemma gives us an estimative of T_n that will be useful to show FA is satisfied by a large set of parameters.

Lemma 2.4.10 *For R sufficiently large, let $\hat{\omega}$ the union of sets $\omega \in \mathcal{E}_{n-1}$ such that $\omega \subset (EX)_{n-1} \cap (BA)'_{n-1}$. Then,*

$$\int_{\hat{\omega}} e^{cT_n(a)} da \leq e^{\delta n}.|\hat{\omega}|$$

and

$$|\{a \in \hat{\omega}; T_n(a) \geq \delta n\}| \leq e^{-\delta n \frac{c}{2}} |\hat{\omega}|.$$

Proof. Firstly we will divide the set $\hat{\omega}$ into subsets with a fixed number of substantial escapes and then subdivide it again considering host intervals, that is, let $\omega^s \subset \hat{\omega}$ be the set of parameters with s substantial escapes and $\omega_{\bar{r}_1,\ldots,\bar{r}_s} \subset \omega^s$ be the set of parameters whose host interval of the i-th substantial return is $I_{\bar{r}_i}$.

Then we have,

$$\int_{\hat{\omega}} e^{cT_n(a)} da = \sum_{s=0}^{\infty} \int_{\omega^s} e^{cT_n(a)} da = \sum_{s=0}^{\infty} \sum_{\bar{r}_1,\ldots,\bar{r}_s} \int_{\omega_{\bar{r}_1,\ldots,\bar{r}_s}} e^{cT_n(a)} da. \tag{2.20}$$

Since $E_0, \ldots, E_{s-2}$ do not depend on $\bar{r}_s$, we may write this integral as

$$\sum_{s=0}^{\infty} \sum_{\bar{r}_1,\ldots,\bar{r}_{s-1}} \sum_{\bar{r}_s} \int_{\omega_{\bar{r}_1,\ldots,\bar{r}_s}} e^{cT_n(a)} da = \sum_{s=0}^{\infty} \sum_{\bar{r}_1,\ldots,\bar{r}_{s-1}} e^{cT_{n-1}(a)} \sum_{\bar{r}_s} \int_{\omega_{\bar{r}_1,\ldots,\bar{r}_s}} e^{cE_{s-1}(a)} da.$$

Furthermore, $E_i > 0$ only if $r_i \geq R$ and $E_{s-1}(a) \leq n - \hat{\nu}_s \leq n - \hat{\nu}_{s-1}$. Hence, we get (with $\bar{r}_1, \ldots, \bar{r}_{s-1}$ fixed),

$$\begin{aligned}
\sum_{\bar{r}_s} \int_{\omega_{\bar{r}_1,\ldots,\bar{r}_s}} e^{cE_{s-1}(a)} da &= \sum_{r_s \geq R} \sum_{r_s'=1}^{r_s^2} \int_{\omega_{\bar{r}_1,\ldots,\bar{r}_s}} e^{cE_{s-1}(a)} da \\
&= \sum_{r_s \geq R} \sum_{r_s'=1}^{r_s^2} \int_{\{a \in \omega_{\bar{r}_1,\ldots,\bar{r}_s}; \frac{6r_s}{c} \leq E_{s-1}(a) \leq n - \hat{\nu}_{s-1}\}} e^{cE_{s-1}(a)} da \\
&\quad + \int_{\{a \in \omega_{\bar{r}_1,\ldots,\bar{r}_s}; E_{s-1}(a) \leq \frac{6r_s}{c}\}} e^{cE_{s-1}(a)} da
\end{aligned}$$

Then, using 2.4.8, 2.19 and 2.4.1, we get

$$\begin{aligned}\sum_{\bar{r}_s}\int_{\omega_{\bar{r}_1,\dots,\bar{r}_s}} e^{cE_{s-1}(a)}da &\leq \sum_{r_s\geq R}\sum_{r'_s=1}^{r_s^2}|\omega_{\bar{r}_1,\dots,\bar{r}_s}|\left(e^{-\frac{r_s}{8}}+e^{\frac{r_s}{6}}\right)\\ &\leq C.\sum_{\bar{r}_s}|\omega_{\bar{r}_1,\dots,\bar{r}_s}|.\end{aligned}$$

Doing the same thing with other $\bar{r}_i's$, we get by induction,

$$\begin{aligned}\sum_{\bar{r}_k,\dots,\bar{r}_s}\int_{\omega_{\bar{r}_1,\dots,\bar{r}_s}} e^{c(E_{s-1}(a)+\dots+E_{k-1}(a))}da &\leq \sum_{r_{k-1}\geq R}\sum_{r'_{k-1}=1}^{r_{k-1}^2}\sum_{\bar{r}_k,\dots,\bar{r}_s}|\omega_{\bar{r}_1,\dots,\bar{r}_s}|\left(e^{-\frac{r_s}{8}}+e^{\frac{r_s}{6}}\right)\\ &\leq \sum_{r_k\geq R}\sum_{r'_k=1}^{r_k^2}C^{s-k}|\omega_{\bar{r}_1,\dots,\bar{r}_s}|\\ &\leq C^s\sum_{\bar{r}_1,\dots,\bar{r}_s}|\omega_{\bar{r}_1,\dots,\bar{r}_s}|\end{aligned}$$

