Pontifícia Universidade Católica
do Rio de Janeiro

David Gonzalez Stolnicki

Regularity theory for non local partial differential equations

Dissertação de Mestrado

Thesis presented to the Programa de Pós–Graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática .

Advisor : Prof. Boyan Slavchev Sirakov
Co-advisor: Prof. Edgard Almeida Pimentel

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Abstract

Stolnicki, Gonzalez David; Sirakov, Slavchev Boyan (Advisor); Pimentel, Almeida Edgard (Co-Advisor). **Regularity theory for non local partial differential equations**. Rio de Janeiro, 2019. 52p. Dissertação de mestrado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

In this work, we put forward a brief introduction to the nonlocal diffusion operators, based on the work of Luis Caffarelli and Luis Silvestre [2]. Our object of study are the fractional Laplacian and some of its variants. We present a number of elementary properties and establish an Alexandroff-Bakelman-Pucci estimate, as well as a Harnack inequality. As an application, we examine the regularity of the solutions in Hölder spaces.

Keywords

Viscosity Solutions; Regularity Theory; Non Local Operators; Weak Solutions; ABP Inequality; Holder Estimates.

Resumo

Stolnicki, Gonzalez David; Sirakov, Slavchev Boyan; Pimentel, Almeida Edgard. **Teoria da Regularidade para Equações Diferenciais Parciais Não-Locais**. Rio de Janeiro, 2019. 52p. Dissertação de Mestrado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

Palavras-chave

Viscosity Solutions; Regularity Theory; Non Local Operators; Weak Solutions; ABP Inequality; Holder Estimates.

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List of Abreviations

B_r: The open ball centered at the origin with radius r.

$B_r(x_0)$: The open ball centered at the point(x_0) with radius r.

$\mathcal{S}(d)$: The space of dxd real symmetric matrices.

1
Introduction

In the past few years, nonlocal operators have received a substantial attention from the mathematical community. This class of operators appears naturally in several fields of pure mathematics, in particular, analysis, probability and geometry. Their relevance is also to be found in a number of models in life and social sciences.

In this thesis, we discuss some of the results established by Luis Caffarelli and Luis Silvestre in [2]. Our exposition covers a few preliminary elements, Alexandroff-Bakelman-Pucci estimates (ABP, for short), Harnack inequalities and regularity results in Hölder spaces. The remainder of this work unfolds as follows.

In Chapter 2 put forward, classical results from convex analysis which unfortunately are not always taught during the formative years of the student.

Once the background is set, we start the analysis of our class of nonlocal PDE's. In Chapter 3, we define our operator of interest and proceed with a discussion regarding the correct way of understanding the Dirichlet problem related to our operator, which culminate in the concept of viscosity solutions for the problem. Associated with that definition, we introduce the extremal operators in a similar way to what is done in the local case; see, for instance, the monograph [1].

In Chapters 4-7 we explore aspects related to the regularity of solutions. In Chapter 4 we state a version of the classical ABP estimate in the nonlocal context. In Chapter 5, we obtain point-wise estimates, the nonlocal analogous to the local L^ε-estimates, in terms of the measure of sets related to the growth of the solution. In Chapter 6, we obtain a Harnack inequality from the previous developments. As usual in the literature, we resort to the Harnack inequality to produce a Hölder-continuity result; this is reported in Chapter 7.

2
Former results and preliminary material

With the finality of guiding the reader through the theory developed in the following chapters, this chapter will present some basic facts of analysis which are not commonly taught at the more mainstream analysis courses, but are nonetheless necessary for the correct understatement of what follows. The following results were in part found and the interested reader may look for deeper results and applications at [3].

Definition 2.1 *A subset* $C \subset \mathbb{R}^d$ *is said to be convex if* $(1-\lambda)x + \lambda y \in C$ *whenever* $x, y \in C$ *and* $\lambda \in [0,1]$

Definition 2.2 *Let* $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$*. The set*

$$\{(x,\mu)\,|\, x \in, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

is called the epigraph of f *and is denoted by* $epi(f)$*.*

Definition 2.3 *A function* $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ *is said to be convex if* $epi(f)$ *is a convex set.*

Remark 2.4 *A convex function must have a convex domain. On the other hand, a locally convex function does not have such a restriction and as such has important applications in regularity theory.*

Definition 2.5 *A function* $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ *is said to be concave if* $-f$ *is a convex function.*

Lemma 2.6 *Every convex function* $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ *can be extended to a convex function* $\bar{f} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$*.*

Proof: Define $\bar{f} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ as

$$\bar{f}(x) = \begin{cases} f(x) & x \in S \\ +\infty & x \notin S \end{cases}$$

Let $(x, \mu), (y, \nu) \in \mathbb{R}^{d+1} \cap epi(\bar{f})$, therefore $(1-t)(x,\mu) + t(y,\nu) \in epi(f) \subset epi(\bar{f})$ for every t between 0 and 1. As a direct consequence $epi(\bar{f})$ is convex and convexity of $\bar{f}$ follows.

■

In light of the previous result, the following lemmas will be stated assuming the function f is defined over the whole space $\mathbb{R}^d$.

Lemma 2.7 *A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if*

$$f((1-\lambda)x + \lambda y) < (1-\lambda)\alpha + \lambda\beta \qquad \forall \lambda \in (0,1)$$

whenever $f(x) < \alpha, f(y) < \beta$.

Proof: Suppose f is convex. Since $\alpha > f(x)$ and $\beta > f(y)$, the pairs (x, α) and (y, β) are in $epi(f)$. Since f is convex, the epigraph of f is a convex set, which implies:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) < (1-\lambda)\alpha + \lambda\beta \qquad \lambda \in (0,1).$$

on the other hand, suppose the inequality is true. Let $(x,\alpha), (y,\beta) \in epi(f)$ and $\lambda \in (0,1)$. Then$f((1-\lambda)x + \lambda y) < (1-\lambda)\alpha + \lambda\beta$ which implies that the epigraph of f is convex and therefore the function f is convex. ■

Corollary 2.8 (Jensen's Inequality) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Then*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \qquad \lambda \in [0,1].$$

Proof: It follows from Lemma 2.7 taking the infimum over α, β such that $f(x) < \alpha, f(y) < \beta$. ■

Definition 2.9 *Let $C \subset \mathbb{R}^d$ be a convex set, and suppose $f : C \to \mathbb{R}$ is concave. A vector $p \in \mathbb{R}^d$ is a supergradient of the function f at the point $x \in \mathbb{R}^d$ if for every $y \in \mathbb{R}^d$*

$$f(y) \leq f(x) + p \cdot (y - x).$$

Analogously, if f is a convex function, we say that $p \in \mathbb{R}^d$ is a subgradient of f at $x \in \mathbb{R}^d$ if

$$f(y) \geq f(x) + p \cdot (y - x).$$

In both cases we denote the set of all supergradients and subgradients of f at the point x as $\partial f(x)$

Definition 2.10 *A convex function satisfying the assumption 2.9 is said to be superdifferentiable at a point $x \in \mathbb{R}^d$ if $\partial f(x)$ is non-void. In the same manner we define that a concave function f as in 2.9 is subdifferentiable at x.*

Theorem 2.11 *A concave function on a convex set in $\mathbb{R}^d$ is superdifferentiable at each interior point.*

Proof: Let f be a concave function defined on a convex set $C \subset \mathbb{R}^d$, and let x be an interior point of C. Consider the strict subgraph of f,S, as:

$$S := \{(y, \alpha) \in C \times \mathbb{R} : \alpha < f(y)\}$$

It follows from the concavity of f that S is a convex set. Also clear is the fact that the pair $(x, f(x))$ does not belong to the set S. By the Separating Hyperplane Theorem we obtain a nonzero pair $(p, \lambda) \in \mathbb{R}^d \times \mathbb{R}$ such that:

$$p \cdot x + \lambda f(x) \geq p \cdot y + \lambda \alpha, \tag{2-1}$$

where the inequality above holds for every $y \in C, \alpha < f(y)$. It follows from letting α tend to infinity that λ must be a non-negative number. We proceed to conclude a stronger fact, namely, that λ is indeed strictly positive. Suppose, in order to obtain a contradiction, that $\lambda = 0$. Since x is an interior point, for some $\epsilon > 0$ the ball $B_\epsilon(x)$ is contained in C. Considering points of the form $y = x \pm \epsilon z$, with $z \in B_1$, in 2-1 we obtain:

$$\begin{cases} 0 \geq & p \cdot z \\ 0 \geq & -p \cdot z \end{cases} \tag{2-2}$$

We conclude that p must be zero, which contradicts the fact that (p, λ) is nonzero, therefore λ is strictly positive.

Since λ is strictly positive, dividing the whole expression in 2-1 by λ we obtain:

$$f(x) + (y - x) \cdot (\frac{-p}{\lambda}) \geq \alpha$$

The result follows from letting α tend to $f(y)$ and noticing that$-\frac{p}{\lambda} \in \partial f(x)$.
■

Theorem 2.12 *Let A be an open convex subset of a finite dimensional vector space over $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a bounded convex function. Then f is continuous on A.*

Proof: Let A and f be as in the theorem, let $x \in A$ an arbitrary point. Consider P the parallelepiped centered at x lying completely inside A, such parallelepiped exists since A is open. Let $y \in \partial P$, for $\lambda \in [0, 1]$, convexity of f implies

$$f((1-\lambda)x+\lambda y) \leq f(x)+\lambda[f(y)-f(x)]. \tag{2-3}$$

Also, for $\alpha \in [0, 1/2]$, it follows that

$$\begin{aligned} f(x) &= f\left((1-\alpha)\left[\frac{(1-2\alpha)x}{1-\alpha}+\frac{\alpha y}{1-\alpha}\right]+\alpha(2x-y)\right) \\ &\leq (1-\alpha)f(\frac{(1-2\alpha)x}{1-\alpha}+\frac{\alpha y}{1-\alpha})+\alpha f(2x-y) \end{aligned}$$

Choosing λ as $\frac{\alpha}{1-\alpha}$ we obtain

$$(1+\lambda)f(x) \leq f((1-\lambda)x+\lambda y)+\lambda f(2x-y). \tag{2-4}$$

Using the two enumerated inequalities we obtain

$$-(\lambda f(2x-y)-f(x)) \leq f(x+\lambda(y-x))-f(x) \leq \lambda(f(y)-f(x)).$$

Since both $y, 2x-y$ are in ∂P, the above inequality implies that a strict maximum of f cannot be attained in the interior, another conclusion of the previous inequality is that for any vector $z \in P_\lambda := \{x+\lambda(y-x) : y \in \partial P\}$, is true that

$$|f(z)-f(x)| \leq \lambda|\sup_{y\in\partial P} f(y)-f(x)|$$

■

3
Further context: the operator L

Motivated by the expression of the infinitesimal generator of a Lévy process consisting only of jumps, without either drift nor diffusion. We consider an operator of the form

$$Lu(x) = \int\limits_{\mathbb{R}^d \setminus \{0\}} [u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{0<|y|<1}(y)] \nu(dy), \tag{3-1}$$

above ν is a Lévy measure, in particular, this means that ν is subjected to the growth condition

$$\int\limits_{\mathbb{R}^d} \frac{|y|^2}{1+|y|^2} d\nu(y) < \infty$$

If we restrict our attention to the case where ν is an absolutely continuous measure with respect to the Lebesgue measure, such that $\nu' s$ Radon derivative $\frac{d\nu}{dy} := K(y)$ is represented by a positive symmetric kernel, one would be able to rewrite (3-2) in the following form:

$$Lu(x) = \int\limits_{\mathbb{R}^d \setminus \{0\}} [u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{0<|y|<1}(y)] K(y) dy. \tag{3-2}$$

As a consequence of $\nu' s$ growth condition $K(y)$ must satisfy:

$$\int\limits_{\mathbb{R}^d} \frac{|y|^2}{1+|y|^2} K(y) dy$$

Depending on the structure of our kernel K, the above integral may present singularities, so (3-2) must be understood in the sense of Cauchy's principal value.

Remark 3.1 *One of the most studied examples in this class of operators is the fractional laplacian defined as:*

$$Lu = PV \int\limits_{\mathbb{R}^d} \frac{u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{0<|y|<1}(y)}{|y|^{d+\sigma}} dy.$$

Noticing the odd symmetry of the last term we may simplify the above expres-

sion and obtain

$$Lu = PV \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\sigma}} dy,$$

furthermore if $u \in C^2(\Omega)$ *by expanding the function* u *through the truncated Taylor series, we may conclude that the above integral can be understood in the usual sense not requiring to be taken in the principal value sense, since there would be no powers of* $|y|$ *smaller than* -1 *after the appropriated simplifications are done.*

In view of the previous remark, we concentrate our efforts in the study of the following Dirichlet problem with special attention to regularity results:

$$\begin{cases} Lu = PV \int_{\mathbb{R}^d} [u(x+y) - u(x)] K(y) dy = f & \text{in } \mathbb{R}^d \\ u = g & \text{in } \mathbb{R}^d \setminus \Omega \end{cases} \tag{3-3}$$

We can further rewrite the integrand as:

$$PV \int_{\mathbb{R}^d} [u(x+y) - u(x)] K(y) dy = PV \int_{\mathbb{R}^d} [u(x-y) - u(x)] K(-y) dy$$

Recalling that K is symmetric we conclude by summing the two identical expressions above and dividing by two that

$$Lu = PV \int_{\mathbb{R}^d} \frac{u(x+y) + u(x-y) - 2u(x)}{2} K(y) dy.$$

In order to ease the reader in the following calculations, we introduce the function:

$$\delta(u, x, y) = \frac{u(x+y) + u(x-y) - 2u(x)}{2} \tag{3-4}$$

With the objective of correctly posing Problem 3-3, we put forward the viscosity solutions framework adapted to the nonlocal setting.

A function $u : \mathbb{R}^d \to \mathbb{R}$, upper(lower) semi continuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to $Lu = f$, and we write $Lu \geq f$ ($Lu \leq f$), if whenever

1. x is any point in Ω.
2. N is a neighborhood of x in Ω
3. φ is some $\mathcal{C}^2$ function in $\overline{N}$.
4. $\varphi(x) = u(x)$.

5. $\varphi(y) > u(y)$ ($\varphi(y) < u(y)$) for every $y \in N \setminus \{x\}$,

and if we construct the function

$$v := \begin{cases} \varphi & \text{in } N \\ u & \text{in } \mathbb{R}^d \setminus N, \end{cases}$$

we have $Lv(x) \geq f(x)$ ($Lv(x) \leq f(x)$). We then define that a solution to the problem (3-3) is a function which is both a subsolution and a supersolution.

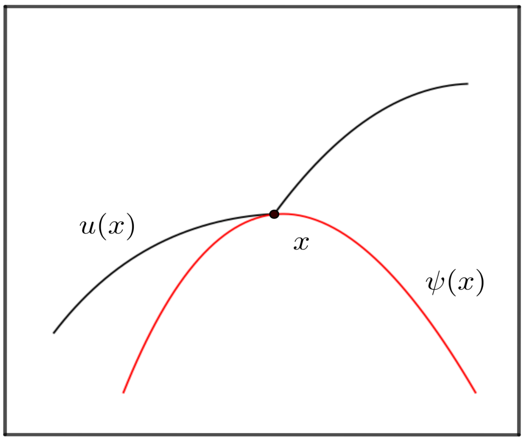

Figure 3.1: On this image the function ψ touches u from below at the point x

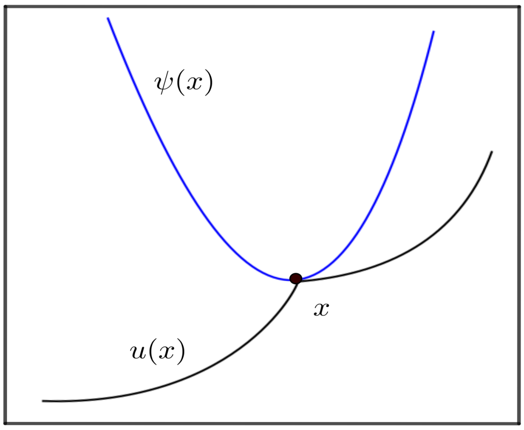

Figure 3.2: On this image the function ψ touches u from below at the point x

Maximal Operators

As in the case for local, Fully Nonlinear PDE's, we bound the operator by a pair of linear operators with simpler structure, in this context those operators are called Maximal and Minimal Operators which play the whole of the Pucci Operators in the aforementioned theory.

Let us consider a collection of linear operators $\mathcal{L}$. The maximal and a minimal operator with respect to $\mathcal{L}$ are defined as:

$$\mathcal{M}^+_{\mathcal{L}} u(x) = \sup_{L\in\mathcal{L}} Lu(x)$$
$$\mathcal{M}^-_{\mathcal{L}} u(x) = \inf_{L\in\mathcal{L}} Lu(x)$$

We will be interested in a class $\mathcal{L}_0$ of operators as in (3-2) such that the kernel K satisfies:

$$(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}$$

The class $\mathcal{L}_0$ may be understood as those operators which are bounded by multiples of the fractional laplacian, for this particular class, we can explicitly describe the maximal and minimal operators:

$$\mathcal{M}^+_{\mathcal{L}_0} u(x) = (2-\sigma)\int_{\mathbb{R}^d} \frac{\Lambda\delta^+(u,x,y) - \lambda\delta^-(u,x,y)}{|y|^{n+\sigma}} dy$$
$$\mathcal{M}^-_{\mathcal{L}_0} u(x) = (2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda\delta^+(u,x,y) - \Lambda\delta^-(u,x,y)}{|y|^{n+\sigma}} dy$$

Remark 3.2 *Unless explicitly stated, the classes $\mathcal{L}_0$ and $\mathcal{L}$ will be used interchangeably, for a friendlier reading.*

In this context we need to define what does it mean for an operator L be elliptic with respect to a class of linear operators $\mathcal{L}$, for a more meaningful reading the next definition is helpful.

Definition 3.3 *We say that a function is punctually $\mathcal{C}^{1,1}$ at the point $x \in \Omega$ if, there exists a vector $v \in \mathbb{R}^d$ and $M > 0$ such that:*

$$|u(x+y) - u(x) - v\cdot y| \leq M|y|^2$$

for sufficiently small y.

Definition 3.4 *Let $\mathcal{L}$ be a class of linear operators, we say that an operator L is elliptic with respect to $\mathcal{L}$ if it satisfies:*

- *If u is any bounded function, $Lu(x)$ is well-defined for all $\mathcal{C}^{1,1}$ continuous functions at x.*

– *If u is a $\mathcal{C}^2(\Omega)$ function for some open set Ω, then Lu is a continuous function over Ω*

– *If u, v are bounded functions, $\mathcal{C}^{1,1}$ continuous at x, then*

$$\mathcal{M}^-_{\mathcal{L}}(u-v) \leq L(u) - L(v) \leq \mathcal{M}^+_{\mathcal{L}}(u-v).$$

Now we will verify that L is indeed elliptic with respect to the class $\mathcal{L}_0$, we will only prove the first point since the other two are immediate.

Lemma 3.5 *If u is any bounded function, $Lu(x)$ is well-defined for all $\mathcal{C}^{1,1}$ continuous functions at x.*

Proof: Since u is a $\mathcal{C}^{1,1}$, there exists $N(0)$ a symmetric neighborhood of 0 such that

$$|u(x+y) - u(x) - v \cdot y| \leq M|y|^2,$$

for all $y \in N(0)$. From the definition of K, one may compare Lu with the following integrals:

$$(2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda(u(x+y)-u(x))}{|y|^{n+\sigma}} dy \leq Lu \leq (2-\sigma)\int_{\mathbb{R}^d} \frac{\Lambda(u(x+y)-u(x)}{|y|^{n+\sigma}} dy \quad (3\text{-}5)$$

Due to (3-5), it is enough to prove that for a fixed $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \frac{u(x+y)-u(x)}{|y|^{n+\sigma}} dy < \infty$$

Now, we will split the above integral in two by splitting our domain and follow by estimating them separately.

At the first moment, we will restrict our attention at $N(x)$.

$$\begin{aligned}\int_{N(x)} \frac{u(x+y)-u(x)}{|y|^{n+\sigma}} dy \quad &\leq \int_{N(x)} \frac{|u(x+y)-u(x)-v\cdot y|}{|y|^{n+\sigma}} dy + \int_{N(x)} \frac{v\cdot y}{|y|^{n+\sigma}} dy \\ &\leq M \int_{N(x)} \frac{|y|^2}{|y|^{n+\sigma}} dy,\end{aligned}$$

which is finite since $\sigma \in (0,2)$. Notice that the third integral vanishes due to symmetry. Outside $N(0)$ the estimate is simpler.

$$\int_{\mathbb{R}^d \setminus N(x)} \frac{u(x+y)-u(x)}{|y|^{n+\sigma}} dy \leq ||u||_\infty \int_{\mathbb{R}^d \setminus N(x)} \frac{1}{|y|^{n+\sigma}} dy < \infty$$

■

Remark 3.6 *In fact we have proved more, we have shown that*

$$\int_{\mathbb{R}^d} \frac{|u(x+y)-u(x)|}{|y|^{n+\sigma}} dy < \infty, \ \textit{instead of} \int_{\mathbb{R}^d} \frac{u(x+y)-u(x)}{|y|^{n+\sigma}} dy < \infty$$

Theorem 3.7 *If we have a subsolution to $Lu \geq f$ in Ω, and φ a $\mathcal{C}^2$ function that touches u from above at x in Ω, then $Lu(x)$ is evaluated in the classical sense.*

Proof: Given r>0, consider the function defined as:

$$v_r(y) = \begin{cases} \varphi(y) \text{ in } B_r(y) \\ u(y) \text{ in } \mathbb{R}^d \end{cases}$$

Since φ touches u from above, we have that $\delta(v_r, x, y) \geq \delta(u, v, y)$ for every $y \in \mathbb{R}^d$ and is increasing with respect to r, thus:

$$\mathcal{M}^+ v_r(x) \geq Lv_r(x) \geq Lu(x) \geq f. \tag{3-6}$$

Since v_r is a $\mathcal{C}^{1,1}$ continuous function at x we may evaluate Lv_r, and $\mathcal{M}^+ v_r(x)$ in the classical sense, also, both $|\delta(v_r, x, y)|/|y|^{n+\sigma}, \delta^+(u, x, y)/|y|^{n+\sigma}$, are integrable by the previous remark. Rewriting (3-6) it follows that:

$$(2-\sigma) \int_{\mathbb{R}^d} \frac{\Lambda\delta^+(v_r, x, y) - \lambda\delta^-(v_r, x, y)}{|y|^{n+\sigma}} dy \geq f(x).$$

Fix $r_0 > 0$, then for every $r < r_0$:

$$(2-\sigma) \int_{\mathbb{R}^d} \frac{\lambda\delta^-(v_r, x, y)}{|y|^{n+\sigma}} dy \leq (2-\sigma) \int_{\mathbb{R}^d} \frac{\Lambda\delta^+(v_{r_0}, x, y)}{|y|^{n+\sigma}} dy - f(x)$$

Notice that the integrand on the left side monotonically converges, and by the Monotonous Convergence Theorem:

$$\begin{aligned} \lim_{r\to 0} (2-\sigma) \int_{\mathbb{R}^d} \frac{\lambda\delta^-(v_r, x, y)}{|y|^{n+\sigma}} dy \quad &= (2-\sigma) \int_{\mathbb{R}^d} \frac{\lambda\delta^-(u, x, y)}{|y|^{n+\sigma}} dy \\ &\leq (2-\sigma) \int_{\mathbb{R}^d} \frac{\Lambda\delta^+(v_{r_0}, x, y)}{|y|^{n+\sigma}} dy - f(x) \end{aligned}$$

Therefore both the positive and negative parts may be understood in the classical sense and it follows analogously that: $\mathcal{M}^+(v_r - u)$ tends to zero and therefore $Lu \geq f$. ■

4
Alexandroff-Bakelman-Pucci Estimate

In this section we present the Alexandroff-Bakelman-Pucci estimate adapted to the context of integro-differential equations. This will enable us to move from measure theory estimates to pointwise estimates. Later in this monograph, such an estimate will be an essential element in the proof of the nonlocal Harnack Inequality. This gives us information over the growth of solutions. This estimate still holds true under the limit of the fractional parameter σ recovering the classical ABP.

In what follows we introduce the definition of concave envelope for a class of functions $u : \mathbb{R}^d \to \mathbb{R}$. First, we put forward the definition of affine function.

Definition 4.1 *A function of the form*

$$\ell(x) = a + b \cdot x,$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ is called affine.

Remark 4.2 *There is also the natural extension of affine functions, namely, vectorial affine maps, which are of the form:*

$$\ell(x) = a + b \cdot x,$$

where $a \in \mathbb{R}^d, b \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$. In this case, ℓ also describes a plane in the ambient space.