Hence, summing every $\bar{r}_i$,

$$\sum_{\bar{r}_1,\dots,\bar{r}_s}\int_{\omega_{\bar{r}_1,\dots,\bar{r}_s}} e^{cT_n(a)}da \leq C^s|\omega^s| = \left(C^{\frac{s}{\delta^2 n}}\right)^{\delta^2 n}|\omega^s| \leq e^{\delta^2 n}|\omega^s|.$$

The last inequality holds because $\frac{s}{n}$ goes to zero as $R\to\infty$. This is true because for every $l\in\mathbb{N}$, there exist $k,\epsilon>0$ such that if $|a-2|<\epsilon$, then every interval L such that $|L|<\sqrt{k}$ has the following property: $s\leq\#\{i\leq l : f_a^i(L)\cap U\neq\emptyset\}\leq 1$.

And finally,

$$\int_{\hat{\omega}} e^{cT_n(a)}da \leq \sum_{s=0}^{\infty}e^{\delta^2 n}|\omega^s| = |\hat{\omega}|e^{\delta^2 n}.$$

Therefore,

$$\begin{aligned}|\{a\in\hat{\omega}; T_n(a)\geq\delta n\}| &= \int_{\{a\in\hat{\omega};cT_n(a)\geq c\delta n\}} 1\,da\\ &\leq \int_{\{a\in\hat{\omega};cT_n(a)\geq c\delta n\}} e^{-c\delta n}.e^{cT_n(a)}\,da\\ &\leq e^{-c\delta n}\int_{\hat{\omega}} e^{cT_n(a)}da \leq e^{-c\delta n+\delta^2 n}|\hat{\omega}|\\ &= e^{-\delta n(c-\delta)}|\hat{\omega}| \leq e^{-\delta n\frac{c}{2}}|\hat{\omega}|\,.\end{aligned}$$

■

By the previous two lemmas, we conclude that

$$\frac{|(FA)_n \cap (EX)_{n-1} \cap (BA)'_{n-1}|}{|(EX)_{n-1} \cap (BA)'_{n-1}|} \geq 1 - e^{-\delta n^{\frac{c}{2}}} .$$

Therefore, there exists a large set of parameters that satisfy FA and since we showed the same for BA, we can conclude that there is a positive measure set of parameters for which the Lyapunov exponent is positive.

To summarize, since we wanted to show that the derivative is most of the time exponentially large, we began the proof by deriving partial results ensuring this property for some parameters. The choice of those parameters was made in such a way that we could extend these partial results to intervals of parameters (see proposition 2.4.1). In order to do this, we defined a partition into intervals that would essentially preserve the properties chosen in the beginning. The partition was chosen such that inside the same interval of the partition, the derivative with respect to the parameter does not vary too much.

This lead us to two sets of good parameters whose intersection has the property that for each element, the derivative of the iterate of the corresponding quadratic map is exponentially large until a given number of iterates n.

We then showed that at each step, we do not lose too much of the previous good set of parameters. This was achieved through a large deviations type argument. Therefore, after taking the intersections of those good sets of parameters, for every step (or number of iterations) n, we are left with a positive measure set (in fact with one where 2 is a Lebesgue density point).

3
Singer's theorem

The theorem we prove in this chapter states that under some assumptions, a certain type of attractor, a periodic sink, must attract a critical point. Together with Benedicks-Carleson's result, we will conclude that for a positive measure set of parameters a, the quadratic map $f_a = 1 - ax^2$ does not have periodic sinks.

For an integer $l \geq 1$, let $C^l([0,1])$ be the set of l times continuously differentiable transformations $F\colon [0,1] \to [0,1]$.

Definition 3.0.1 *Let $F \in C^1([0,1])$. The (forward) orbit of a point x (with respect to the transformation F) is the set*

$$\{x, F(x), \ldots, F^k(x), \ldots\}\,.$$

The set of all limit points of the orbit of x is called the ω-limit of x.

Definition 3.0.2 *Let p be a periodic point of period k for a transformation $F \in C^1([0,1])$, that is, $F^k(p) = p$.*

We call

$$\lambda(p) := \frac{dF^k}{dx}(p)$$

an eigenvalue.

If $|\lambda(p)| < 1$, we say that p is a periodic sink of F.