Definition 4.3 *Let $u : \mathbb{R}^d \to \mathbb{R}$ be such that*

$$u(x) \leq 0\,,\ x \in \mathbb{R}^d \setminus B_1.$$

The concave envelope of u in B_3 at x is denoted by $\Gamma(x)$ and defined as follows:

$$\Gamma(x) = \begin{cases} \min\{\ell(x),\ \ell \text{ affine } \mid \ell(x) \geq u^+(x) \text{ in } B_2\} & \text{for } x \in B_3 \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B_3 \end{cases}$$

Next we establish some proprieties of the function Γ.

Lemma 4.4 *Let* $\Gamma : \mathbb{R}^d \to \mathbb{R}$ *be as in Definition 4.3. The restriction of* Γ *to* B_3 *is concave.*

Proof: The proof follows from the fact that the minimum of concave functions is concave. Notice that when restricted to B_3, Γ is the minimum of affine functions. Recalling that every affine function is concave, we conclude that Γ is concave. ∎

As a corollary of the previous lemma, we prove the continuity of Γ in B_3 as a direct consequence of Theorem 2.12.

Given a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the conditions in Definition 4.3, one may consider the concave envelope Γ as on above and define the contact set of u as follows.

Definition 4.5 *If* u *satisfies the conditions of Definition 4.3, we define the contact set of* u*, denoted by* Σ_u *as:*

$$\Sigma_u := \{y \in B_1 \,|\, u(y) = \Gamma(y)\}.$$

Remark 4.6 *Unless we are dealing simultaneously with more than one concave envelope, we will not use the subindex in order to preserve the clarity of the notation.*

Lemma 4.7 *The set* $\{y \in B_3 | u = \Gamma\}$ *is closed.*

Proof: Since u is lower semicontinuous and Γ is continuous, $u - \Gamma$ is lower semicontinuous and therefore $\{u = \Gamma\}$ is the preimage of a closed set by a lower semicontinuous function. The result follows from classic consideration in Analysis. ∎

With the above results at hand, we are going to present a number of lemmas. Those are combined to yield the ABP estimate for a class of nonlocal operators.

Lemma 4.8 *Let* u *satisfy the conditions of Definition 4.3. Furthermore suppose*

$$\mathcal{M}^+ \geq -f \ \text{in } B_1.$$

Let:

– $\rho_0 := 1/8\sqrt{d}$,

– $r_k =: \rho_0 2^{-\frac{1}{2-\sigma}-k}$,

– $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}$.

There is a constant $C_0(d, \lambda, \Lambda)$ such that for any x in the contact set, and any $M > 0$, there exists $k \in \mathbb{N}$ for which

$$\left| R_k(x) \cap \{u(x) < u(y) + (y-x)\nabla\Gamma(x) - Mr_k^2\} \right| \leq C_0 \frac{f(x)}{M} |R_k(x)|, \quad (4\text{-}1)$$

where $\nabla\Gamma$ is any element of the superdifferential of Γ at x.

Remark 4.9 *Such a superdifferential is always nonempty in B_3 since Γ restricted to B_3 is concave.*

Remark 4.10 *If u is differentiable the superdifferential of Γ coincides with Du. On the other hand if Γ is differentiable $D\Gamma$ coincides with the superdifferential of the function Γ.*

Proof of Lemma 4.8: Since u can be touched from above by a plane, Lemma (3.5) implies that $\mathcal{M}^+u(x)$ can be evaluated in the classical sense. Recalling the definition of such operators,

$$\mathcal{M}^+u(x) = (2-\sigma) \int_{\mathbb{R}^n} \frac{\Lambda\delta^+ - \lambda\delta^-}{||y||^{d+\sigma}} dy$$

.

we now verify that if x is in the contact set Σ then for all points y in $\mathbb{R}^n$, $\delta(u, x, y) \leq 0$. First, suppose that both $x + y$, $x - y$ are elements of B_3.

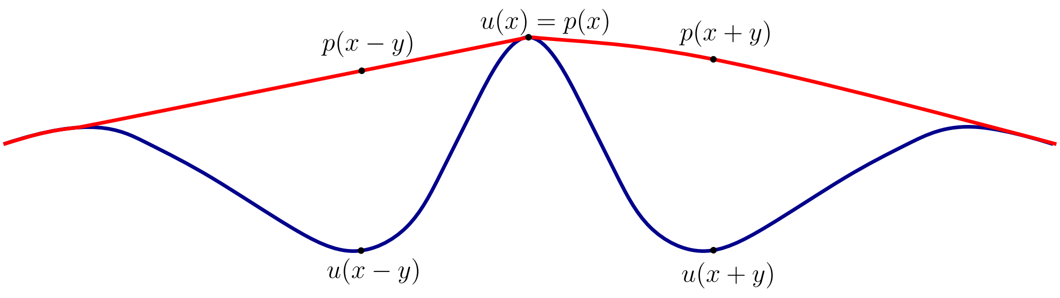

Figure 4.1: Visual comparison between p and u

So as illustrated above we get:

$$\delta(u, x, y) = u(x+y) + u(x-y) - 2u(x) \leq p(x+y) - p(x-y) - 2p(x) = 0, \quad (4\text{-}2)$$

where the last equality follows from p being an affine function.

Next, after going through the first case, suppose that either of $x+y$ or $x-y$ does not belong to B_3. Suppose without loss of generality that $x+y$ is outside B_3, In a straightforward manner we obtain:

$$
\begin{aligned}
&|x| + |y| \geq |x+y| \geq 3, \\
&|y| \geq 2, \\
&|x-y| \geq |x| - |y| \geq 1.
\end{aligned}
$$

Therefore neither $x+y$ nor $x-y$ belong to B_1. This implies that both $u(x+y)$ and $u(x-y)$ are negative and in particular we obtain $\delta(u,x,y) \leq 0$. As a consequence we get:

$$
\begin{aligned}
-f(x) \leq \mathcal{M}^+ u(x) &= (2-\sigma) \int_{\mathbb{R}^d} \frac{-\lambda \delta^-(u,x,y)}{|y|^{n+\sigma}} dy \qquad (4\text{-}3)\\
&\leq (2-\sigma) \int_{B_{r_0}(0)} \frac{-\lambda \delta^-(u,x,y)}{|y|^{d+\sigma}} dy, \qquad (4\text{-}4)
\end{aligned}
$$

where (4-4) follows from integrate a negative function over a smaller domain.

Decompose the domain of integration on (4-3) as the disjoint union of rings $R_k(x)$:

$$
B_{r_0} = \bigcup_{k \in \mathbb{N}} R_k(0).
$$

Then

$$
\begin{aligned}
(2-\sigma) \int_{B_{r_0}} \frac{-\lambda \delta^-(u,x,y)}{|y|^{d+\sigma}} dy &= (2-\sigma) \int_{\bigcup_{k \in \mathbb{N}} R_k(0)} \frac{-\lambda \delta^-(u,x,y)}{|y|^{d+\sigma}} dy \\
&= (2-\sigma) \sum_{k \in \mathbb{N}} \int_{R_k(0)} \frac{-\lambda \delta^-(u,x,y)}{|y|^{d+\sigma}} dy.
\end{aligned}
$$

Replacing the previous identity in 4-4

$$
f(x) \geq (2-\sigma) \sum_{k \in \mathbb{N}} \int_{R_k(0)} \frac{\lambda \delta^-(u,x,y)}{|y|^{d+\sigma}} dy. \qquad (4\text{-}5)
$$

Suppose by contradiction that (4-1) does not hold. In other words it means that there exists a point $x \in \{u = \Gamma\}$ such that for every $C_0 > 0$ there will be $M > 0$ satisfying

$$
|R_k(x) \cap \{u(x) < u(y) + (y-x)\nabla\Gamma(x) - M r_k^2\}| > C_0 \frac{f(x)}{M} |R_k(x)| \qquad (4\text{-}6)
$$

for every $k \in \mathbb{N}$. As a consequence of (4-6) taking $y = x \pm z$ we obtain:

$$|z \in R_k(0) \cap \{u(x+z) < u(x) + (z)\nabla\Gamma(x) - Mr_k^2\}| > C_0 \frac{f(x)}{M}|R_k(x)|$$

Recall now that since Γ is concave, $\nabla\Gamma$ satisfies:

$$\Gamma(x-z) \leq \Gamma(x) + \nabla\Gamma(x)\cdot(-z)$$

It follows from the definition of concave hull and the fact that x is in the contact set that,

$$u(x-z) \leq \Gamma(x-z) \text{ and } u(x) = \Gamma(x).$$

In turn, it implies,

$$|z \in R_k(0) \cap \{2u(x) > u(x-z) + u(x+z) + Mr_k^2\}| > cC_0 \frac{f(x)}{M}|R_k(x)|$$

or in another notation,

$$|z \in R_k(0) \cap \{\delta^-(u,x,z) > Mr_k^2\}| > C_0 \frac{f(x)}{M}|R_k(x)|$$

Then we can further manipulate (4-5) as follows:

$$\begin{aligned}
f(x) &\geq (2-\sigma)\sum_{k\in\mathbb{N}} \int_{R_k(x)} \frac{\lambda\delta^-}{|y|^{d+\sigma}}dy \\
&\geq (2-\sigma)\sum_{k\in\mathbb{N}} \int_{R_k(x)\cap\{\delta^- > 2Mr_k^2\}} \frac{\lambda\delta^-}{|y|^{d+\sigma}}dy \\
&\geq (2-\sigma)\sum_{k\in\mathbb{N}} 2M\frac{Mr_k^2}{r_k^{d+\sigma}}\frac{cC_0 f(x)}{M} \\
&\geq c(2-\sigma)\frac{\rho_0^2}{1-2^{-(2-\sigma)}}C_0 f(x) \\
&\geq cC_0 f(x),
\end{aligned}$$

the last inequality follows from the fact that the expression is uniformly bounded for the fractional parameter $\sigma \in (0,2)$. Since C_0 is arbitrary, letting $C_0 > \frac{1}{c}$ would lead us to a contradiction.

$\blacksquare$

Remark 4.11 *At this point the choice of ρ_0 is not necessary: the above lemma holds for any choice of $\bar{\rho}_0$ adjusting $\bar{C}_0 = \frac{C_0\bar{\rho}_0{}^2}{\rho_0^2}$*

Corollary 4.12 *Suppose that for some point $x \in \mathbb{R}^d, \mathcal{M}^+u(x) \geq g(x)$ then x cannot be in the contact set if $g(x) > 0$*

Proof: Suppose that the inequality holds for some point x in the contact set, then as in the above, one would obtain:

$$0 < g(x) \leq \mathcal{M}^+u(x) = (2-\sigma)\int_{\mathbb{R}^d} \frac{-\lambda\delta^-}{|y|^{n+\sigma}}dy \leq 0 \tag{4-7}$$

which is a contradiction, and as such the result follows. ■

Lemma 4.13 *Let Γ be a concave function in B_r. Assume that for a small ε,*

$$|\{y : \Gamma(y) < \Gamma(x) + (y-x)\cdot\nabla\Gamma(x) - h\} \cap (B_r(x) \setminus B_{r/2}(x)| \leq \varepsilon|B_r(x) \setminus B_{r/2}(x)|$$

then $\Gamma(y) \geq \Gamma(x) + (y-x)\cdot\nabla\Gamma(x) - h$ in $B_{r/2}$.