Lemma 3.0.1 *If p is a periodic sink for $F \in C^1([0,1])$ then there exists a neighbourhood U of p such that for all $x \in U$,*

$$\lim_{n\to\infty} |F^n(x) - F^n(p)| = 0\,,$$

with an exponential rate of convergence.

In other words, each point $x \in U$ is attracted to the orbit of p (a finite set, as p is periodic).

Proof. We know that $|(F^k)'(p)| < 1$ and $(F^k)'$ is continuous (as $F \in C^1([0,1])$). Thus there exist $0 < \rho < 1$ and an open interval V containing p such that

$$|(F^k)'(x)| \leq \rho < 1 \quad \text{for all } x \in V\,.$$

Since $(F^k)'$ is continuous and $F^k(p) = p$, there is an open interval U containing p such that

$$F^k(U) \subset V\,.$$

Using the mean value theorem, the fact that p is k-periodic and that U, V are intervals, we obtain inductively that for all integers $q \geq 1$,

$$|F^{qk}(x) - F^{qk}(p)| \leq \rho^q \quad \text{for all } x \in U\,.$$

Let $n \in \mathbb{N}$ and write it as $n = qk + r$, where $0 \leq r < k$.

Put

$$M := \max\{(F^r)'(x)\colon 0 \leq r < n \;\text{ and } x \in [0,1]\}\,.$$

Again by the mean value theorem, for every $x \in U$ there is $c_x \in [0,1]$ such that

$$\begin{aligned}|F^n(x) - F^n(p)| &= |F^{qk+r}(x) - F^{qk+r}(p))| \\ &= |F^r(F^{qk}(x)) - F^r(F^{qk}(p))| \\ &= |(F^r)'(c_x)|\,|F^{qk}(x) - F^{qk}(p)| \leq M\,\rho^q\,.\end{aligned}$$

As $n \to \infty$, also $q \to \infty$, so $M\,\rho^q \to 0$, which completes the proof. ■

Now we will introduce the concept of a stable manifold.

Definition 3.0.3 *Let p be a periodic point of period k for a transformation $F \in C^1([0,1])$ and denote $G = F^k$. The stable manifold of p is defined as*

$$S(p) = \{x : G^n(x) \to F^j(p) \textit{ for some } j \leq k\}\,.$$

Proposition 3.0.2 *The stable manifold $S(p)$ is an open set.*

Proof. Take the set U as in Lemma 3.0.1, i.e., an open interval around p for which every point is attracted to the orbit of p.

Define $V_n = G^{-n}(U)$. Since G is continuous,

$$V := \bigcup_{n \geq 0} V_n$$

is an open set. We will show that $V = S(p)$.

Indeed, if $x \in V$, then $G^m(x) \in U$ for some $m \in \mathbb{N}$. Therefore, by Lemma 3.0.1, we have:

$$0 = \lim_{n\to\infty} |F^n(G^m(x)) - F^n(p)| = \lim_{n\to\infty} |F^{nk}(G^m(x)) - F^{nk}(p)|$$

$$= \lim_{n\to\infty} |G^n(G^m(x)) - G^n(p)| = \lim_{n\to\infty} |(G^{n+m}(x)) - p| = \lim_{n\to\infty} |(G^n(x)) - p|\,,$$

which implies that $x \in S(p)$.

On the other hand, if $x \in S(p)$, since U is open and $p \in U$, it follows that $G^n(x) \in U$ for some n. Hence, $x \in V$.

Therefore $V = S(p)$, showing that $S(p)$ is open. ∎

We know introduce the concept of Schwarzian derivative.

Definition 3.0.4 *Let $f \in C^3([0,1])$. The Schwarzian derivative of f at a regular point x is defined as*

$$\{f, x\} = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

We describe below the basic properties of the Schwarzian derivative of a function f. A useful property of the function f, which will show up later, is having negative Schwarzian derivative at all points.

Lemma 3.0.3 *The Schwarzian derivative $\{f, x\} = 0$ for all regular point x if and only if f is a fractional linear transformation, that is,*

$$f(x) = \frac{ax+b}{cx+d}$$

for some $a, b, c, d \in \mathbb{R}$.