Proof: Let $y \in B_{r/2}(x)$. We may create two points as in the figure:

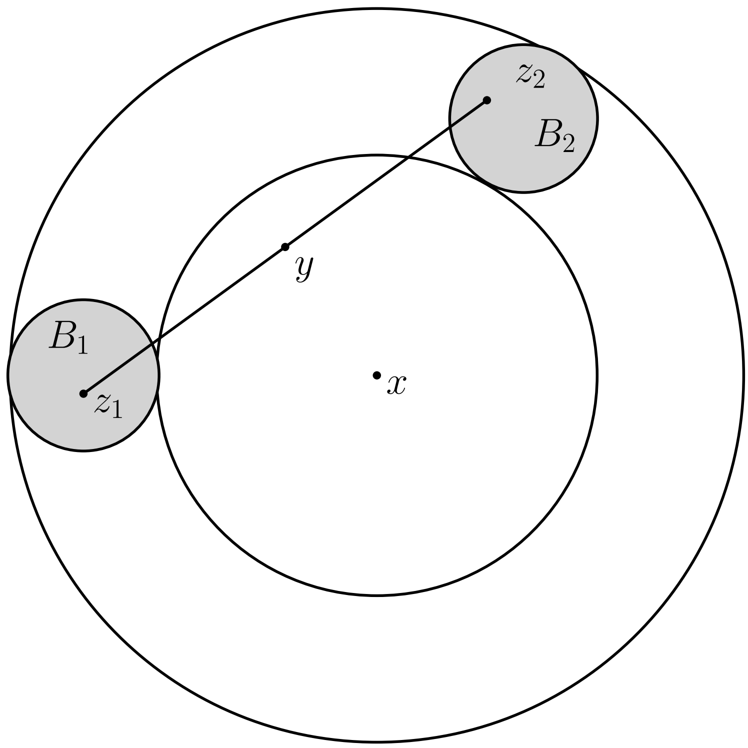

Figure 4.2: Geometric construction

To construct these points, consider the straight line ℓ_1 going through x and y, now construct a line orthogonal ℓ_2 to the one previously created. Now choose two points y_1, y_2, in ℓ_2 symmetric with respect to y such that

$$|y_1 - y| = |y_2 - y| = \sqrt{\frac{9}{16}r^2 - |y-x|^2} \tag{4-8}$$

from the above construction,

1. $y_1, y_2 \in B_r(x) \setminus B_{r/2}(x)$
2. $|y_1 - x| = |y_2 - x| = \frac{3}{4}r$
3. $y = \frac{y_1+y_2}{2}$

Consider now the balls $B_1 = B_{r/4}(y_1), B_2 = B_{r/4}(y_2)$, from the previous considerations we have that both balls are symmetric with respect to y and are entirely contained in the ring $B_r \setminus B_{r/2}$. Taking ε sufficiently small we would have:

$$|B1| > \varepsilon|B_r \setminus B_{r/2}| \geq |\{y : \Gamma(y) < \Gamma(x) + (y - x) \cdot \nabla\Gamma(x) - h\} \cap (B_r(x) \setminus B_{r/2}(x)|$$
$$|B2| > \varepsilon|B_r \setminus B_{r/2}| \geq |\{y : \Gamma(y) < \Gamma(x) + (y - x) \cdot \nabla\Gamma(x) - h\} \cap (B_r(x) \setminus B_{r/2}(x)|$$

Thus, both sets must intersect the set

$$\{y : \Gamma(y) \geq \Gamma(x) + (y - x) \cdot \nabla\Gamma(x) - h\} \cap (B_r(x) \setminus B_{r/2}(x),$$

and therefore

$$|\{y : \Gamma(y) \geq \Gamma(x) + (y - x) \cdot \nabla\Gamma(x) - h\} \cap B_1| > 0$$
$$|\{y : \Gamma(y) \geq \Gamma(x) + (y - x) \cdot \nabla\Gamma(x) - h\} \cap B_2| > 0.$$

The former inequalities imply the existence of two points $z_1 \in B_1, z_2 \in B_2$ such that:

1. $z = \frac{z_1+z_2}{2}$
2. $\Gamma(z_1) \geq \Gamma(x) + (z_1 - x) \cdot \nabla\Gamma(x) - h$
3. $\Gamma(z_2) \geq \Gamma(x) + (z_2 - x) \cdot \nabla\Gamma(x) - h$

by the concavity of Γ

$$\Gamma(z) = \Gamma(\frac{z_1 + z_2}{2}) \geq \frac{\Gamma(z_1) + \Gamma(z_2)}{2} \geq \Gamma(x) + (z - x) \cdot \nabla\Gamma(x) - h,$$

and the proof is complete. ■

Corollary 4.14 *For any $\varepsilon_0 > 0$ there is an universal constant C, such that for any function u satisfying the same hypothesis as in Lemma (4.8), there is* *a* $r \in (0, \rho_0 2^{-\frac{1}{2-\sigma}})$ *for which :*

$$\frac{|\{y \in B_r(x) \setminus B_{r/2}(x) : u(y) < u(x) + (y-x) \cdot \nabla\Gamma(x) - Cf(x)r^2\}|}{|B_r(x) \setminus B_{r/2}(x)|} \leq \varepsilon_0$$
$$|\nabla\Gamma(Br/4)| \leq Cf(x)^d |B_{r/4}(x)|.$$

Above, ρ_0 *is the same as in Definition* 4.3.

Proof: The first statement follows immediately from Lemma 4.8 choosing $M = \frac{Cf(x)}{\varepsilon_0}$. The second fact follows from the previous one, together with convexity. ■

Lemma 4.15 *Let* $B_r(x)$ *be the family of balls* B_{r_k} *in Lemma 4.8. Assume that* x *is a point in the contact set. Then:*

$$\left|\bigcup_{x\in\Sigma} B_r(x)\right| \geq C(\sup u)^d \tag{4-9}$$

Lemma 4.16 *The measure of the image of* $\nabla\Gamma(B_3 \setminus \Sigma)$ *is* 0.

Proof: Notice that the collection $B_r(x)_{x\in\Sigma}$ is a cover of $\overline{\Sigma}$, in particular due to the compacity of $\overline{\Sigma}$ we may extract a countable subcovering of Σ, $B_j = B_{r_j}(x_j)$ such that:

1. For every $j \in \mathbb{N}$, the ball $B_j \subset B_{r/4}(x)$ where r is given by corollary 4.14.

2. The subcover has finite overlapping.

Recall that when restricted to B_j, the function Γ is at most quadratic, and thus:

$$|\nabla\Gamma(B_j)| \leq C\,|B_j|\,.$$

It follows from lemma 4.16 that for every set $\Sigma \subset X \subset B_3$ we obtain:

$$|\nabla\Gamma(B_3)| = |\nabla\Gamma(X)| = |\nabla\Gamma(\Sigma)|\,.$$

A consequence of choosing $X = \bigcup_{j\in\mathbb{N}} B_j$ is:

$$(\sup u)^d = (\sup \Gamma)^d \leq C\,|\nabla\Gamma(B_3)| \leq C \sum_{j=0}^{\infty} |\nabla\Gamma(B_j)| \leq C \left|\bigcup_{j\in\mathbb{N}} \nabla\Gamma(B_j)\right|.$$

■

Even though the previous result gives us an $\mathcal{L}^\infty$ estimate for the solution u, the estimate is still too imprecise. In what follows, we will present a refinement of the estimate which goes a step further and generalize the celebrated ABP-estimate.

Theorem 4.17 (ABP Estimate) *Let u and Γ as in Definition 4.3 then there exists a finite tilling of cubes Q_j, with radius d_j satisfying the following:*

1. $Q_i \cap Q_j = \emptyset$.
2. $\Sigma \subset \bigcup_{j=1}^{m} \overline{Q}_j$.
3. *The tilling is minimal, in the sense that $\Sigma \cap \overline{Q}_j$ is nonempty.*
4. $d_j \leq \rho_0 2^{\frac{-1}{2-\sigma}}$
5. $|\nabla\Gamma(\overline{Q}_j)| \leq C(\max_{\overline{Q}_j} f)^d |Q_j|$.
6. $|\{y \in 8\sqrt{n}Q_j | u(y) > \Gamma(y) - C\max_{Q_j} f d_j^2\}| \geq \mu |Q_j|$

Proof: We start by covering B_1 with a finite amount of disjoint cubes with diameter equal or less than $\rho_0 2^{\frac{-1}{2-\sigma}}$. This is always possible for a fixed radius r, since one can cover $\overline{B_1}$ with open cubes of radius r centered at every point of B_1 and due to compacity extract a finite subcover.

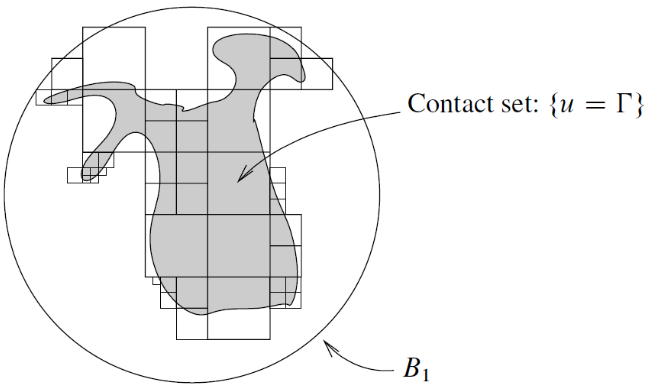

Figure 4.3: Covering the contact set with cubes [2]

By now our construction attends conditions *2* and *4*, in order to conclude the proof we will iterate the following procedure: Whenever a cube does not satisfy conditions *5* and *6*, split that cube in 2^n cubes with half of the diameter and proceed by discarding all of those whose closure do not intersect the contact set.

It remains to show that after a finite amount of iterations, we obtain the desired collection of cubes with the exception they may not be disjoint.

Suppose for contradiction, that the algorithm produces an infinite amount of cubes i.e for every iteration, there is a cube which is split into

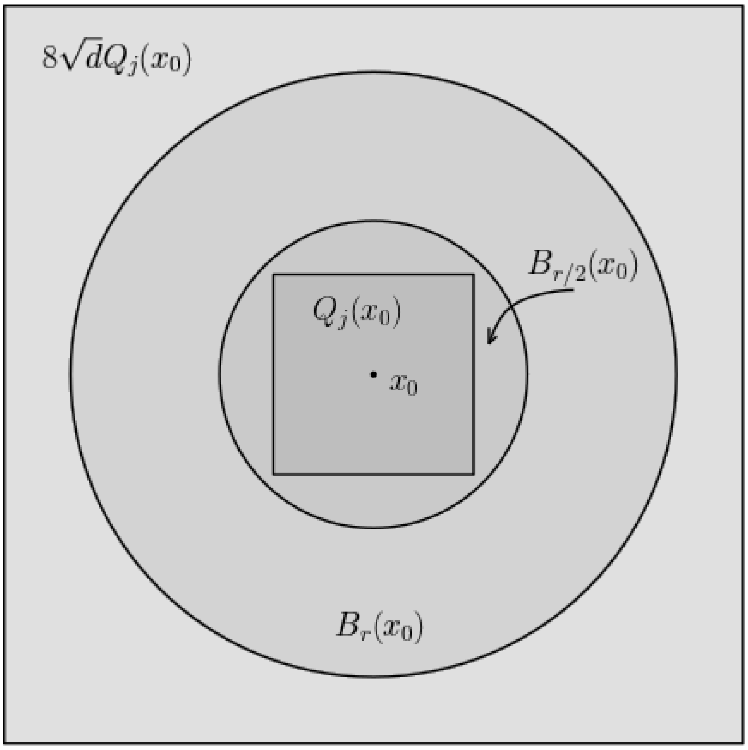

Figure 4.4: Relation between the cubes and the balls

smaller cubes. If that is the case, then we would obtain a sequence of nested cubes.

Recall that given a nested sequence of closed nonempty subsets of a compact metric space X, the intersection $\bigcap_i X_i$ is nonempty.

Therefore the intersection of the closure of our nested cubes must be a point x_0 in Σ since all the closed cubes intersect Σ, in particular it implies that $u(x_0) = \Gamma(x_0)$.

The contradiction will follow from showing that one of the cubes containing x_0 will not split. It follows from corollary 4.14 that for some $r \in (0, \rho_0 2^{-\frac{1}{2-\sigma}})$ for which holds true that:

$$\frac{|\{y \in B_r(x) \setminus B_{r/2}(x) : u(y) < u(x) + (y - x) \cdot \nabla\Gamma(x) - Cf(x)r^2\}|}{|B_r(x) \setminus B_{r/2}(x)|} \leq \varepsilon_0$$

$$|\nabla\Gamma(B_{r/4})| \leq Cf(x)^d |B_{r/4}(x)|.$$

Let Q_n be the nested sequence of cubes with diameter d_n, whose closure contains x_0. Since the diameters of the nested sequence of cubes decrease by half at each iteration, we have that for some integer $j \in \mathbb{N}$ it holds true that:

$$\overline{Q}_j \subset B_{r/2}(x_0) \qquad B_r(x_0) \subset 8\sqrt{d}Q_j$$

Recall that in B_2, Γ is a concave function as a consequence of Lemma 4. This implies, following definition 2.10, that:

$$\Gamma(y) \leq \Gamma(x_0) + (y - x_0)\nabla\Gamma(x_0)$$

Using the fact that $x_0 \in \Sigma$ we obtain:

$$\Gamma(y) \leq u(x_0) + (y - x_0)\nabla\Gamma(x_0) \tag{4-10}$$

With all the previous estimates at hand, we may show that condition *5* holds.

$$|\{y \in 8\sqrt{d}Q_j \mid u(y) \geq \Gamma(y) - C \max_{\overline{Q}_j} f(x)d_j^2\}| \geq \tag{4-11}$$
$$|\{y \in 8\sqrt{d}Q_j \mid u(y) \geq \Gamma(x_0) + (y - x_0)\nabla\Gamma(x_0) - C \max_{\overline{Q}_j} f(x)d_j^2\}| \geq \tag{4-12}$$
$$|\{y \in 8\sqrt{d}Q_j \mid u(y) \geq \Gamma(x_0) + (y - x_0)\nabla\Gamma(x_0) - Cf(x_0)r^2\}| \geq \tag{4-13}$$
$$(1 - \varepsilon_0)|B_r(x_0) \setminus B_r/2(x_0)| \geq \mu|Q_j|. \tag{4-14}$$

On the above, inequality (4-11) is an immediate consequence of (4-10), the next estimate follows from the fact that $C > 0$, so we are only weakening the bound. To proceed, we recall that d_j and r are comparable due to figure 4.4, and conclude (4-13). The last line follows from Corollary 4.14 and simple euclidean geometry.

From the last chain of inequalities we may conclude that *6* holds and since Q_j is contained in B_r *5* is true as well. ■

Remark 4.18 *At a first moment it may not be clear that Theorem 4.17 should be called the nonlocal ABP estimate. But the name is well-deserved if we notice that as $\sigma \to 0$ the size of the squares tends to zero. In particular, we have for every positive σ:*

$$\sum_j |\nabla(Q_j)| \leq \sum_j C \max_{Q_j} f^+ \tag{4-15}$$

Now, letting σ goes to zero, the right-hand side converges to a Riemann integral while the left side converges to the measure of $\nabla\{\Sigma\}$ i.e:

$$|\nabla\{\Sigma\}| \leq C \int_\Sigma f^+ \tag{4-16}$$

5
Preliminary regularity: L^ε-estimates

In what follows, we are interested in obtaining point-wise estimates from the measure of some sets. This kind of analysis plays an important role when demonstrating a Harnack Inequality. The results proven here are analogous to those found in [1]. The first step in producing this result is the creation of an explicit subsolution, we proceed with the construction of such object.

Proof: We start noticing that proving for $x = e_1 = (1, 0, \ldots, 0)$ is enough. Suppose that x is not a unitary vector, then a rotation brings us back to the previous situation, otherwise we may consider the function $\tilde{f}(y) = |x|^p f(|x|y) \geq f(y)$, hence the proof may be simplified by only concerning ourselves with the case $x = e_1$.

We follow computing an estimate of $\delta(f, x, y)$ for $y \in B_{1/2}$. Recall the $|y| < 1/2$ implies that neither $x + y$ nor $x - y$ is in $B_{1/2}$ therefore $f(z) = |z|^{-p}$ for $z \in \{x, x + y, x - y\}$.

Recall the following Bernoulli-like inequality

Lemma 5.1 *Let $x > -1$, then for every $q > 0$*

$$(1 + x)^{-q} \geq 1 + qx + \frac{q(q + 1)}{2} x^2.$$

From the above inequality we obtain a pair of inequalities, which will be used on what follows.

Lemma 5.2 *Let $a > b > 0$, then for every $q > 0$*

$$(a + b)^{-q} \geq a^{-q}(1 + q\frac{b}{a}).$$

Proof:[Lemma 5.2] Consider the expression $(a + b)^{-q}$, rewrite it as $a^q(1 + \frac{b}{a})^{-q}$ and apply the previous inequality with $x = \frac{b}{a}$. Hence, $(a + b)^{-q} \geq a^{-q}(1 + q\frac{b}{a} + \frac{q(q+1)}{2}\frac{b}{a}^2)$. Since the last term is positive we may discard it and obtain the desired inequality. ■

Lemma 5.3 *Let $a > b > 0$, then for every $q > 0$*

$$(a+b)^{-q} + (a-b)^{-q} \geq 2a^{-q} + q(q+1)b^2a^{-q-2}.$$

Proof:[Lemma 5.3] As in the proof before, we may consider $x = \frac{b}{a}$, use the first inequality and obtain:

$$\begin{aligned}(1+x)^{-q} &\geq 1 + qx + \frac{q(q+1)}{2}x^2\\(1-x)^{-q} &\geq 1 - qx + \frac{q(q+1)}{2}(-x)^2\end{aligned}$$

The result follows from summing both inequalities and multiplying by a^{-q} and explicitly writing the value of x as b/a. ■

Since the above inequalities have been established, we may proceed estimating $\delta(f,x,y)$ for $y \in B_{1/2}$.

$$\begin{aligned}\delta(f,x,y) = &\quad |x+y|^{-p} + |x-y|^{-p} - 2|x|^{-p}\\ = &\quad (|x+y|^2)^{-p/2} + (|x-y|^2)^{-p/2} - 2(|x|^2)^{-p/2}\\ = &\quad (|x|^2 + 2\langle x,y\rangle + |y|^2)^{-p/2} + (|x|^2 - 2\langle x,y\rangle + |y|^2)^{-p/2} - 2|x|^{-p}\\ = &\quad (1 + 2y_1 + |y|^2)^{-p/2} + (1 - 2y_1 + |y|^2)^{-p/2} - 2\end{aligned}$$

We may now estimate the above in view of Lemma (5.3), choosing $a = 1+|y|^2, b = 2y_1, q = -p/2$. Hence,

$$\delta(f,x,y) \geq 2(1+|y|^2)^{-p/2} + p(p+2)y_1^2(1+|y|^2)^{-p/2-2} - 2$$

We may continue our estimates using Lemma (5.2) on both terms.

$$\delta(f,x,y) \geq p(-|y|^2 + (p+2)y_1^2 - \frac{1}{2}(p+2)(p+4)y_1^2|y|^2)$$

Now we proceed estimating $\mathcal{M}^- f(e_1)$. The key idea is to estimate the integral splitting it over a small ball of radius smaller than 1/2 and over the complement of such ball. On the former set, we will use the above estimate and on the later we may use that $0 < f(z) < 2^p$.

$$\mathcal{M}^- f(e_1) = (2-\sigma)\int_{B_r} \frac{\lambda\delta^+ - \Lambda\delta^-}{||y||}^{d+\sigma} dy + (2-\sigma)\int_{\mathbb{R}^d\setminus B_r} \frac{\lambda\delta^+ - \Lambda\delta^-}{||y||^{d+\sigma}} dy$$

We may choose p large enough in order that

$$(p+2)\lambda \int_{\partial B_1} y_1^2 d\sigma(y) - \Lambda|\partial B_1| =: \delta_0 > 0$$

With the above definition we may bound the above integrals as

$$\mathcal{M}^- f(e_1) = (2-\sigma)\int_{B_r} \frac{\lambda\delta^+ - \Lambda\delta^-}{||y||}^{d+\sigma} dy + (2-\sigma)\int_{\mathbb{R}^d \setminus B_r} \frac{\lambda\delta^+ - \Lambda\delta^-}{||y||^{d+\sigma}} dy$$
$$\geq (2-\sigma)\int_0^r \frac{\lambda p\delta_0 s^2 - \frac{1}{2}p(p+2)(p+4)C\Lambda s^4}{s^{1+\sigma}} ds - (2-\sigma)\int_{\mathbb{R}^d \setminus B_r} \Lambda \frac{2^p}{|y|^{n+\sigma}}$$

Evaluating the above integrals we conclude

$$\mathcal{M}^- f(e_1) \geq cr^{2-\sigma} p\delta_0 - p(p+2)(p+4)C\frac{2-\sigma}{4-\sigma} r^{4-\sigma} - \frac{2-\sigma}{\sigma} C 2^{p+1} r^{-\sigma}$$

Now, fix r positive but smaller than $1/2$ and then choose $\sigma_0 < 2$ such that for every $\sigma \in (\sigma_0, 2)$ the last two terms are small to the point that the above expression is positive.

■

Corollary 5.4 *Given any $\sigma_0 \in (0,2)$ and $r > 0$, there exist $p > 0$ and $\delta > 0$ such that*

$$f(x) = \min(\delta^{-p}, |x|^{-p})$$

is a subsolution to

$$\mathcal{M}^- f(x) \geq 0$$

outside B_r.

Proof: The idea behind the proof is controlling the negative part of $\mathcal{M}^- f(x)$ choose δ sufficiently small in order to have the positive part of $\mathcal{M}^- f(x)$ big enough to overcome the negative one. As before let $x = e_1$, and without loss of generality fix $r = 1$. From the previous corollary we obtain the existence of σ_1 and p_0 such that choosing $\delta = 1/2$, f is a subsolution for every $2 > \sigma > \sigma_1$ outside of B_1. Suppose now that $\sigma_1 > \sigma > \sigma_0$, otherwise there is nothing to be done. Define p as the maximum between p_0 and the dimension d.

Writing

$$\mathcal{M}^- f(e_1) = (2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda\delta^+}{||y||^{d+\sigma}} dy - (2-\sigma)\int_{\mathbb{R}^d} \frac{\Lambda\delta^-}{||y||^{d+\sigma}} dy =: I_1 + I_2.$$

Since f is bounded the rightmost term is finite i.e $I_2 > -C$ for some positive C. Notice that $||x+y||^{-p} + ||x-y||^{-p} - 2||x||^{-p}$ is not an integrable function since for every $M > 0$ there exist δ_0 such that

$$\int_{\mathbb{R}^d \setminus B_{\delta_0}} \frac{||x+y||^{-p} + ||x-y||^{-p} - 2||x||^{-p}}{||y||^{d+\sigma}} dy > M.$$

So choosing $\delta = \delta_0$ given by setting $M = 2C$ we obtain:

$$\mathcal{M}^- f(e_1) = I_1 + I_2 > 2C - C > 0.$$

■

Corollary 5.5 *Given any $\sigma \in (0,2)$ there exists a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that*

1. *φ is continuous over $\mathbb{R}^d$.*
2. *φ is compactly supported in $B_{2\sqrt{d}}$.*
3. *$\varphi(x) > 2$ for every $x \in Q_3$.*
4. *$\mathcal{M}^-\varphi(x) > -\psi(x)$ for some positive function ψ supported in $\overline{B_{1/4}}$.*

Proof: Fix $r = 1/4$, let $p, \delta > 0$ being the ones given by the previous lemma. Consider the function described by:

$$\varphi(x) = c \begin{cases} 0 & \text{in } \mathbb{R}^d \setminus B_{2\sqrt{d}} \\ ||x||^{-p} - (2\sqrt{d})^{-p} & \text{in } B_{2\sqrt{d}} \setminus B_\delta \\ q & \text{in } B_\delta \end{cases}$$

Above, q is a paraboloid chosen in order to make φ continuous over $B_{2\sqrt{d}}$, c is such that the third affirmative is true. It follows directly from the definition of the above function the veracity of the first two affirmations, the fourth holds due direct computation. ■

With the above subsolution constructed we may proceed in obtaining a pointwise estimate.

Lemma 5.6 *Let $\sigma > \sigma_0 > 0$. There exists $\varepsilon_0 > 0, 0 < \mu < 1$, and $M > 1$ (depending only on $\sigma_0, \lambda, \Lambda, d$) such that if*

1. *$u \geq 0$ in $\mathbb{R}^d$*
2. *$\inf_{Q_3} u \leq 1$*
3. *$\mathcal{M}^- u \leq \varepsilon_0$ in $Q_{4\sqrt{d}}$*

then $|\{u \leq M\} \cap Q_1| > \mu$.

Proof: Consider the function $v = \varphi - u$, such that φ is as given by corollary (5.5). Notice that:

1. $v \leq 0$ in $\mathbb{R}^d \setminus B_{2\sqrt{d}}$.

2. $\mathcal{M}^+ v(x) \geq -\psi(x) - \varepsilon_0$. in $B_{2\sqrt{d}}$.

The former point follows from the fact that outside $B_{2\sqrt{d}}$ φ is identically zero and u is nonnegative at the whole space. The latter, follows from the computation

$$\mathcal{M}^+ v \geq \mathcal{M}^- v = \mathcal{M}^- \varphi - \mathcal{M}^- u \geq -\psi - \varepsilon_0.$$

Consider the function $\tilde{v} : \mathbb{R}^d \to \mathbb{R}$ the scaled version of f i.e $\tilde{v}(x) = f(2\sqrt{d}x)$ therefore.