Proof. ($\Rightarrow$) Without loss of generality, assume that $f'(x) > 0$ and define $g = \frac{1}{\sqrt{f'}}$. Then

$$g''(x) = -\frac{1}{2\sqrt{f'(x)}}\left(\frac{f'''(x)}{f'(x)} - \left(\frac{3}{2}\frac{f''(x)}{f'(x)}\right)^2\right) = -\frac{g(x)}{2}\{f, x\} .$$

Therefore, if $\{f, x\} = 0$, then $g''(x) = 0$. Hence,

$$g(x) = ax + b = \frac{1}{\sqrt{f'(x)}}.$$

Then, $f'(x) = \left(\frac{1}{ax+b}\right)^2$ and,

$$f(x) = -\frac{1}{a}\frac{1}{ax+b} + C = \frac{Ca^2x + Cab - 1}{a^2x + ab}.$$

($\Leftarrow$) A simple calculation shows that

$$f'(x) = \frac{ad - bc}{(cx + d)^2}$$
$$f''(x) = -2\,\frac{adc - bc^2}{(cx + d)^3}$$
$$f'''(x) = 6\,\frac{adc^2 - bc^3}{(cx + d)^4}\ .$$

Therefore,

$$\{f, x\} = \frac{6c^2}{(cx + d)^2} - \frac{3}{2}\left(\frac{-2c}{cx + d}\right)^2 = 0\,,$$

which completes the proof. ■

The following lemma describes how the Schwarzian derivative behaves under compositions of functions (and thus under iterations of a function).

Theorem 3.0.4 *If $f, g \in C^3$, then $\{f \circ g, x\} = \{f, g(x)\}g'(x)^2 + \{g, x\}$.*

Proof. The proof is a straightforward calculation:

$$\begin{aligned}
\{f \circ g, x\} &= \frac{(f \circ g)'''(x)}{(f \circ g)'(x)} - \frac{3}{2}\left(\frac{(f \circ g)''(x)}{(f \circ g)'(x)}\right)^2 = \frac{(f'(g(x)).g'(x))''}{f'(g(x)).g'(x)} - \frac{3}{2}\left(\frac{(f'(g(x))g'(x))'}{f'(g(x)).g'(x)}\right)^2 \\
&= \frac{[f''(g(x)).(g'(x))^2 + f'(g(x)).g''(x)]'}{f'(g(x)).g'(x)} - \frac{3}{2}\left(\frac{f''(g(x)).(g'(x))^2 + f'(g(x)).g''(x)}{f'(g(x)).g'(x)}\right)^2 \\
&= \frac{f'''(g(x)).(g'(x))^3 + 3f''(g(x)).g'(x).g''(x) + f'(g(x)).g'''(x)}{f'(g(x)).g'(x)} \\
&- \frac{3}{2}\left[\left(\frac{f''(g(x))g'(x)}{f'(g(x))}\right)^2 + 2\left(\frac{f''(g(x))g'(x)g''(x)}{f'(g(x))g'(x)}\right) + \left(\frac{g''(x)}{g'(x)}\right)^2\right] \\
&= \frac{f'''(g(x))}{f'(g(x))}g'(x)^2 - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2 g'(x)^2 + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 \\
&= \{f, g(x)\}g'(x)^2 + \{g, x\}.
\end{aligned}$$

■

Corollary 3.0.1 *If f is a fractional linear transformation, then*

$$\{f \circ g, x\} = \{g, x\}\,.$$

Corollary 3.0.2 *If $\{g, x\} < 0$ (respectively > 0) for all x and $\{f, x\} \leq 0$ (respectively ≥ 0) for all x, then $\{f \circ g, x\} < 0$(respectively > 0) for all x.*

Finally we conclude that the sign of the Schwarzian derivative does not change as we iterate the function.

Corollary 3.0.3 *If $\{f, x\} < 0$ for all x, then $\{f^n, x\} < 0$ for all n and x.*

Singer's theorem requires that the function has finitely many critical points (which is true for the quadratic family).

Lemma 3.0.5 *If $\{F, x\} < 0$ for all x and if F has finitely many critical points, then for all $N \geq 1$,*

$$\#\mathrm{Per}_N(F) < \infty\,,$$

where $\mathrm{Per}_N(F)$ denotes the set of all periodic points of period N of F, that is, the set of all p such that $F^N(p) = p$.

Proof. Let $G = F^N$. If $G(x) = x$ has infinitely many solutions, then by the mean value theorem, $G'(x) = 1$ also has infinitely many solutions. Since G' has no positive local minimum value, it vanishes infinitely often and this contradicts the hypothesis that F has only finitely many critical points. ■

We may now formulate the main result of this chapter.

Theorem 3.0.6 (Singer) *Let $F \in C^3([0, 1])$ be a transformation. Assume that F has negative Schwarzian derivative, that is, $\{F, x\} < 0$ for all regular point $x \in [0, 1]$ and that F has finitely many critical points.*

If p is a periodic sink of F, then there exists a critical point c of F whose ω-limit is the orbit of p.

In particular, there are at most finitely many periodic sinks.

Proof. Let p be a periodic sink for F. Let k be the period of p and put $G := F^k$.

We define the semi-local stable manifold $slsm(p)$ to be the connected component of the stable manifold $S(p)$ which contains p.