1. $\tilde{v} \leq 0$ in $\mathbb{R}^d \setminus B_1$.

2. $\mathcal{M}^+ \tilde{v}(x) \geq -\psi(2\sqrt{d}x) - \varepsilon_0$. in B_1.

The first point follows from the previous observation *mutatis mutandis* while the second may be shown computing

$$\mathcal{M}^+ \tilde{v}(x) = (2\sqrt{d})^{\sigma} \mathcal{M}^+ v(2\sqrt{d}x)$$

Hence,

$$\mathcal{M}^+ \tilde{v}(x) \geq (2\sqrt{d})^{-\sigma}[-\psi(2\sqrt{d}x) - \varepsilon_0] \geq -\psi(2\sqrt{d}x) - \varepsilon_0.$$

With the above estimate at hand, we may apply the results from the previous section and obtain

$$\max_{B_1} \tilde{v} \leq C|\nabla\tilde{\Gamma}(B_1)|^{1/d} \leq C\left(\sum_j |\nabla\tilde{\Gamma}(\overline{Q_j})|\right)^{1/d} \leq C\left(\sum_j \max_{Q_j}(\psi(2\sqrt{d}x) + \varepsilon_0)^+ |Q_j|\right)^{1/d}$$

for a family $\{Q_j\}_{j=1}^n$ of cubes given by Theorem (4.17). Rewriting the above in terms of v:

$$\max_{B_2\sqrt{d}} v \leq C|\nabla\Gamma(2\sqrt{d})|^{1/d} \leq C\left(\sum_j |\nabla\Gamma(\overline{Q_j})|\right)^{1/d}$$

for Γ the concave envelope of v over $B_{6\sqrt{d}}$. It follows from the fact that $Q_3 \subset B_{2\sqrt{d}}$ that $\max_{B_{2\sqrt{d}}} \geq \max_{Q_3}$. Notice that φ is greater than 2 in Q_3 and the infimum of u over Q_3 is bounded from above by 1. Hence, $\max_{B_2\sqrt{d}} v \geq 1$

so,

$$1 \leq C \left(\sum_j |\nabla\Gamma(\overline{Q_j})| \right)^{1/d} \leq C \left(\sum_j (\max_{Q_j}(\psi(2\sqrt{d}x) + \varepsilon_0)^+)^d |Q_j| \right)^{1/d}$$

Recalling that $(A+B)^+ \leq A^+ + B^+$,

$$1 \leq C\varepsilon_0 + +C \left(\sum_j (\max_{Q_j} \psi(2\sqrt{d}x)^+)^d |Q_j| \right)^{1/d}$$

Choosing ε_0 smaller than $1/2C$

$$\frac{1}{2} \leq C \left(\sum_j (\max_{Q_j} \psi(2\sqrt{d}x)^+)^d |Q_j| \right)^{1/d}$$

Since ψ is supported in $B_{1/4}$, we may only consider the cubes Q_j such that $\overline{B}_{1/4} \cap Q_j \neq \emptyset$. Also, since ψ is bounded, an estimate about the measure of such cubes.

$$\sum_{\substack{Q_j \\ \overline{B}_{1/4} \cap Q_j \neq \emptyset}} |Q_j| > \frac{1}{2C|\psi|_\infty} =: c > 0 \tag{5-1}$$

The diameters of all cubes Q_j are all smaller than $\rho_0 2^{\frac{-1}{(2-\sigma)}}$ which is uniformly bounded by $\frac{\rho_0}{\sqrt{2}}$.

Lemma 5.7 *Let $Q(x_0)$ be a cube such that the diameter of $Q(x_0)$ is smaller than $\frac{\rho_0}{\sqrt{2}}$. If $Q(x_0) \cap B_{1/4} \neq \emptyset$, then $4\sqrt{d}Q(x_0) \subset B_{1/2}$.*

Proof:[Lemma 5.7] Since $Q(x_0) \cap B_{1/4} \neq \emptyset$ it follows that

$$||x_0|| \leq \frac{1}{4} + \frac{diam(Q(x_0))}{2} \leq \frac{1}{4} + \frac{1}{16\sqrt{d}}.$$

Let $z \in 4\sqrt{d}Q(x_0)$ then

$$||z|| \leq ||x_0|| + \frac{4\sqrt{d} \cdot diam(Q_j)}{2} = ||x_0|| + \frac{1}{4\sqrt{2}} \leq \frac{1}{4} + \frac{1}{16\sqrt{d}} + \frac{1}{4\sqrt{2}} \leq \frac{1}{2}.$$

■

Applying Theorem 4.17 to $\tilde{v}$ and rescaling we conclude

$$|\{x \in 4\sqrt{d}Q_j | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq |\{x \in 4Q_j | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq c|Q_j|,$$

with $d_j \leq \rho_0$. Consider now the collection $\{4\sqrt{d}Q_j\}$, for Q_j such that $B_{1/4} \cap Q_j \neq \emptyset$, an open cover for the union $\bigcup_{Q_j \cap B_{1/4} \neq \emptyset} \overline{Q}_j$, we may extract a subcover with finite overlapping.

$$\sum_j |\{x \in 4\sqrt{d}Q_j | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq \sum_j c|Q_j|,$$

Let $p = \max_{i \leq n} |\{j \in \mathbb{N} | Q_i \cap Q_j \neq \emptyset\}|$ from the estimate (5-1) and the above, ■

$$p|\{x \in B_{1/2} | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq \sum_j |\{x \in 4\sqrt{d}Q_j | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq \sum_j c|Q_j| \geq \tilde{c},$$

In particular since p does not depend on the original cover we obtain an uniform bound:

$$|\{x \in B_{1/2} | v(x) \geq \Gamma(x) - Cd_j^2\}| \geq \frac{\tilde{c}}{p} =: c > 0,$$

recalling the definition of v we may rewrite the above as

$$|\{x \in B_{1/2} | \varphi(x) - u(x) \geq \Gamma(x) - Cd_j^2\}| \geq c,$$

Defining M_0 as the supremum of φ over the ball of radius $1/2$ and recalling that $d_j \leq \rho_0$ and that Γ is positive we conclude

$$|\{x \in B_{1/2} | u(x) \leq M_0 + C\rho_0^2\}| \geq c,$$

choosing $M = \max(1, M_0 + C\rho_0^2)$.

Definition 5.8 *Let Q_1 be the unit cube in $\mathbb{R}^d$. We split this cube into 2^d cubes of half side. We do the same thing with each of these 2^d cubes, and we iterate this process. The cubes obtained this way are called dyadic cubes.*

Definition 5.9 *If Q is a dyadic cube different from Q_1, we say that $\tilde{Q}$ is the predecessor of Q if Q is one of the 2^d cubes generated by splitting $\tilde{Q}$.*

Lemma 5.10 *Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ such that*

- $|A| \leq \delta$
- *If Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$.*

Then $|A| \leq \delta|B|$.

Proof: It is clear that

$$\frac{|Q_1 \cap A|}{Q_1} = \frac{|A|}{1} = |A| \leq \delta$$

Now, we proceed dividing Q_1 into 2^d dyadic cubes. At least one of those smaller cubes Q satisfies

$$\frac{|Q \cap A|}{Q} \leq \delta$$

We then divide Q into 2^d dyadic cubes and iterate the process. In this way we produce a sequence of cubes Q^i such that

$$\frac{|Q^i \cap A|}{|Q^i|} > \delta, \qquad \forall i \in \mathbb{N}.$$

Suppose that $x \notin \cup Q^i$, then there exist a family of closed dyadic cubes Q, with sides tending to zero, but still satisfy

$$\frac{|Q \cap A|}{Q} \leq \delta < 1.$$

Applying the Lebesgue differentiation theorem we obtain that $\chi_A(x) \leq \delta < 1$ for almost every x outside the union $\cup Q^i$. This of course implies that$A \subset \cup Q^i$ except for a set of null measure. Consider $\tilde{Q}_i$ the family of predecessors of the cubes Q^i and extract a disjoint sub collection that still covers the original family $\{Q^i\}$. From the choice of Q^i we obtain

$$\frac{|\tilde{Q}^i \cap A|}{|\tilde{Q}^i|} \leq \delta, \qquad \forall i \in \mathbb{N}.$$

The above fact implies that $\tilde{Q}^i$ is in fact a subset of B, hence

$$A \subset \bigcup Q^i \subset \bigcup \tilde{Q}_i \subset B$$

Estimating the measure of the above sets give us

$$|A| \leq |\bigcup Q^i \cap A| \leq |\bigcup \tilde{Q}_i \cap A| = \sum |\tilde{Q}_i \cap A| \leq \delta \sum |\tilde{Q}_i| \leq \delta |B|.$$

■

Lemma 5.11 *Let u be as in lemma (5.6). Then*

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k$$

for $k \in \mathbb{N}$ where, M and μ are as in 5.6. As a consequence, we have that

$$|\{u > t\} \cap Q_1| \leq dt^{-\varepsilon} \qquad \forall t > 0.$$

where d and ε are universal positive constants.

Proof: We will prove the result using strong induction. For $k = 1$, the above result is the same as (5.6). Suppose we are able to establish the result for $k-1$, define the sets

$$A = \{u > M^k\} \cap Q_1 \qquad\qquad B = \{u > M^{k-1}\} \cap Q_1.$$

It suffices to show that $|A| \leq (1-\mu)|B|$. Since $M > 1$ it follows

$$|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$$

In view of lemma 5.10 it suffices to prove that if Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$. In fact, let $Q = 2^{-i}Q_1(x_0)$ be a dyadic cube such that $|A \cap Q| > \delta|Q|$ and suppose in order to obtain a contradiction that $\tilde{Q}$ is not contained in B, therefore exists a point $\tilde{x} \in \tilde{Q}$ such that

$$u(\tilde{x}) \leq M^{k-1}.$$

Consider now the function defined as

$$v(x) = \frac{u(x_0 + \frac{x}{2^i})}{M^{k-1}}$$

Clearly v is positive in the whole $\mathbb{R}^d$, also

$$\inf_{Q_3} v(x) = \frac{\inf_{2^{-i}Q_3(x_0)} u(x)}{M^{k-1}} \leq \frac{u(\tilde{x})}{M^{k-1}} \leq 1,$$

and

$$\mathcal{M}^- v(x) = \frac{\mathcal{M}^- u(x_0 + \frac{x}{2^i})}{M^{k-1}2^{i\sigma}} \leq \varepsilon_0$$

Therefore by lemma 5.6 we obtain

$$\mu < |\{v(x) \leq M\} \cap Q_1| = 2^{id}|\{u(x) \leq M^k\} \cap Q|$$

The above is a clear contradiction with the fact that $|A \cap Q| > (1-\mu)|Q|$ ■

Lemma 5.12 *Let $u \geq 0$ in $\mathbb{R}^d, u(0) \leq 1$, and $\mathcal{M}^- u \leq \varepsilon_0$ in B_2. Assume $\sigma \geq \sigma_0 > 0$. Then*

$$|\{u > t\} \cap B_1| \leq Ct^{-\varepsilon} \qquad \forall t > 0.$$

where the constant C and ε are dependent on $\lambda, \Lambda, d, and \sigma_0$.

Scaling the above theorem, we obtain the more general formulation

Lemma 5.13 *Let $u \geq 0$ in $\mathbb{R}^d$ and $\mathcal{M}^- u \leq C_0$ in B_{2r}. Assume $\sigma \geq \sigma_0 > 0$. Then*

$$|\{u > t\} \cap B_1| \leq C r^d (u(0) + C_0 r^\sigma)^\varepsilon t^{-\varepsilon} \qquad \forall t > 0.$$

where the constants C and ε are dependent on $\lambda, \Lambda, d,$ and σ_0.

6
Harnack Inequality

The Harnack Inequality, named after Carl Gustav Axel von Harnack, first appeared in the context of harmonic functions in the plane and much later became a fundamental tool in the general theory of harmonic functions and PDE's as a whole. At a first glance, this inequality states that the value of a sub and super solution is bound by the value of the function on a point, a more careful use of this kind of inequality generally produces a proof of Hölder regularity of solutions of elliptic problems.