Since $S(p)$ is open, $slsm(p)$ is an interval (r, s). Furthermore, since G is continuous, $G((r, s)) \subset (r, s)$ and $G(r), G(s)$ do not belong to $slsm(p)$. Therefore, there are 3 possibilities:

1. $G(r) = r$ and $G(s) = s$
2. $G(r) = s$ and $G(s) = r$
3. $G(r) = G(s)$

Case 3 is easy. By Rolle's theorem (r, s) must contain a critical point c of G. Moreover, as $c \in (r, s) \subset S(p)$, it follows that

$$F^{nk}(c) = G^n(c) \to p \quad \text{as } n \to \infty\,.$$

Since F is continuous, for all $0 \leq i < k$ we then have

$$F^{nk+i}(c) = F^i(p) \quad \text{as } n \to \infty\,,$$

showing that the ω-limit of c is the orbit of the periodic point p.

Case 2 becomes analogous to Case 1 by considering the transformation $H := G^2$.

What is left is proving the existence of a critical point for G in Case 1 (the same argument as above will then show that its ω-limit is the orbit of p). For that, we must exclude cases like the following.

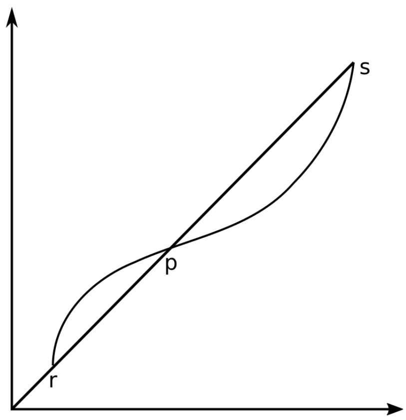

Figure 3.1: Graph

A way to do this is to require that the derivative G' has no positive local minimum. More generally, we could assume that if $G'' = 0$, then $\frac{G'''}{G'} < 0$, but since the statement we want to prove does not assume some specific period for p, we should use some property of F that holds for any iterate.

This is accomplished in the following two lemmas.

Lemma 3.0.7 *If $\{G, x\} < 0$ for all x, then G' cannot have both a positive local minimum value and a negative local maximum value.*

Proof. Since $G''(x) = 0$, G' and G''' must have opposite signs, because $\{G, x\} < 0$. ■

Lemma 3.0.8 *If $a < b < c$ are consecutive fixed points of $G = F^N$ and $[a, c]$ contains no critical points of G, then $G'(b) > 1$.*

Proof. By mean value theorem, there exist u, v such that $a < u < b < v < c$ and $G'(u) = G'(v) = 1$. By intermediate value theorem, $G' > 0$ on $[a, c]$ and by the previous lemma, $G'(b) > 1$. ■

From here we conclude the existence of a critical point of G in Case 1. Therefore this is also true in Case 2 as we observed. Hence, there exists a critical point c' of G in $slsm(p)$. Since $G'(c') = F'(F^{k-1}(c')).\ldots.F'(c')$, then there exists a critical point $c \in slsm(F^i(p))$ of F, for some $i < K$. Thus, the $\omega-$limit of c is the orbit of p.

■

The next lemma will make the connection between the Singer's theorem and the Benedicks-Carleson's theorem.

Lemma 3.0.9 *Assume that x is attracted to a periodic sink, that is, there exists a periodic sink z with period p, such that*

$$\lim_{n\to\infty} f^{nk}(x) = p\,.$$

Then $L(f, x) \leq 0$.

Proof. Define $g := f^k$. Since

$$\lim_{n\to\infty} g^n(x) = p\,,$$

and since g' is continuous, we have

$$\lim_{n\to\infty} g'(g^n(x)) = g'(p).$$

Then, if $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$.

$$|g'(g^n(x)) - g'(g(p))| < \epsilon\,.$$

It follows that for all $n > N$,

$$|g'(g^n(x))| \leq |g'(g^n(x)) - g'(g(p))| + |g'(g(p))| < \epsilon + 1\,.$$

Let $K \in \mathbb{N}$. Then we have,

$$\frac{\sum_{i=N}^{N+K-1} \log(|g'(g^i(x))|)}{N+K} \leq \frac{Klog(\epsilon+1)}{N+K} \leq \log(\epsilon+1)\,.$$

We can take K large enough to get

$$\frac{\sum_{i=0}^{N-1} \log(|g'(g^i(x))|)}{N+K} < \epsilon,$$

so we have:

$$\frac{\sum_{i=0}^{N+K-1} \log(|g'(g^i(x))|)}{N+K} < \log(\epsilon+1)+\epsilon\,,$$

showing that $L(g,x) \leq 0$.

Therefore,

$$\begin{aligned}\liminf \frac{1}{n}\log(|(g^n)'(x)|) &= k \liminf \frac{1}{nk}\log|(f^{nk})'(x)| \\ &\geq k \liminf \frac{1}{n}\log(|(f^n)'(x)|) \\ &= kL(f,x)\end{aligned}$$

which proves the statement. ■

Therefore, we have established that for a positive measure set of parameters, the quadratic family has no periodic sinks.

4
Quasi-periodic cocycles and Herman's theorem

In this chapter, we introduce the concept of linear cocycle and that of Lyapunov exponent for this type of transformation. Moreover, we present a proof Herman's theorem, which establishes a positive lower bound on the Lyapunov exponent for a certain type of quasi-periodic Schrödinger cocycle.

A linear cocycle is a skew-product map acting on a vector bundle. Below we define this and other basic concepts in a formal way.

Definition 4.0.1 *A measure preserving dynamical system (MPDS) is a triple (X, μ, T), where (X, μ) is a probability space and $T : X \to X$ is a measure preserving transformation, i.e., T is measurable and $\mu(T^{-1}(A)) = \mu(A)$ for every measurable set A. We say that (X, μ, T) is ergodic if for every measurable set A such that $A = T^{-1}(A)$, we have $\mu(A) \in \{0, 1\}$.*

Definition 4.0.2 *Let (X, μ, T) be an ergodic MPDS. A linear cocycle over (X, μ, T) is a transformation of the form*

$$\begin{aligned} F_A : X \times \mathbb{R}^m &\to X \times \mathbb{R}^m \\ (x, v) &\mapsto (Tx, A(x)v)\,, \end{aligned}$$

where $A : X \to \mathrm{GL}_m(\mathbb{R})$ is a measurable function.

From this definition, it follows that the n-th iterate of F_A is

$$F_A^n(x, v) = (T^n x, A^{(n)}(x)v)\,,$$

where

$$A^{(n)}(x) := A(T^{n-1}x) \ldots A(Tx)A(x)\,.$$

To simplify the notations, we will refer to the matrix valued function A as the cocycle (instead of the transformation F_A).

Definition 4.0.3 *Let (X, μ, T) be an ergodic MPDS. A sub-additive sequence over (X, μ, T) is a sequence $\{f_n\}_{n\geq 1}$ of measurable functions $f_n \colon X \to \mathbb{R}$ such that for all integers n and m we have*

$$f_{n+m} \leq f_n + f_m \circ T^n\,.$$

Considering the operator norm on $\mathrm{GL}_m(\mathbb{R})$ we note that for any two matrices $A_1, A_2 \in \mathrm{GL}_m(\mathbb{R})$, we have

$$\|A_2\, A_1\| \leq \|A_2\|\, \|A_1\|\,.$$

It follows that given a linear cocycle $A : X \to \mathrm{GL}_m(\mathbb{R})$, the sequence of functions

$$f_n(x) := \log \|A^{(n)}(x)\| \quad \text{for all } n \geq 1$$

is sub-additive over (X, μ, T).

Indeed, for all $x \in X$,

$$\begin{aligned} f_{n+m}(x) = \log \|A^{(n+m)}(x)\| &= \log \|A^{(m)}(T^n x) A^{(n)}(x)\| \\ &\leq \log \|A^{(m)}(T^n x)\| + \log \|A^{(n)}(x)\| \\ &= f_m(T^n(x)) + f_n(x). \end{aligned}$$

The sub-additivity of the sequence $\{f_n\}$ defined above allows us to define the Lyapunov exponent of the linear cocycle A via Kingman's ergodic theorem. Let us first state Kingman's theorem (see [13] or [14]).

Theorem 4.0.1 *(Kingman) Let (X, μ, T) be an MPDS and let $\{f_n\}_{n\geq 1}$ be a sub-additive sequence of $L^1(\mu)$ functions. Then the limit*

$$\lim_{n\to\infty} \frac{f_n(x)}{n}$$

exists for μ-a.e. $x \in X$ and it defines a $T-$invariant measurable function.

In particular, if the system is ergodic then the limit is μ-a.e. constant. Moreover, in this case,

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \lim_{n\to\infty} \int_X \frac{f_n(x)}{n} d\mu(x) = \inf_{n\geq 1} \int_X \frac{f_n(x)}{n} d\mu(x) =: L(f)\,.$$

Definition 4.0.4 *Let $A : X \to \mathrm{GL}_m(\mathbb{R})$ be a linear cocycle over the ergodic system (X, μ, T). The μ-a.e. limit*

$$L(A) = \lim \frac{1}{n} \log \|A^{(n)}(x)\|$$

is called the Lyapunov exponent of A.