Theorem 6.1 *Let $u \geq 0$ in $\mathbb{R}^d, \mathcal{M}^- u \leq C_0$, and $\mathcal{M}^+ u \geq -C_0$ in B_2. Assume $\sigma \geq \sigma_0 > 0$. Then $u(x) \leq C(u(0) + C_0)$ for every $x \in B_{1/2}$.*

Proof: We may start simplifying the problem by considering $\tilde{u}$ as $u/(u(0)+C_0)$, and prove the result for $\tilde{u}$ assuming $\tilde{u}(0) \leq 1, C_0 = 1$. Let $\gamma = \frac{d}{\varepsilon}$, where ε is the one given by lemma 5.13. Let suppose that for some $t_0 \in \mathbb{R}$

$$u(x) \leq t_0(1 - ||x||)^{-\gamma} \qquad \forall x \in B_1.$$

Let t be the smallest value of t_0 such that the above inequality is valid, i.e,

$$\Big(u(x) \leq t(1 - ||x||)^{-\gamma}\, \forall x \in B_1\Big) \wedge \Big(\forall t_1 < t \;\; \exists x_1 \in B_1 \;\; u(x_1) > t_1(1 - ||x_1||)^{-\gamma}\Big) \tag{6-1}$$

As a consequence of the minimality of t there must be a point x_0 in B_1 such that $u(x_0) = t(1 - ||x_0)||)^{-\gamma}$, otherwise we could make t smaller and still have a valid inequality which in turn would contradict the definition of t. Denote by D the distance between x_0 to ∂B_1 and consider the set $A = \{u > u(x_0)/2\}$. By the L^ε estimates in 5.13 we may obtain the bound

$$|A \cap B_1| \leq C \left| \frac{2}{u(x_0)} \right|^\varepsilon \leq C t^{-\varepsilon} D^d.$$

Let $r = D/2$ therefore $B_r(x_0) \subset B_1$ and $|B_r(x_0)| = CD^d$, so if t is big the above inequality would imply that A could not intersect a big part of $B_r(x_0)$, or in other words,

$$\left|\{u > u(x_0)/2\} \cap B_{\theta r(x_0)}\right| \leq C t^{-\varepsilon} \left|B_r\right| \tag{6-2}$$

Let $\theta > 0$ small such that $B_{\theta r}(x_0) \subset B_1$. For every point inside $B_{\theta r}(x_0)$ we have

$$1 - ||x|| \geq 1 - |x_0| - \theta r = D(1 - \frac{\theta}{2}) \geq D\frac{1-\theta}{2}$$
$$(1 - ||x||)^{-\gamma} \leq \left(D\tfrac{1-\theta}{2}\right)^{-\gamma}$$

By inequality 6-1, we then obtain

$$u(x) \leq t(1 - ||x||)^{-\gamma} \leq \left(D\frac{1-\theta}{2}\right)^{-\gamma} \leq u(x_0)\left(\frac{1-\theta}{2}\right)^{-\gamma},$$

whenever $x \in B_{\theta r}(x_0)$. In view of the previous inequality we proceed defining the function

$$v(x) = \left(1 - \frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x).$$

At a first glance we would be tempted to apply the $\mathcal{L}^{\varepsilon}$ inequality as in lemma (5.13) to the function v. Unfortunately, v is a priori only positive in $B_{\theta r}(x_0)$. The solution for this quandary is considering the function $w(x) = v(x)^+$ and incorporating the truncation error into the right-hand side. In the next few pages, we will go through a series of delicate estimates in order to derive the result. We proceed by estimating $\mathcal{M}^- w(x)$, for $x \in B_{\frac{\theta}{2}r}(x_0)$.

$$\mathcal{M}^- w(x) = 2(2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda(w(x+y) - w(x))^+ - \Lambda(w(x+y) - w(x))^- dy}{|y|^{d+\sigma}}$$

We first start by splitting our domain in two,

$$\begin{aligned}\mathcal{M}^- w(x) =& 2(2-\sigma)\int_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(w(x+y) - w(x))^+ - \Lambda(w(x+y) - w(x))^- dy}{|y|^{d+\sigma}} + \\ & 2(2-\sigma)\int_{\mathbb{R}^d \cap \{v(x+y)\geq 0\}} \frac{\lambda(w(x+y) - w(x))^+ - \Lambda(w(x+y) - w(x))^- dy}{|y|^{d+\sigma}}\end{aligned}$$

Rewriting the definion of w,

$$\begin{aligned}\mathcal{M}^- w(x) =& 2(2-\sigma)\int_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v^+(x+y) - v^+(x))^+ - \Lambda(v^+(x+y) - v^+(x))^- dy}{|y|^{d+\sigma}} + \\ & 2(2-\sigma)\int_{\mathbb{R}^d \cap \{v(x+y)\geq 0\}} \frac{\lambda(v^+(x+y) - v^+(x))^+ - \Lambda(v^+(x+y) - v^+(x))^- dy}{|y|^{d+\sigma}}\end{aligned}$$

Since $v+(x+y)$ is null over the set $\mathbb{R}^d \cap \{v(x+y)<0\}$

$$\begin{aligned}\mathcal{M}^- w(x) =& 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(-v^+(x))^+ - \Lambda(-v^+(x))^- dy}{|y|^{d+\sigma}} + \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)\geq 0\}} \frac{\lambda(v(x+y)-v^+(x))^+ - \Lambda(v(x+y)-v^+(x))^- dy}{|y|^{d+\sigma}}\end{aligned}$$

Proceeding by summing and subtracting the quantity,

$$2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v(x+y)-v(x))^+ - \Lambda(v(x+y)-v(x))^- dy}{|y|^{d+\sigma}}$$

,

we may rewrite the above as

$$\begin{aligned}\mathcal{M}^- w(x) =& 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(-v^+(x))^+ - \Lambda(-v^+(x))^- dy}{|y|^{d+\sigma}} - \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v(x+y)-v(x))^+ - \Lambda(v(x+y)-v(x))^- dy}{|y|^{d+\sigma}} + \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v(x+y)-v(x))^+ - \Lambda(v(x+y)-v(x))^- dy}{|y|^{d+\sigma}} + \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)\geq 0\}} \frac{\lambda(v(x+y)-v^+(x))^+ - \Lambda(v(x+y)-v^+(x))^- dy}{|y|^{d+\sigma}}.\end{aligned}$$

Since the last sum may be majored by $\mathcal{M}^- v(x)$,

$$\begin{aligned}\mathcal{M}^- w(x) \leq & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(-v^+(x))^+ - \Lambda(-v^+(x))^- dy}{|y|^{d+\sigma}} - \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v(x+y)-v(x))^+ - \Lambda(v(x+y)-v(x))^- dy}{|y|^{d+\sigma}} + \mathcal{M}^- v(x)\end{aligned}$$

also it follows from the fact that $\mathcal{M}^+ u \geq 1$, that $\mathcal{M}^- v \leq 1$ in particular,

$$\begin{aligned}\mathcal{M}^- w(x) \leq 1 + & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(-v^+(x))^+ - \Lambda(-v^+(x))^- dy}{|y|^{d+\sigma}} - \\ & 2(2-\sigma) \int\limits_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\lambda(v(x+y)-v(x))^+ - \Lambda(v(x+y)-v(x))^- dy}{|y|^{d+\sigma}}\end{aligned}$$

Recalling the identity $(-v^+)^+ = v^-$ and $(-v)^- = v^+$ and discarding the negative terms we may simplify the above as

$$\mathcal{M}^- w(x) \leq 1 + 2(2-\sigma) \int_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{-\Lambda(v^+(x) - (v(x+y) - v(x))^-)dy}{|y|^{d+\sigma}}$$

Since $v(x+y)$ is negative on the domain of integration it follows that

$$-\Lambda(v^+(x) - (v(x+y) - v(x))^-) \leq -\Lambda v(x+y),$$

and as such

$$\mathcal{M}^- w(x) \leq 1 + 2(2-\sigma) \int_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{-\Lambda v(x+y)dy}{|y|^{d+\sigma}}.$$

Due to the fact that the integrand is a positive function, we may write

$$\mathcal{M}^- w(x) \leq 1 + 2(2-\sigma) \int_{\mathbb{R}^d \cap \{v(x+y)<0\}} \frac{\Lambda(-v(x+y))^+ dy}{|y|^{d+\sigma}}$$

Since the integrand is a positive function over $\mathbb{R}^d$ the above integral increases if we exchange our domain for a bigger one, so

$$\mathcal{M}^- w(x) \leq 1 + 2(2-\sigma) \int_{\mathbb{R}^d \setminus B_{\theta r/2}(x_0 - x)} \frac{\Lambda(-v(x+y))^+ dy}{|y|^{d+\sigma}}$$

Rewriting in terms of u, we obtain

$$\mathcal{M}^- w(x) \leq 1 + 2(2-\sigma) \int_{\mathbb{R}^d \setminus B_{\theta r/2}(x_0 - x)} \Lambda \frac{(u(x+y) - (1-\frac{\theta}{2})^{-\gamma} u(x_0))^+ dy}{|y|^{d+\sigma}}$$

We may not only use the fact that u is a positive function in order to obtain a bound here. In order to continue our estimates, let us consider the function $g_\tau(x) = \tau(1 - |4x|^2)^+$. There exists a maximum value of t such that $u \geq g_t$, in particular this implies that for some $x_1 \in B_{1/4}$ we have $u(x_1) = g_t(x_1)$, also, t cannot be greater than 1 since $1 = u(0) \geq g_t(0) = t$. Thus,

$$\int_{\mathbb{R}^d} \frac{\delta(u, x_1, y)^-}{|y|^{d+\sigma}} dy \leq \int_{\mathbb{R}^d} \frac{\delta(g_t, x_1, y)^-}{|y|^{d+\sigma}} dy = \int_{\mathbb{R}^d} \frac{\delta(g_t, x_1, y)^-}{|y|^{d+\sigma}} dy \leq C,$$

for a bound C which is independent of σ. Since $\mathcal{M}^{-}u(x_1) \leq 1$ we obtain

$$(2-\sigma)\int_{\mathbb{R}^d} \frac{\delta(u,x_1,y)^+}{|y|^{d+\sigma}}dy \leq \Lambda^{-1}\left((2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda\delta(u,x_1,y)^-}{|y|^{d+\sigma}}dy + \mathcal{M}^{-}u(x_1)\right) \leq C.$$

Also,

$$(2-\sigma)\int_{\mathbb{R}^d} \frac{\delta(u,x_1,y)^+}{|y|^{d+\sigma}}dy \leq (2-\sigma)\int_{\mathbb{R}^d} \frac{(u(x_1+y)-2)^+}{|y|^{d+\sigma}}dy$$

In the above we used the fact that $u(x_1) \leq 1$ and that since u is a positive function, $u(x-y) \geq 0$. Notice now, that we may suppose that $u(x_0)$ is greater than 2, if that was not the case, we would have:

$$2 \geq u(x_0) = h_t(x_0) = t(1-|x_0|)^{-\gamma},$$

which in turn would imply

$$t \leq 2(1-|x_0|)^{\gamma},$$

and we would finish the proof. So, assuming that $u(x_0) \geq 2$ we may conclude that

$$(2-\sigma)\int_{\mathbb{R}^d} \frac{\left(u(x_1+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0)\right)^+}{|y|^{d+\sigma}}dy \leq (2-\sigma)\int_{\mathbb{R}^d} \frac{(u(x_1+y)-2)^+}{|y|^{d+\sigma}}dy \leq C$$

where we used the fact that $u(x_0) \geq 2$ implies $(1-\frac{\theta}{2})^{-\gamma}u(x_0) \geq 2$. And with the above estimate we are finally able to bound $\mathcal{M}^{-}w(x)$. Recall that,

$$\mathcal{M}^{-}w(x) \leq 1 + 2(2-\sigma)\int_{\mathbb{R}^d \setminus B_{\theta r/2}(x_0-x)} \Lambda\frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+ dy}{|y|^{d+\sigma}}$$

If we rewrite the above inequality we obtain that $\mathcal{M}^{-}w(x)$ is less than or equal to:

$$1+2(2-\sigma)\int_{\mathbb{R}^d \setminus B_{\theta r/2}(x_0-x)} \Lambda\frac{(u(x_1+x+y-x_1)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y+x-x_1|^{d+\sigma}}\frac{|y+x-x_1|^{d+\sigma}}{|y|^{d+\sigma}}dy$$

Notice that since $x \in B_{\theta r}(x_0)$ we have that for some $\xi < 1$, $|x-x_0| = \xi\frac{\theta r}{2}$, therefore $B_{\eta\theta r} \subset B_{\theta r/2}(x-x_0)$ whenever $\eta < 1-\xi/2$. This follows from the computation: Let $z \in B_{\eta r}$, then $|x-x_0-z| \leq |x-x_0|+|z| \leq \xi\frac{\theta r}{2}+\eta\theta r \leq \frac{\theta r}{2}$.