Moreover, we have that

$$L(A) = \lim_{n\to\infty} \int_X \frac{1}{n} \log \|A^{(n)}(x)\| d\mu(x) = \inf_{n\geq 1} \int_X \frac{1}{n} \log \|A^{(n)}(x)\| d\mu(x)\,.$$

The type of linear cocycle that we will be focused on is the Schrödinger cocycle. These cocycles are important due to their connections with mathematical physics. Their iterates formally solve the eigenvalue equation associated to a discrete Schrödinger operator.

More precisely, to define a Schrödinger cocycle, consider an invertible MPDS (X, μ, T) and a bounded observable $\phi : X \to \mathbb{R}$. For every $n \in \mathbb{Z}$,

$$v_n(x) := \phi(T^n(x))$$

is called the potential at site n.

The discrete Schrödinger operator $H(x)\colon l^2(\mathbb{Z}, \mathbb{R}) \to l^2(\mathbb{Z}, \mathbb{R})$ (we are denoting by $l^2(\mathbb{Z}, \mathbb{R})$ the set of sequences $\phi_{n\in\mathbb{Z}} \in \mathbb{R}$ such that $\sum_{n\in\mathbb{Z}} |\phi_n|^2 < \infty$), with potential $\{v_n(x)\}_{n\geq 1}$ and coupling constant λ is defined as follows.

For $\psi = \{\psi_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{R})$,

$$[H(x)\psi]_n = -(\psi_{n+1} + \psi_{n-1}) + \lambda\, v_n(x)\psi_n \tag{4.1}$$

Consider the eigenvalue equation, also called the Schrödinger equation:

$$H(x)\psi = E\psi \tag{4.2}$$

for some state ψ and energy $E \in \mathbb{R}$.

Definition 4.0.5 *Given an energy parameter E and a coupling constant λ, the cocycle $A_{E,\lambda}\colon X \to \mathrm{SL}_2(\mathbb{R})$, where*

$$A_{E,\lambda}(x) := \begin{bmatrix} \lambda(\phi(x) - E) & -1 \\ 1 & 0 \end{bmatrix}$$

is called the Schrödinger cocycle associated to the Schrödinger equation (4.2)*.*

Instead of any Schrödinger cocycle, we will also require that T be an irrational translation over the torus.

Definition 4.0.6 *A quasi-periodic cocycle is a cocycle $A : \mathbb{T} \to \mathrm{SL}_2(\mathbb{R})$ over an irrational (hence ergodic, see [13]) torus translation $T : \mathbb{T} \to \mathbb{T}$, $Tx = x + \omega \mod 1$, where $\omega \in \mathbb{R} \setminus \mathbb{Q}$.*

The proof of Herman's theorem uses Jensen's formula and the concept of subharmonic function. We describe these notions below.

Theorem 4.0.2 *(Jensen) Given a holomorphic function f in a region of the complex plane which contains a disk of centre z_0 and radius r, we have the following:*

$$\log |f(z_0)| \leq \int_0^1 \log |f(z_0 + re^{2\pi i\theta})| d\theta.$$

Definition 4.0.7 *Let $\Omega \subset \mathbb{C}$ be an open set and let $u\colon \Omega \to [-\infty, \infty)$ be a continuous function. We say that u is subharmonic if for every $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0, r) \subset \Omega$, the following sub-mean value inequality holds:*

$$u(z_0) \leq \int_0^1 u(z_0 + re^{2\pi i\theta})d\theta \,.$$

Example 4.0.1 *By Jensen's formula, if f is a holomorphic function in a domain $\Omega \subset \mathbb{C}$, then $\log |f|$ is subharmonic.*

Example 4.0.2 *Let $\Omega \subset \mathbb{C}$ be an open set and let $A\colon \Omega \to \mathrm{Mat}_m(\mathbb{C})$ ($\mathrm{Mat}_m(\mathbb{C})$ is the set of $m \times m$ complex matrices) be a holomorphic function (meaning that each matrix entry is holomorphic as a function of $z \in \Omega$). Then*

$$u(z) := \log \|A(z)\|$$

is subharmonic.

Proof. First note that $u(z)$ is continuous, since it is a composition of continuous functions. Moreover, for each $v \in \mathbb{R}^m$ with $\|v\| = 1$, the function $z \mapsto \langle A(z)v, v\rangle$ is holomorphic, because it is a linear combination of the entries of $A(z)$, which are holomorphic. Hence $f_v\colon \Omega \to \mathbb{R}$, defined by

$$f_v(z) := \log \Big|\langle A(z)v, v\rangle\Big|$$

is subharmonic. Then $u(z)$ is also subharmonic, since

$$u(z) = \log \|A(z)\| = \sup_{\|v\|=1} f_v(z) \,.$$

Indeed, given $z_0 \in \Omega$ and $r > 0$ such that $\bar{D}(z_0, r) \subset \Omega$, for all v we have

$$f_v(z_0) \leq \int_0^1 f_v(z_0 + re^{2\pi i\theta})d\theta \,.$$

Then

$$\begin{aligned} u(z_0) &= \sup_{\|v\|=1} f_v(z_0) \\ &\leq \sup_{\|v\|=1} \int_0^1 f_v(z_0 + re^{2\pi i\theta})d\theta \\ &\leq \int_0^1 \sup_{\|v\|=1} f_v(z_0 + re^{2\pi i\theta})d\theta \\ &= \int_0^1 u(z_0 + re^{2\pi i\theta})d\theta \,, \end{aligned}$$

which shows that u satisfies the sub-mean value property. ■

We may now formulate and prove Herman's theorem.

Theorem 4.0.3 *(Herman) Consider the quasi-periodic Schrödinger operator* (4.1) *with potential*

$$v_n(x) = \cos(T^n x) = \cos(x + n\omega)$$

for all $n \in \mathbb{Z}$ and for a given irrational number ω.

Then for all energies $E \in \mathbb{R}$ and coupling constants $\lambda > 0$, the corresponding Lyapunov exponent satisfies the following inequality:

$$L(A_{E,\lambda}) \geq \log \frac{\lambda}{2}.$$

Therefore, a coupling constant greater than 2 *is sufficient to have the positivity of the Lyapunov exponent.*

Proof. We may write the n-th iterate of the Schrödinger cocycle $A_{E,\lambda}$ as

$$A^{(n)}_{E,\lambda}(x) = \begin{bmatrix} \lambda(\cos(x + (n-1)\omega) - E) & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} \lambda(\cos(x) - E) & -1 \\ 1 & 0 \end{bmatrix}$$

Since

$$\cos(x + jw) = \frac{e^{ix}e^{jw} + e^{-ix}e^{-jw}}{2},$$

the norm of iterates may be written as follows:

$$\|A^{(n)}_{E,\lambda}(x)\| = \left\| \prod_{j=0}^{n-1} \begin{bmatrix} \frac{\lambda}{2}(ze^{i(n-1-j)w} + \frac{1}{z}e^{-i(n-1-j)w} - 2E) & -1 \\ 1 & 0 \end{bmatrix} \right\|$$

where $z = e^{ix}$.

Multiplying every term by z, which has norm 1, we obtain

$$\|A^{(n)}_{E,\lambda}(x)\| = \left\| \prod_{j=0}^{n-1} \begin{bmatrix} \frac{\lambda}{2}(z^2e^{i(n-1-j)w} + e^{-i(n-1-j)w} - 2Ez) & -z \\ z & 0 \end{bmatrix} \right\|.$$

Also, the function $M_n \colon \mathbb{C} \to \mathrm{Mat}_2(\mathbb{C})$,

$$M_n(z) := \prod_{j=0}^{n-1} \begin{bmatrix} \frac{\lambda}{2}(z^2e^{i(n-1-j)w} + e^{-i(n-1-j)w} - 2Ez) & -z \\ z & 0 \end{bmatrix}$$

is well defined and holomorphic everywhere. Moreover, if $z = e^{ix}$,

$$\|A^{(n)}_{E,\lambda}(x)\| = \|M_n(z)\|.$$

Then the function $\mathbb{C} \ni z \mapsto \log \|M_n(z)\|$ is subharmonic, implying the following:

$$\int_0^1 \frac{1}{n} \log \|A_{E,\lambda}^{(n)}(x)\| \, dx = \int_{\partial D(0;1)} \frac{1}{n} \log \|M_n(z)\| \, dz \geq \frac{1}{n} \log \|M_n(0)\|$$
$$= \frac{1}{n} \log \left(\frac{\lambda}{2}\right)^n = \log \frac{\lambda}{2}\,.$$

Since this holds for every $n \in \mathbb{N}$, by letting $n \to \infty$ we obtain the stated lower bound on the Lyapunov exponent. ■

The argument above may also be used to extend this result to trigonometric polynomial observables.

Sorets and Spencer generalized Herman's theorem to the case of non-constant real analytic observables (see [15] and Bourgain [9]).

Still in the analytic case, Duarte and Klein provided another argument which lead to an optimal lower bound (see [7]).

Klein obtained a similar lower bound for the Lyapunov exponent if we suppose that the observable belongs to a Gevrey class and it satisfies a generic transversality condition, while the frequency ω satisfies a Diophantine condition (see [16]).

Bjerklöv proved a similar result (under a Diophantine assumption on the frequency ω) for observables that are *cosine*-like: C^2 functions with exactly two non-degenerated critical points (see [10]).

The approach used by Bjerklöv in this work is based on ideas in the proof of Benedicks-Carleson theorem presented in chapter 2 of this dissertation.

In a future project we will consider the problem of establishing the positivity of the Lyapunov exponent for other types of cocycles, e.g. Schrödinger but with more general observables.

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