Hence we may estimate the value of $\mathcal{M}^{-}w(x)$ being less than or equal to:,

$$1+2(2-\sigma)\left(1+\frac{1/6}{\eta\theta r}\right)^{d+\sigma}\int\limits_{\mathbb{R}^d\setminus B_{\theta r/2}(x_0-x)}\Lambda\frac{(u(x_1+x+y-x_1)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|y+x-x_1|^{d+\sigma}}dy$$

Doing a change of variables $z=y+x-x_1$, we obtain:

$$\mathcal{M}^{-}w(x)\leq 1+2(2-\sigma)\left(1+\frac{1/6}{\eta\theta r}\right)^{d+\sigma}\int\limits_{\mathbb{R}^d\setminus B_{\theta r/2}(x_0-x)}\Lambda\frac{(u(x_1+z)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))^+}{|z|^{d+\sigma}}dz$$

where the last is integral is bounded due to the last calculations, hence for small r,

$$\mathcal{M}^{-}w(x)\leq C(\theta r)^{-d-\sigma}.$$

$$\left|\{u<u(x_0)/2\}\cap B_{\theta r/4(x_0)}\right|=\left|\left\{w>u(x_0)\left(\left(1-\frac{\theta}{2}\right)^{-\gamma}-\frac{1}{2}\right)\right\}\cap B_{\theta r/4(x_0)}\right|$$

Applying lemma (5.13) to a translated version of the function w, we obtain,

$$\begin{gathered}\left|\left\{w>u(x_0)\left(\left(1-\tfrac{\theta}{2}\right)^{-\gamma}-\tfrac{1}{2}\right)\right\}\cap B_{\theta r/4(x_0)}\right|\leq\\ C\left(\tfrac{\theta r}{4}\right)^d\left(\left(\left(1-\tfrac{\theta}{2}\right)^{-\gamma}-1\right)u(x_0)+C(\theta r)^{-d-\sigma}\left(\tfrac{\theta r}{4}\right)^{\sigma}\right)^{\varepsilon}\left(u(x_0)\left(1-\tfrac{\theta}{2}\right)^{-\gamma}-\tfrac{1}{2}\right)^{-\varepsilon}\leq\\ C\left(\tfrac{\theta r}{4}\right)^d\left(\left(\left(1-\tfrac{\theta}{2}\right)^{-\gamma}-1\right)^{\varepsilon}+\tfrac{\theta^{-d\varepsilon}}{t^{\varepsilon}}\right).\end{gathered}$$

Now, choosing θ sufficiently close to 0, and then, t sufficiently large we would obtain:

$$C\left(\frac{\theta r}{4}\right)^d\left(\left(1-\frac{\theta}{2}\right)^{-\gamma}-1\right)^{\varepsilon}\leq\frac{1}{4}\left|B_{\theta r/4}\right|$$

and,

$$C\left(\frac{\theta r}{4}\right)^d\left(\frac{\theta^{-d\varepsilon}}{t^{\varepsilon}}\right)\leq\frac{1}{4}\left|B_{\theta r/4}\right|.$$

This in turn would imply that

$$\left|\{u<u(x_0)/2\}\cap B_{\theta r/4(x_0)}\right|\leq\frac{1}{2}\left|B_{\theta r/4}\right|$$

But for t sufficiently large would imply then that

$$\left|\{u>u(x_0)/2\}\cap B_{\theta r/4(x_0)}\right|\geq c\left|B_r\right|$$

And that is a direct contradiction with the bound in (6-2). ■

7
Hölder Estimates

With the results previously proven we will produce a regularity result, in particular we are going to demonstrate how any bounded function, whose Pucci extremal operators are also uniformly bounded in B_1 must be an alpha-Hölder continuous function at the origin.

Lemma 7.1 *Let $\sigma > \sigma_0 > 0$. Let u be a function such that*

$$-1/2 \leq u \leq 1/2 \ \textit{in}\ \mathbb{R}^d,$$
$$\mathcal{M}^+ u \geq -\varepsilon_0 \ \textit{in}\ B_1,$$
$$\mathcal{M}^- u \leq \varepsilon_0 \ \textit{in}\ B_1;$$

Proof: The idea behind the proof is that we will create two sequences m_k, M_k such that

$$m_k \leq u \leq M_k \text{ in } B_{4^{-k}},$$
$$M_k - m_k = 4^{-\alpha k} \text{ for some positive } \alpha.$$

If we create such a pair of sequences then given x in a neighborhood of the origin, then for some natural number k we would have $x \in B_{4^{-k}}, x \notin B_{4^{-k-1}}$, hence,

$$u(x) - u(0) \leq M_k - m_k = 4^{-\alpha k} \leq 4^{\alpha}|x|^{\alpha}$$
$$u(x) - u(0) \geq m_k - M_k = -4^{-\alpha k} \geq -4^{\alpha}|x|^{\alpha}$$

This of course gives
$$|u(x) - u(0)| \leq 4^{\alpha}|x|^{\alpha}$$

as desired, and we would finish the proof, therefore it only remains to construct these sequences, we will proceed by induction. For $k = 0$ we may set $M_0 = 1/2$ and $m_0 = -1/2$. Suppose we have created our pair of sequences up to m_k, M_k, we will now show that it is possible to construct the next elements m_{k+1}, M_{k+1}. Notice that in $B_{4^{-k-1}}$ the function u is either greater or smaller than the average of M_k, m_k for at least half of the space, in a measure sense. In other words either $\left|\left\{w \geq \frac{m_k+M_k}{2}\right\} \cap B_{4^{-k-1}}\right| \geq \frac{1}{2} \mid B_{4^{-k-1}}|$ or $\left|\left\{w \leq \frac{m_k+M_k}{2}\right\} \cap B_{4^{-k-1}}\right| \geq \frac{1}{2} \mid B_{4^{-k-1}}|$. Let us suppose that the first case is true. Define the function

$$v(x) = 2\frac{u(4^{-k}x) - m_k}{M_k - m_k}$$

Due to the inductive hypothesis $v \geq 0$ in B_1, since $m_k \leq u$ in $B_{4^{-k}}$, also it follows from $v \geq 1$ implies $u(4^{-k}x) \geq \frac{m_k+M_k}{2}$, that $\left|\{v \geq 1\} \cap B_{1/4}\right| \geq \frac{1}{2} \left| B_{1/4}\right|$.

We proceed by estimating $\mathcal{M}^{-}v(x)$.

$$\mathcal{M}^{-}v(x) = \frac{4^{-k\sigma}\mathcal{M}^{-}u(4^{-k}x)}{(M_k - m_k)/2} \leq \frac{2\varepsilon_0 4^{-k\sigma}}{4^{-\alpha k}} \leq 2\varepsilon_0$$

if we choose $\alpha < \sigma$. Let j be a positive integer smaller than k. Then for every $x \in B_{4^j}$,

$$v(x) \geq 2\frac{m_{k-j} - m_k}{M_k - m_k} \geq 2\frac{M_{k_j} - M_k + m_{k-j} - m_k}{M_k - m_k} \geq 2(1 - 4^{\alpha j}).$$

Hence, for x outside B_1 ,

$$v(x) \geq -2(|4x|^{\alpha} - 1).$$

Let $w = v^{+}$, then as in the proof of theorem (6.1) we obtain, for $x \in B_{3/4}, \alpha$ small enough.

$$\mathcal{M}^{-}w(x) \leq M^{-}v(x) + 2\varepsilon_0.$$

Let z be an arbitrary point in $B_{1/4}$. Applying lemma (5.13) to a translated version of w, we obtain:

$$\left|\{w > 1\} \cap B_{1/2}(z)\right| \leq C(w(z) + 2\varepsilon_0)^{\varepsilon}.$$

Notice that $B_{1/4} \subset B_{1/2}(z) \subset B_{3/4}$, therefore

$$\left|\{w > 1\} \cap B_{1/2}(z)\right| \geq \left|\{w \geq 1\} \cap B_{1/4}\right| = \left|\{v \geq 1\} \cap B_{1/4}\right| \geq \frac{1}{2} \left| B_{1/4}\right|.$$

Combining the two inequalities above we conclude

$$\frac{1}{2} \left| B_{1/4}\right| \leq C(w(z) + 2\varepsilon_0)^{\varepsilon}.$$

Now, if ε_0 is small enough we conclude that w is a uniformly positive function in $B_{1/4}$, i.e, exists some $\theta > 0$ such that $w(x) > \theta > 0$ for every $x \in B_{1/4}$. Now it only remains to define $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta(M_k - m_k)/2$ and verify that this choice satisfies our requirements. Observe that $w(x) > \theta > 0$ for every $x \in B_{1/4}$ implies that $w(x) = v(x)$ for every $x \in B_{1/4}$.

$$v(x) \geq \theta \text{ in } B_{1/4} \text{ implies } \frac{u(4^{-k}x) - m_k}{(M_k - m_k)/2} \geq \theta$$

hence for $x \in B_{1/4}$,

$$u(4^{-k}x) \geq \theta(M_k - m_k)/2 + m_k = m_{k+1}$$

Or in other words $u \geq m_{k+1}$ in $B_{4^{-k+1}}$. The upper bound is directly from the hypothesis since $M_k = M_0 \geq u$. To finish the construction we choose α, θ small enough such that $\left(1 - \frac{\theta}{2}\right) = 4^{-\alpha}$. Thus, $M_{k+1} - m_{k+1} = 4^{-\alpha(k+1)}$. If we had at the start supposed that $\left|\left\{w \leq \frac{m_k + M_k}{2}\right\} \cap B_{4^{-k-1}}\right| \geq \frac{1}{2} \mid B_{4^{-k-1}} \mid$ we would still be able to follow the same general idea just changing our function v for

$$v(x) = \frac{M_k - u(4^{-k}x)}{(M_k - m_k)/2}$$

end using the bound $\mathcal{M}^+ u \geq -\varepsilon_0$. ■

Theorem 7.2 *Let $\sigma > \sigma_0 > 0$. Let u be a bounded function in $\mathbb{R}^d$ such that*

$$\begin{aligned} \mathcal{M}^+ u &\geq -C_0 \text{ in } B_1, \\ \mathcal{M}^- u &\leq C_0 \text{ in } B_1; \end{aligned}$$

then there is an $\alpha = \alpha(\lambda, \Lambda, d, \sigma_0) > 0$ such that $u \in \mathcal{C}^\alpha(B_{1/2})$ and

$$|u|_{\mathcal{C}^\alpha(B_{1/2})} \leq C(\sup_{\mathbb{R}^d} |u| + C_0)$$

for a positive constant C.

Proof: The result will follow straightly from a scaling argument. Let x_0 be an arbitrary point in $B_{1/2}$ Consider the function

$$v(x) = u(x/2 + x_0) \frac{\min(1, \varepsilon_0)}{2 \max(C_0, |u|_\infty)}$$

An easy calculation shows.

$$|v(x)| \leq |u(x/2 + x_0)| \frac{\min(1, \varepsilon_0)}{2 \max(C_0, |u|_\infty)} \leq |u|_\infty \frac{\min(1, \varepsilon_0)}{2 \max(C_0, |u|_\infty)} \leq \frac{1}{2},$$

also for $x \in B_1$,

$$\mathcal{M}^- v(x) = M^- u(x/2 + x_0)|\frac{\min(1, \varepsilon_0)}{2^{\sigma+1}\max(C_0, |u|_\infty)} \leq C_0 \frac{\min(1, \varepsilon_0)}{2\max(C_0, |u|_\infty)} \leq \varepsilon_0.$$

and analogously,

$$\mathcal{M}^+ v(x) = M^+ u(x/2 + x_0)|\frac{\min(1, \varepsilon_0)}{2^{\sigma+1}\max(C_0, |u|_\infty)} \geq -C_0 \frac{\min(1, \varepsilon_0)}{2\max(C_0, |u|_\infty)} \geq -\varepsilon_0.$$

Hence, we may apply the previous lemma to v and obtain:

$$|v(x) - v(0)| \leq C|x|^\alpha$$

but equivalently we may write

$$|u(x/2 + x_0) - u(x_0)|\frac{\min(1, \varepsilon_0)}{2\max(C_0, |u|_\infty)} \leq C|x|^\alpha$$

therefore, since x_0 is arbitrary we obtain

$$|u|_{\mathcal{C}^\alpha(B_{1/2})} \leq \frac{2\max(C_0, |u|_\infty)}{\min(1, \varepsilon_0)} + |u|_\infty \leq C(\sup_{\mathbb{R}^d} |u| + C_0)$$

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